COMPLEX NUMBERS – WORKSHEET #4

COURSE/LEVEL

NSW Secondary High School Year 12 HSC Mathematics Extension 2.

TOPIC

Complex Numbers: Curves and regions. (Syllabus Ref: 2.5)

1. Sketch the locus of the point P representing the complex number z, on an Argand diagram, for each of the following:

   (a) $\operatorname{Re}(z) = 2$

   (b) $\operatorname{Re}(z) = \operatorname{Im}(z)$

   (c) $|z-1-3i| = 2$

   (d) $|z-1| = |z+2-i|$

   (e) $\arg(z-i) = \frac{\pi}{4}$

   (f) $\arg\left(\frac{z-i}{z-2}\right) = \frac{\pi}{3}$

   (g) $|z-2| = \operatorname{Re}(z)$

   (h) $|z-2| + |z+2| = 6$

   (i) $|z-2| - |z+2| = 2$

   (j) $|z-3| = 2|z|$

2. Sketch the locus of z if $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| = 1$.

3. If $(z-\bar{z})^2 + 8(z+\bar{z}) = 16$ show that the locus of z is a parabola.

   (i) Sketch the parabola

   (ii) Show that $\frac{-\pi}{4} \le \arg z \le \frac{\pi}{4}$

4. Sketch the locus of the points P representing z in each case:

   (a) $1 < \operatorname{Re}(z) < 2$

   (b) $\left|\frac{z-1}{z+1}\right| \le 1$

   (c) $1 \le |z+2+i| \le 2$

   (d) $|z-1| < |z-i|$

   (e) $\operatorname{Im}\left(\frac{1}{z}\right) < \frac{-1}{2}$

   (f) $(1-i)\bar{z} = (1+i)z$

   (g) $|z-1| = |z+2| = |z-i|$

   (h) $|z+2i| = \operatorname{Re}(z) + 2i$

   (i) $\frac{\pi}{3} < \arg(z-i) \le \frac{\pi}{2}$

   (j) $\frac{1}{4} < \operatorname{Re}\left(\frac{1}{z}\right) + \operatorname{Im}\left(\frac{1}{z}\right) < \frac{1}{2}$

5. If z satisfies $|z+3i|^2 - |z-3i|^2 = 12$, prove that the locus is the line $y = 1$.

6. If z satisfies $|z+ik|^2 + |z-ik|^2 = 10k^2,\ k > 0$, prove that the locus of z is a circle, centre 0, radius $2k$.

7. P_1, P_2, P_3 represent the complex numbers z_1, z_2, z_3 respectively where $z_1 z_3 = z_2{}^2$. Show geometrically that OP_2 bisects the angle P_1OP_3.

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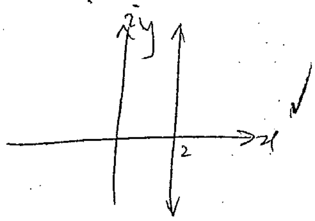
Tricia.

Complex Numbers – Worksheet #4.

1a. $Re(z) = 2$

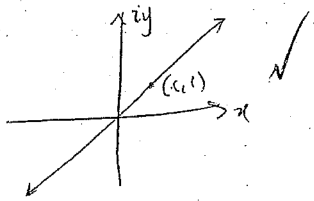
$Re(x + iy) = 2$

$\therefore x = 2$



b. $Re(z) = Im(z)$

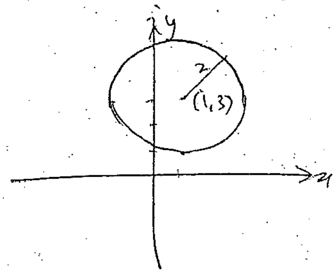
$\therefore x = y$



c. $|z - (1+3i)| = 2$

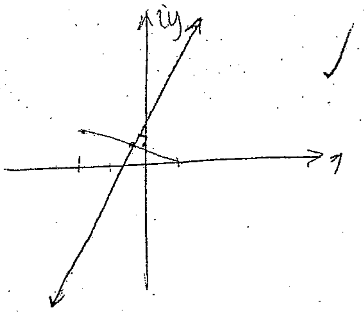
centre $(1, 3)$

radius $= 2$

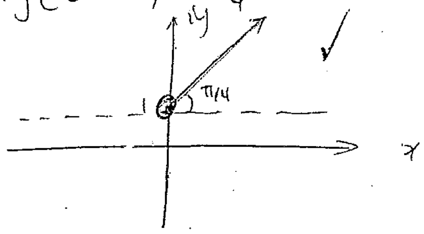


d. $|z - 1| = |z + 2 - i|$

$|z - 1| = |z - (-2+i)|$

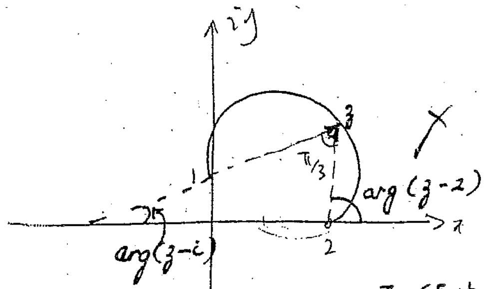


e. $\arg(z - i) = \frac{\pi}{4}$



f. $\arg\left(\frac{z-i}{z-2}\right) = \frac{\pi}{3}$

$= \arg(z-i) - \arg(z-2) = \frac{\pi}{3}$



$\therefore \arg(z-2) = \arg(z-i) + \frac{\pi}{3}$ (Ext. ∠ of Δ)

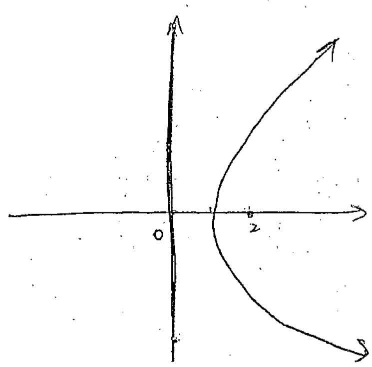
$\therefore \arg(z-2) - \arg(z-i) = \frac{\pi}{3}$ (Not the same as above

∴ Minor arc.

$\frac{\pi}{3} > 0$, ∴ arc is ~~above~~ below. interval

θ is acute, ∴ ~~major~~ minor arc

g. $|z - 2| = Re(z)$ Need to show that the locus is a parabola i.e. $y^2 = 4x - 4$



j. $|z-3| = 2|z|$

~~x+iy~~

let $z = x+iy$

$|x+iy-3| = 2|x+iy|$

$\cancel{((x-3)+iy)^2 = 4(x+iy)^2}$

$\sqrt{(x-3)^2+y^2} = \sqrt{x^2+y^2}$

$(x-3)^2+y^2 = 4(x^2+y^2)$ ✓

$x^2-6x+9+y^2-4x^2-4y^2=0$

$-3x^2-3y^2-6x+9=0$ ✓

$\therefore 0 = 3x^2+6x-9+3y^2$

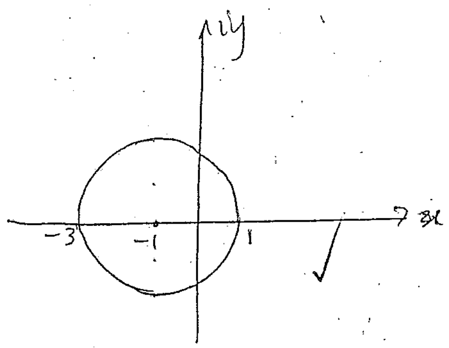
$= x^2+2x-3+y^2$

$x^2+2x+1^2+y^2 = 3+1$

$(x+1)^2+y^2 = 4$ ✓

$\therefore$ centre $(-1,0)$

radius $= 2$



✓

2. $|Re(z)| + |Im(z)| = 1$

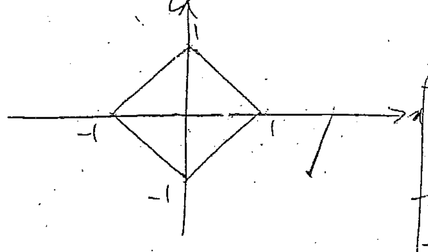
let $x+iy = z$

$|Re(x+iy)| + |Im(x+iy)| = 1$

$|x| + |y| = 1$ ✓

$\cancel{\sqrt{x^2} + \sqrt{y^2} = 1}$

$\cancel{y = -x+1}$



✓

$x+y=1$

$x-y=1$

$-x+y=1$

$-x-y=1$

$-1 \le x \le 1$

$-1 \le y \le 1$

3. $(z-\bar{z})^2 + 8(z+\bar{z}) = 16$

let $z = x+iy$ ✓

$(x+iy-x+iy)^2 + 8(x+iy+x-iy) = 16$

$(2iy)^2 + 8(2x) = 16$

$-4y^2 + 16x = 16$ ✓

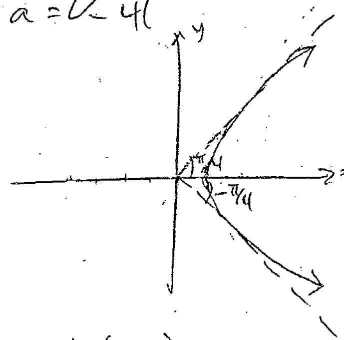
$-y^2+4x = 4$

$y^2 = 4x-4$

which is a parabola.

i) $\cancel{y^2 = -16x = 4ax, \ a = -4}$ $\quad y^2 = 4(x-1)$

$\therefore a = 1$



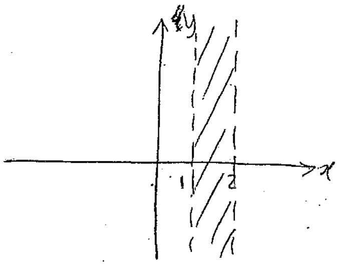
✓

ii) $\therefore$ parabola is between this range ✓

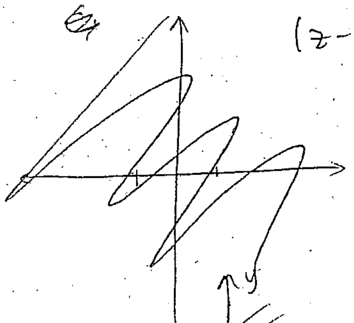
4a. $1 < Re(z) < 2$

let $z = x + iy$

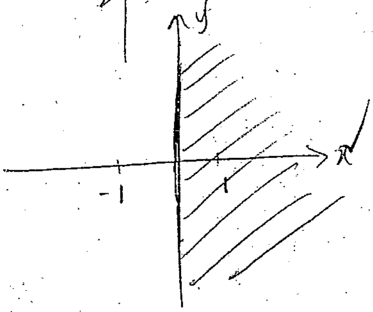
$1 < x < 2$



b. $\left| \frac{z-1}{z+1} \right| \leq 1$

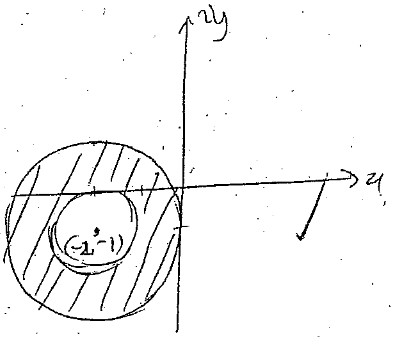


$|z-1| \leq |z+1|$

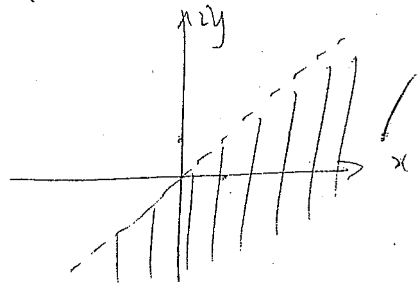


c. $1 \leq |z+2+i| \leq 2$

$1 \leq |z-(-2-i)| \leq 2$



d. $|z-1| < |z-i|$



e. $Im\left(\frac{1}{z}\right) < -\frac{1}{2}$

$Im\left(\frac{1}{x+iy}\right) < -\frac{1}{2}$

$\frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$= \frac{x-iy}{x^2+y^2}$

$\therefore Im\left(\frac{1}{x+iy}\right) = \frac{-y}{x^2+y^2}$

$\frac{-y}{x^2+y^2} < -\frac{1}{2}$

$\frac{y}{x^2+y^2} > \frac{1}{2}$

$2y > x^2+y^2$

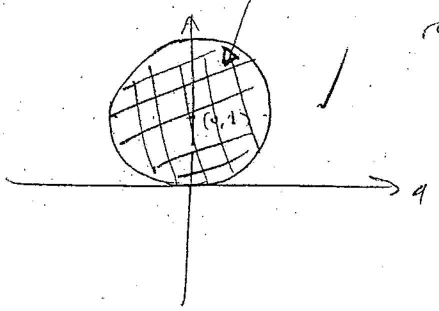
$x^2+y^2-2y < 0$

$x^2+y^2-2y+(-1)^2 < 1$

$x^2+(y-1)^2 < 1$

centre $(0,1)$

radius $= 1$



f. $(1-i)\bar{z} = (1+i)z$

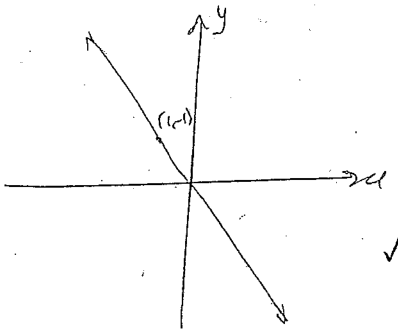
let $x+iy = z$

$(1-i)(x-iy) = (1+i)(x+iy)$

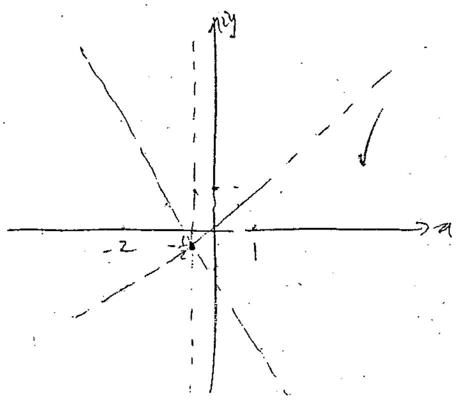
$\cancel{x} - iy - ix - \cancel{y} = \cancel{x} + ix + iy - \cancel{y}$

$0 = 2ix + 2iy$

$= x+y$



g. $|z-1| = |z+2| = |z-i|$



$\therefore$ ~~|z+2| = |z-i|~~

$|z-1| = |z+2|$

$|z+2| = |z-i|$

$|z-1| = |z-i|$

pt of intersection is the locus.

h. $|z+2i| = Re(z) + 2i$

let $x+iy = z$

$|x+iy+2i| = x+2i$

$\sqrt{x^2+(y+2)^2} = x+2i$

$x^2+(y+2)^2 = (x+2i)^2$

$\cancel{x^2}+y^2+4y+4 = \cancel{x^2}+4ix-4$

$y^2+4y+8-4ix = 0$

Equate real & im. parts

$y^2+4y+8=0$

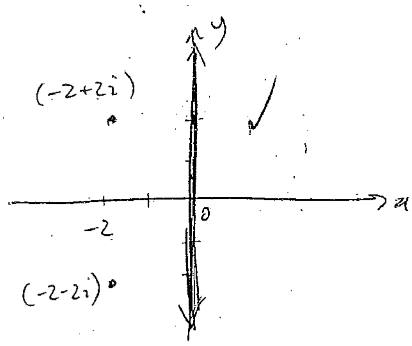
$-4x = 0$

$\therefore x = 0$

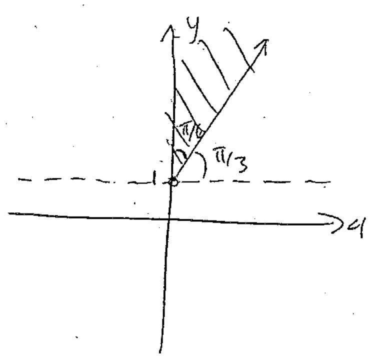
$y = \dfrac{-4\pm\sqrt{4^2-4(8)}}{2}$

$= \dfrac{-4\pm4i}{2}$

$= -2\pm2i$



i. $\frac{\pi}{3} < \arg(z-i) \le \frac{\pi}{2}$



j. $\frac{1}{4} < Re\left(\overline{\frac{1}{z}}\right) + Im\left(\overline{\frac{1}{z}}\right) < \frac{1}{2}$

let $z = x + iy$

$\frac{1}{4} < Re\left(\overline{\frac{1}{x+iy}}\right) + Im\left(\overline{\frac{1}{x+iy}}\right) < \frac{1}{2}$

$\overline{\frac{1}{x+iy} \times \frac{x-iy}{x-iy}}$

$= \overline{\frac{x-iy}{x^2+y^2}}$

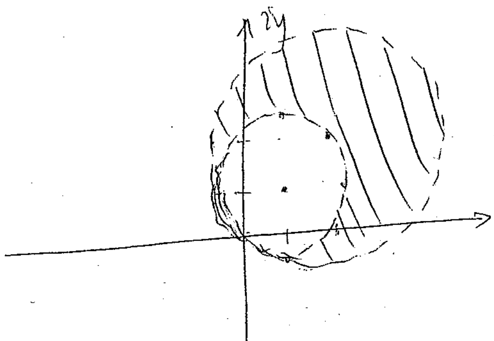
$= \overline{\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}}$

$\therefore \cancel{\frac{1}{4}zz} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$

$\therefore \frac{1}{4} < \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2} < \frac{1}{2}$

$\frac{1}{4}(x^2+y^2) < x+y < \frac{1}{2}(x^2+y^2)$

$(x^2+y^2) < 4x+4y < 2x^2+2y^2$



① $x^2+y^2 < 4x+4y$

$x^2-4x+y^2-4y < 0$

$\cancel{x^2-4x}$

$x^2-4x+(2)^2+y^2-4y+(-2)^2 < 8$

$(x-2)^2+(y-2)^2 < 8$

② $2x+2y < x^2+y^2$

$0 < x^2-2x+y^2-2y$

$2 < (x-1)^2+(y-1)^2$

5. let $z = x + iy$

$$|x+2y+3i|^2 - |x+iy-3i|^2 = 12$$

$$\left(\sqrt{x^2+(y+3)^2}\right)^2 - \left(\sqrt{x^2+(y-3)^2}\right)^2 = 12$$ ✓

$$x^2+(y+3)^2 - x^2 - (y-3)^2 = 12$$

$$\cancel{x^2} + \cancel{y^2} + 6y + 9 - \cancel{x^2} - \cancel{y^2} + 6y - 9 = 12$$

$$12y \cancel{4} = 12$$ ✓

$$\therefore y = 1$$

6. let $z = x + iy$

$$|x+iy+ik|^2 + |x+iy-ik|^2 = 10k^2$$

$$x^2+(y+k)^2 + x^2 + (y-k)^2 = 10k^2$$ ✓

$$x^2+y^2+2yk+k^2+x^2+y^2-2yk+k^2 = 10k^2$$

$$2x^2+2y^2+2k^2 = 10k^2$$

$$2x^2+2y^2-8k^2 = 0$$

$$x^2+y^2-4k^2 = 0$$ ✓

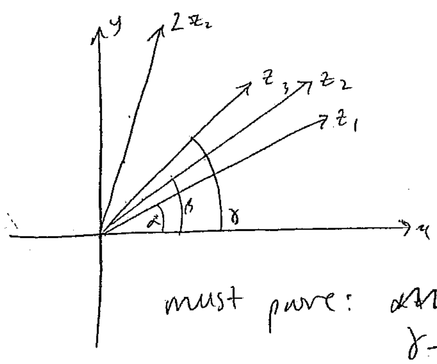
$$x^2+y^2 = 4k^2$$

$\therefore$ centre $(0,0)$ ✓

radius $= 2k$

7. $\arg(z_1 z_3) = \arg(z_2)^2$

$\arg z_1 + \arg z_3 = 2\arg z_2$ ✓



must prove: $\gamma - \beta = \beta - \alpha$

but $\alpha + \gamma = 2\beta$ ✓

$\alpha + \gamma - \beta = \beta$

$\gamma - \beta = \beta - \gamma$