COURSE/LEVEL
NSW Secondary High School Year 12 HSC Extension 2 Mathematics.

HSC TRIAL EXAMINATION

MATHEMATICS

Extension 2

Time allowed - Three hours

DIRECTIONS

- Attempt ALL questions
- EACH question is out of 15 marks
- All necessary working should be shown. Marks may be deducted for careless or poorly arranged work
- Start each question on a new page

QUESTION 1.

(a) *(3 marks)*

Find $\int \sin^3 x\,dx$

(b) *(4 marks)*

Using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, or otherwise, show that

$$\int_0^{\pi/2} \frac{1}{1+\sin\theta}\,d\theta = 1.$$

(c) *(4 marks)*

Evaluate $\int_0^1 \tan^{-1} x\,dx$

(d) *(4 marks)*

(i) Express

$$\frac{3-x}{\left(1+2x^2\right)\left(1+6x\right)}$$

in partial fractions.

(ii) Show that

$$\int_0^2 \frac{3-x}{\left(1+2x^2\right)\left(1+6x\right)}\,dx = \frac{1}{2}\ln\left(\frac{13}{3}\right).$$

QUESTION 2.

(a) *(3 marks)*

Given that $(2+3i)p-q=1+2i$, find p and q if

(i) p and q are real

(ii) p and q are complex conjugate numbers

(b) *(3 marks)*

If $z=\cos\theta+i\sin\theta$, show that

$$\frac{1}{1+z}=\frac{1}{2}\left(1-i\tan\frac{\theta}{2}\right)$$

(c) *(4 marks)*

(i) On an Argand diagram, shade in the region for which

$$0\le|z|\le 2 \quad \text{and} \quad 1\le \operatorname{Im} z\le 2$$

(ii) Write down the complex number with largest argument that satisfies the inequalities of (i). Express your answer in the form $a+ib$.

(d) *(5 marks)*

(i) Find the two square roots of $5-12i$ in the form $x+iy$ where x and y are real.

(ii) Show the points P and Q representing the square roots on an Argand diagram. Find the complex numbers represented by points R_1, R_2 such that the triangles PQR_1 and PQR_2 are equilateral.

QUESTION 3.

(a) *(5 marks)*

The rate of change, with respect to x, of the gradient of a curve is constant and the curve passes through the points $(1,2)$ and $(-3,0)$, the gradient at the former point being $-\tfrac{1}{2}$. Find the equation of the curve and sketch the curve.

(b) *(10 marks)*

For the ellipse $x^2+4y^2=100$,

(i) Write down the eccentricity, the co-ordinates of the foci and the equations of the directrices.

(ii) Sketch a graph of the ellipse showing the above features.

(iii) Find the equation of the tangent and normal to the ellipse at the point $P(8,3)$.

(iv) If the normal at P meets the major axis at G and the perpendicular from the centre O to the tangent at P meets that tangent at K, prove that $PG.OK$ is equal to the square of the minor semi-axis.

QUESTION 4.

(a) *(6 marks)*

(i) If $P(x)=x^3-9x^2+24x+c$ for some real number c, find the values of x for which $P'(x)=0$. Hence find the two values of c for which the equation $P(x)=0$ has a repeated root.

(ii) Sketch the graphs of $y=P(x)$ for these values of c. Hence write down the values of c for which the equation $P(x)=0$ has three distinct real roots.

(b) *(6 marks)*

Let $f(x)=x-2+\dfrac{3}{x+2}$.

(i) Find the points at which $f(x)=0$.

(ii) Find the turning points of $f(x)$, if any, and identify them.

(iii) Find the asymptotes.

(iv) Sketch the curve, marking all the features you have found in parts (i) - (iii) above.

(c) *(3 marks)*

The polynomial $x^3+x^2+3x-2=0$ has roots α, β and γ. Find the equation with roots $\alpha^2\beta\gamma$, $\alpha\beta^2\gamma$ and $\alpha\beta\gamma^2$.

QUESTION 5. *(15 marks)*

A particle of mass m is projected vertically upwards under gravity in a medium which exerts a resisting force of magnitude $mg(v/k)^2$, where v is the speed of the particle and k is a constant.

(i) For the upward motion of the particle, draw a diagram showing the forces acting on the particle and write down the equation of motion.

(ii) If U is the speed of projection, show that the greatest height of the particle above the point of projection is

$$\frac{k^2}{2g}\ln\left(\frac{k^2+U^2}{k^2}\right).$$

(iii) Repeat part (i) for the downward motion of the particle and hence write down the particle's terminal velocity.

(iv) If V is the speed of the particle on returning to the point of projection, show that

$$\frac{1}{V^2}-\frac{1}{U^2}=\frac{1}{k^2}.$$

QUESTION 6.

(a) *(3 marks)*

Let $\min(a,b)$ denote the minimum of the numbers a and b. Sketch the function $y=\min(2,x)$ over the interval $0\le x\le 3$ and evaluate $\int_0^3 \min(2,x)\,dx$.

(b) *(3 marks)*

Find the area enclosed between the curves $y=x^3$ and $y^3=16x$.

(c) *(9 marks)*

(i) Sketch the curves $y=\tan x$ and $y=2\cos\left(x+\frac{\pi}{12}\right)$ between $x=0$ and $x=\frac{\pi}{2}$

(ii) Verify that $x=\frac{\pi}{4}$ is a solution of the equation $\tan x-2\cos\left(x+\frac{\pi}{12}\right)=0$.

(iii) Find the area enclosed by these curves and the y-axis.

(iv) If this area is rotated through one revolution about the x-axis, find the volume of the solid formed.

QUESTION 7.

(a) *(7 marks)*

Two circles intersect at A and B. The tangents from a point on BA produced meet the circles at P and Q.

If P, A and Q are collinear,

(i) Draw a diagram showing this information.

(ii) Prove that $\Delta TAP \;|||\; \Delta TBP$ and $\Delta TAQ \;|||\; \Delta TBQ$.

(iii) Prove that T, Q, B, P are concyclic.

(iv) Prove that $TP=TQ$.

(b) (*8 marks*)

For a given integer $n \geq 1$, let the positive integers $c_0, c_1, \ldots, c_n$ be defined by the equation, valid for all (real and) complex numbers *z*:

$$(1+z)^n = c_0 + c_1 z + \ldots + c_n z^n.$$

(You are **not** required to establish this identity.)

Prove that

(i) $c_0 = 1$,

(ii) $c_0 - c_1 + c_2 - c_3 + \ldots + (-1)^n c_n = 0$,

(iii) if *n* is odd then $c_1 + c_3 + \ldots + c_{n-2} + c_n = 2^{n-1}$,

(iv) if n is divisible by 4 then $c_0 - c_2 + c_4 - \ldots - c_{n-2} + c_n = (-1)^{n/4} 2^{n/2}$.

QUESTION 8.

(a) (2 marks)

If the functions $f(x)$ and $g(x)$ are such that $f(x) > g(x) \geq 0$ for $a \leq x \leq b$, by using a sketch (or otherwise) explain why $\int_a^b f(x)dx > \int_a^b g(x)dx.$

(b) (13 marks)

Let

$$u_n = \int_0^1 \left(1-t^2\right)^{(n-1)/2} dt$$

where *n* is a non-negative integer.

(i) Using integration by parts, or otherwise, show that $n u_n = (n-1)u_{n-2}$ if $n \geq 2$.

(ii) Let $v_n = n u_n u_{n-1}$, $n \geq 1$. Show that $v_n = \frac{1}{2}\pi$, for all values of $n \geq 1$.

(iii) Using part (a), or otherwise, show that $0 < u_n < u_{n-1}$. Prove that

$$\sqrt{\frac{\pi}{2n+2}} < u_n < \sqrt{\frac{\pi}{2n}}$$

1.(a) $\int \sin^3 x \cdot dx$

$= \int \sin x (1-\cos^2 x) \cdot dx$

$=$ Let $u =$ ~~[illegible]~~ $\cos x$

$\frac{du}{dx} = -\sin x$ ✓

$= -\int du (1-u^2)$

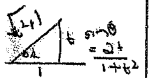
$= \int u^2 - 1 \, du$

$= \frac{u^3}{3} - u + C$ ✓

$= \frac{\cos^3 x}{3} - \cos x + C$

(b). $\int_0^{\pi/2} \frac{1}{1+\sin\theta} \cdot d\theta$

Let $t = \tan\theta/2$



$\sin\theta = \frac{2t}{1+t^2}$

$\frac{dt}{d\theta} = \frac{1}{2}(1+t^2)$

$\frac{2dt}{1+t^2} = d\theta$

At $\theta = \pi/2, t = 1$ ✓

$\theta = 0, t = 0$.

$I = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$

$= \int_0^1 \frac{2dt}{1+t^2+2t}$ ✓

$= \int_0^1 \frac{2dt}{(1+t)^2}$

$= 2\int_0^1 (1+t)^{-2} \cdot dt$

$= -2\left[(1+t^2)^{-1}\right]_0^1$

$= -2\left[(2)^{-1} - (1)^{-1}\right]$

$= -2\left[\frac{1}{2} - 1\right]$ ✓

$= -2 \times -\frac{1}{2}$

$= 1.$ ✓

(c). $\int_0^1 \tan^{-1} x \cdot dx$

Let $u = \tan^{-1} x$ $\quad v = x$

$u' = \frac{1}{1+x^2}$ $\quad v' = 1$ ✓

$I = \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$

$= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2} \cdot dx$

$= \frac{\pi}{4} - \left[\frac{1}{2} \ln |1+x^2|\right]_0^1$ ✓

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

(d)(i) $\frac{3-x}{(1+2x^2)(1+6x)} = \frac{Ax+B}{2x^2+1} + \frac{C}{1+6x}$

$3-x = (Ax+B)(6x+1) + C(2x^2+1)$

compare x^2

$0 = 6A + 2C$

compare constants

$3 = B + C$.

Let $x = \frac{1}{-6}$

$3\frac{1}{6} = C\frac{19}{18}$.

$C = 3$ ✓

$\therefore A = -1$ ✓

$B = 0$. ✓

$I = \int_0^2 \frac{-x}{2x^2+1} + \frac{3}{1+6x} dx$

$= \frac{-1}{4}\int \frac{4x}{2x^2+1} + \frac{1}{2}\int \frac{6}{1+6x}$

$= \left[\frac{-1}{4} \ln |2x^2+1| + \frac{1}{2} \ln |1+6x|\right]_0^2$

~~$= \left[\frac{1}{2} \ln \left|\frac{1+6x}{x^2+1}\right|\right]_0^2$~~

~~$= \frac{1}{2} \ln \left|\frac{13}{5}\right| - \frac{1}{2} \ln 0.$~~

~~$= \frac{1}{2} \ln \left|\frac{13}{5}\right|$~~

$= \left[-\frac{1}{4} \ln 9 + \frac{1}{2} \ln 13\right] - \left[0\right]$

$= \left[-\frac{1}{2} \ln 3 + \frac{1}{2} \ln 13\right]$

$= \frac{1}{2} \ln \left(\frac{13}{3}\right)$.

Question 2.

(a)(i). $(2+3i)p - q = 1+2i$

$2p + 3ip - q = 1+2i$

$2p - q = 1$ $\quad 3p = 2$

$p = 2/3$ ✓

$4/3 - q = 1$

$q = 1/3$ ✓

(ii). $(2+3i)p - q = 1+2i$

$(2+3i)(a+ib) - (a-ib) = 1+2i$

$2a + 2ib + 3ia - 3b - a + ib = 1+2i$

$2a - 3b - a = 1$ ✓

$a - 3b = 1$

$a - 1 = 3b$

$2b + 3a + b = 2i$

$3b + 3a = 2$.

$(a-1) + 3a = 2$

$4a - 1 = 2$.

$4a = 3$ ✓

$a = 3/4$.

$3b = -\frac{1}{4}$.

$b = -\frac{1}{12}$.

$\therefore p = -\frac{1}{4} + \frac{1}{12}i$

$q = \frac{1}{4} - \frac{1}{12}i$ ✓

(b). $\frac{1}{1+z} = \frac{1}{1+\text{cis}\theta}$

$= \frac{1}{1+\cos\theta + i\sin\theta}$

Let $t = \tan\theta/2$

$\cos\theta = \frac{1-t^2}{1+t^2}$ ✓

$\sin\theta = \frac{2t}{1+t^2}$

$= \frac{1}{1+\frac{1-t^2+2t}{1+t^2}}$ ✓

$= \frac{1+t^2}{1+t^2+1-t^2+i2t}$

$= \frac{1+t^2}{2+2ti}$

$= \frac{1}{2}\left[\frac{1+t^2}{1+ti}\right]$

$= \frac{1}{2}\left[\frac{1+t^2}{1+ti} \times \frac{1-it}{1-it}\right]$

$= \frac{1}{2}\left[\frac{1-it+t^2-it^3}{1+t^2}\right]$

$= \frac{1}{2}\left[\frac{\cancel{1+t^2}}{\cancel{1+t^2}} - \frac{it(\cancel{1+t^2})}{\cancel{1+t^2}}\right]$ ✓

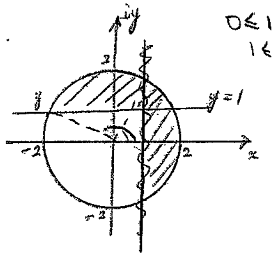
$= \frac{1}{2}\left[1 - it\right]$

$= \frac{1}{2}\left[1 - i\tan\theta/2\right]$

QED.

(c).

(c).



$0 \leq |z| \leq 2$

$1 \leq \text{Im} z \leq 2$

$1 \leq y \leq 2$.

Int of $x^2+y^2 = 4 \cap y = 1$

~~$3 + y^2 = 3$~~

~~$y^2 = 3$~~

~~$y = \sqrt{3}$.~~

~~$\therefore 1 + i\sqrt{3}$.~~

$x^2 = 3$

$x = \pm\sqrt{3}$

$\therefore (\text{Arg } z)_{max} = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$.

(a)(i) $\sqrt{5-12i} = (a+ib)$

$5-12i = a^2 - b^2 + 2iab$

$5 = a^2 - b^2 \qquad -12 = 2ab$

$-\frac{6}{a} = b$

$5 = a^2 - \frac{36}{a^2}$

$a^4 - 5a^2 - 36 = 0.$

$a^2 = 9, -4.$ ✓

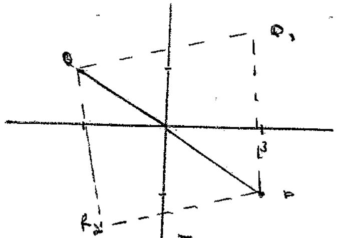
$\therefore a = \pm 3, \ b = \mp 2$ ✓

$\pm(3-2i)$

$q - p = -3 + 2i - 3 + 2i$

$= -6 + 4i$ ✓

(ii).



$\vec{PQ} \times cis\frac{\pi}{3} = \vec{PR_1}$

$(q-p)(\frac{1}{2} + \frac{\sqrt{3}i}{2}) = R_2 - P$

$= \frac{1}{2}(1+\sqrt{3}i)(-6+4i) = r - p$ ✓

$\frac{1}{2}[-6+4i-6\sqrt{3}i + \overset{4\sqrt{3}}{\cancel{4\sqrt{3}}})$

$= -3 - 2\sqrt{3} + i(2-3\sqrt{3}) = r - (3-2i)$

$-3+3-2\sqrt{3} + i(2-3\sqrt{3}-2) = r_2$

$\underline{-2\sqrt{3} - 3\sqrt{3}i = r_2}$ ✓

$\underline{\therefore r_1 = 2\sqrt{3} + 3\sqrt{3}i}$ ✓

Q3. ——————

Q4. (a)(i) $P(x) = x^3 - 9x^2 + 24x + c$

$P'(x) = 3x^2 - 18x + 24 = 0.$

$x^2 - 6x + 8 = 0.$

$x = 2, x = 4$ ✓

$P(2) = 8 - 36 + 48 + C = 0.$

$c = -20.$ ✓

$P(4) = 64 - 144 + 96 + c = 0$

$16 + c$ ✓

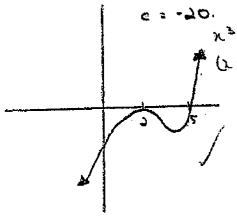
$c = -16.$

(ii).

$c = -20.$

$x^3 - 9x^2 + 24x + 20$

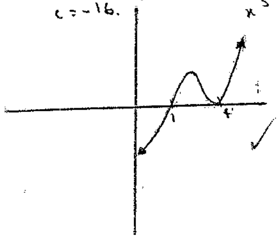
$(x-2)^2(x+5)$ ✓



✓

$x^3 - 9x^2 + 24x - 16$

$(x-4)^2(x-1)$ ✓



✓

$P(x) = x^3 - 9x^2 + 24x + C.$

$P'(x) = 0.$

T.P at $x = 2, x = 4.$

$p(2) = 20 + c = 0, \ P(4) = 16 + C = 0.$ ✓

$\therefore$ TP at $(2, 20+c)$ & $(4, 16+c)$

For Three Distinct Roots then

TPs on opposite sides i.e $(20+c)(16+c) < 0.$ ✓

$\underline{-20 < c < -16}$

(b).(i) $f(x) = x - 2 + \frac{3}{x+2}$

$f(x) = 0 \qquad x - 2 + \frac{3}{x+2} = 0.$ ✓

$x^2 - 4 + 3 = 0.$

$x^2 - 1 = 0.$

$x = \pm 1$ ✓

$\underline{(1, 0) \ \& \ (-1, 0).}$

(b)(ii). $f(x) = x - 2 + \frac{3}{x+2}$

$= (x-2) + 3(x+2)^{-1}$

$f'(x) = 1 - 3(x+2)^{-2}$

$= 1 - \frac{3}{(x+2)^2} = 0.$

$\frac{3}{(x+2)^2} = 1$

$3 = (x+2)^2$

$3 = x^2 + 4x + 4$

$x^2 - 4x + 1 = 0.$

$x = \frac{-4 \pm \sqrt{16-4}}{2}$

$= -2 \pm \sqrt{3}$

$f(-2+\sqrt{3}) = -0.535 \qquad f(-2-\sqrt{3}) = -7.464$ ✓

x	-2	$-2+\sqrt{3}$	0
$f'(x)$	$-$	0	$+$
	\	—	/

x	-4	$-2-\sqrt{3}$	$\sqrt{3}$
$f'(x)$	$+$	0	$-$
	/	—	\

✓

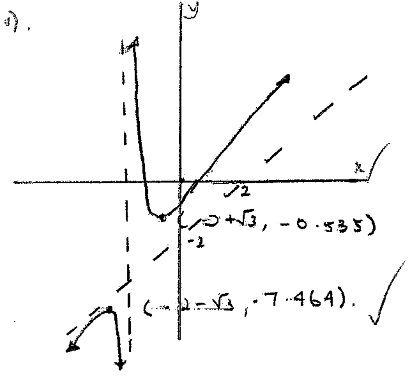
$\therefore \underline{(-2+\sqrt{3}, -0.535)}$ $\underline{\text{is Min TP.}}$ $\qquad \underline{(-2-\sqrt{3}, -7.464)}$ $\underline{\text{is Max TP.}}$

(iii). Asymptotes. ✓

$\underline{y = (x-2)}$

$\underline{x = -2.}$ ✓

(iv).



✓ ✓

(c) $x^3 + x^2 + 3x - 2 = 0$

α, β, γ

For $\alpha^2\beta\gamma = \frac{(\alpha\beta\gamma)^2}{\beta\gamma}$

if β, γ are roots

then: $\frac{(\alpha\beta\gamma)^2}{x^2}$

Let $y = \frac{4}{x^2}$ ✓

$x^2 = \frac{4}{y}$

Let $x = \frac{2}{\sqrt{y}}$ ✓

$\frac{8}{y\sqrt{y}} + \frac{4}{y} + \frac{6}{\sqrt{y}} - 2 = 0.$

$8 + 4\sqrt{y} + 6y - 2y\sqrt{y} = 0.$ ✓

$8 + 6y = 2y\sqrt{y} - 4\sqrt{y}$

$8 + 6y = \sqrt{y}(2y - 4)$

$64 + 96y + 36y^2 = y(4y^2 - 16y + 16)$

$36y^2 + 96y + 64 = 4y^3 - 16y^2 + 16y$

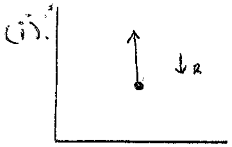
$4y^3 - 52y^2 - 80y + 64 = 0.$

$\underline{\text{OR}}$ ✓

$\underline{x^3 - 13x^2 - 20x + 16 = 0.}$

Questions 5.

(i).



$m\ddot{x} = -F - R$

$m\ddot{x} = -g - mg\left(\frac{v}{k}\right)^2$

$\ddot{x} = -g - g\left(\frac{v}{k}\right)^2$ ✓

(ii). $v\frac{dv}{dx} = -g - g\left(\frac{v}{k}\right)^2$

$= -g - \frac{gv^2}{k^2}$

$v\frac{dv}{dx} = \frac{-gk^2 - gv^2}{k^2}$

$\frac{dv}{dx} = \frac{-gk^2 - gv^2}{vk^2}$ ✓

$\int \frac{dx}{dv}\,dv = \int \frac{vk^2}{-gk^2 - gv^2}\,dv$

$= \frac{-k^2}{2g}\int \frac{+2v}{k^2+v^2}\,dv$

$x = \frac{-k^2}{2g}\ln|k^2+v^2| + c.$

At $x=0$ $v=U$

$0 = \frac{-k^2}{2g}\ln(k^2+U^2) + c$ ✓

$\therefore x = \frac{-k^2}{2g}\ln|k^2+v^2| + \ln|k^2+U^2|$

$= \frac{k^2}{2g}\ln\left|\frac{k^2+U^2}{k^2+v^2}\right|$

At Greatest Height

$v = 0.$ ✓

$\therefore X = \frac{k^2}{2g}\ln\left|\frac{k^2+U^2}{k^2}\right|$

(iii). Downward:

$m\ddot{x} = mg - R$

$m\ddot{x} = mg - \frac{mgv^2}{k^2}$ Terminal when $\ddot{x}=0.$ $\therefore mg = \frac{mgV^2}{k^2}$

$1 = \frac{V^2}{k^2}$ $\therefore k^2 = V^2$

(iv). $\ddot{x} = \frac{gk^2 - gv^2}{k^2}$

$v\frac{dv}{dx} = \frac{gk^2 - gv^2}{k^2}$

$\int \frac{dx}{dv}\,dv = \int \frac{vk^2}{gk^2 - gv^2}\,dv$ ✓

$x = \frac{-k^2}{2g}\int \frac{-2v}{k^2 - v^2}\,dv$

$= \frac{-k^2}{2g}\ln|k^2 - v^2| + c.$

At $x=0$, $v=0$

$c = \frac{k^2}{2g}\ln|k^2|$

$\therefore x = \frac{-k^2}{2g}\ln|k^2 - v^2| + \frac{k^2}{2g}\ln|k^2|$

$= \frac{k^2}{2g}\ln\left|\frac{k^2}{k^2 - v^2}\right|$ ✓

At $x = H$, $v = V$

$H = \frac{k^2}{2g}\ln\left|\frac{k^2}{k^2 - V^2}\right|$

(iv).

$\frac{k^2}{2g}\ln\left|\frac{k^2}{k^2 - V^2}\right| = \frac{k^2}{2g}\ln\left|\frac{k^2+U^2}{k^2}\right|$

$\frac{k^2}{k^2 - V^2} = \frac{k^2+U^2}{k^2}$ ✓

$k^4 = (k^2+U^2)(k^2-V^2)$

$k^4 = k^4 - k^2V^2 + U^2k^2 - V^2U^2$

$k^2U^2 - k^2V^2 - V^2U^2 = 0.$

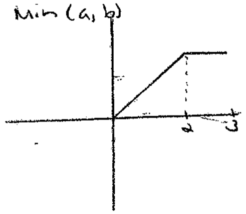
$k^2U^2 - k^2V^2 = V^2U^2$ ✓

$\frac{U^2 - V^2}{V^2U^2} = \frac{1}{k^2}$

$\frac{1}{V^2} - \frac{1}{U^2} = \frac{1}{k^2}.$ ✓

Question 6.

(a). Min (a, b)



✓

$\int_0^3 \min(2, x)\,dx$

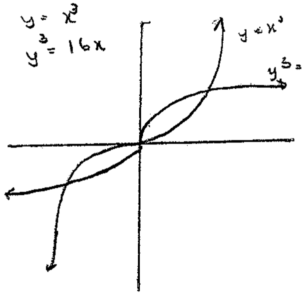
$= \frac{2\times1}{2} + 3\times1$ ✓

$= 1 + 3$

$= 4.$

b) $y = x^3$

$y^3 = 16x$



$y^3 = 16x$

$\frac{y^3}{16} = x$

$y = \left(\frac{y^3}{16}\right)^3$ ✓

$y = \frac{y^9}{4096}$

$4096y = y^9$ ✓

$4096 = y^8$

$y = 2\sqrt{2}.$

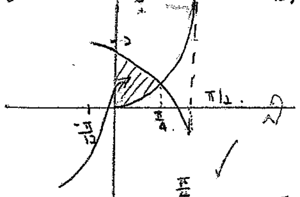
$x = 16\sqrt{2}.$

Odd F(x) ∴ 0 Times

$I = 0\int_0^{16\sqrt{2}} F(x) - g(x)\,dx$

$I = 0$

(c) (i) $y = \tan x$, $y = 2\cos\left(x + \frac{\pi}{12}\right)$



✓

(ii) $\tan x = 2\cos\left(x + \frac{\pi}{12}\right)$

Sub $x = \frac{\pi}{4}$.

LHS: 1 ✓ RHS $= 2\cos\left(\frac{\pi}{4} + \frac{\pi}{12}\right)$

$= 2\cos\left(\frac{\pi}{3}\right)$

$= 2\times\frac{1}{2}$ ✓

$= 1$

LHS = RHS

∴ $\frac{\pi}{4}$ is a solution.

(iii) Area req'd. $= \int_0^{\frac{\pi}{4}} 2\cos\left(x + \frac{\pi}{12}\right) - \tan x\,dx$

$= \left[2\sin\left(x + \frac{\pi}{12}\right) + \ln|\cos x|\right]_0^{\pi/4}$

$= \left(2\sin\frac{\pi}{3} + \ln\left|\frac{1}{\sqrt{2}}\right|\right) - \left(2\sin\frac{\pi}{12} + 0\right)$

$= \sqrt{3} - \frac{1}{2}\ln 2 - 2\sin\frac{\pi}{12} \doteqdot 1.13$ (to 2 dp)

(iii) $A(x) = \int_0^{\frac{\pi}{4}} \tan x\,dx + 2\int_{\pi/4}^{\frac{5\pi}{12}} \cos\left(x + \frac{\pi}{12}\right)dx$

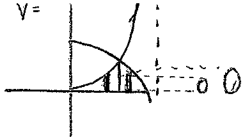
$= -\ln|\cos x|\Big|_0^{\frac{\pi}{4}} + 2\left[\sin\left(x + \frac{\pi}{12}\right)\right]_{\pi/4}^{\frac{5\pi}{12}}$

$= -\ln\frac{1}{\sqrt{2}}$ $+ 2\left[\sin\frac{\pi}{2} - \sin\pi/6\right]$

$= -\ln(2)^{-\frac{1}{2}}$ $+ 2\times\frac{1}{2}$

$= \frac{1}{2}\ln 2$ $+ 1$

(iv) V =



$\therefore V = \pi \sum_{0}^{\pi/4} \lim_{\delta x \to 0} (\tan x)^2 . dx \quad + \quad \pi \sum_{\pi/4}^{5\pi/12} \lim_{\delta x \to 0} 4\cos^2(x + \frac{\pi}{12}) . dx$

$V = \pi \int_{0}^{\pi/4} \tan^2 . dx \quad + \pi \int_{\pi/4}^{5\pi/12} 4\cos^2(x + \frac{\pi}{12}) . dx$

$= \pi \int_0^{\pi/4} \sec^2 x - 1 . dx \quad + \frac{\pi}{2} \int_{\pi/4}^{5\pi/12} \underline{\cos 2(x + \frac{\pi}{12})} + 1 . dx$

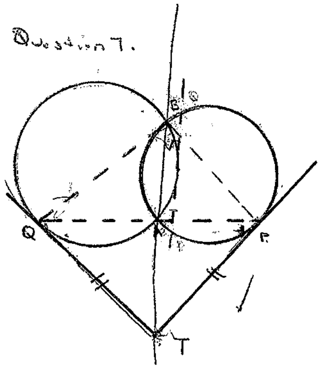
$= \pi \left[\tan x - x \right]_0^{\pi/4} \quad + \frac{\pi}{2} \left[\frac{\sin(2x + \frac{\pi}{6})}{2} + x \right]_{\pi/4}^{\frac{5\pi}{12}}$

$= \pi [1 - \pi/4] \quad + \frac{\pi}{2} \left[\frac{5\pi}{12} - \frac{\sqrt{3}}{4} - \frac{\pi}{4} \right]$

$= \pi - \frac{\pi^2}{4} + \frac{5\pi^2}{24} - \frac{\sqrt{3}\pi}{4} - \frac{\pi^2}{8}$

$= \pi \left[(1 - \frac{\sqrt{3}}{4}) - \frac{\pi}{6} \right]$

Question 7.



(i) In Δ's TAP & TPB

$B\hat{T}P$ is common ✓

$T\hat{P}A = T\hat{B}P$ (alternate segment)

∴ <u>ΔTAP ||| ΔTBP.</u> ✓

Similarly <u>ΔTBQ ||| ΔTAQ</u> ✓

(ii). Let $T\hat{P}A = \alpha$

$T\hat{B}P = \alpha$ (alternate segment) as above.

Since Δ's TAP ||| ΔTBP & ΔTAQ ||| ΔTBQ

$\frac{TB}{TP} = \frac{TB}{TQ}$ ✓

∴ TP = TQ

∴ $T\hat{Q}P = \alpha$ (base angles of iso Δ's)

∴ $T\hat{Q}P = T\hat{B}P$ (angles in the same segment TP)

∴ TQBP is cyclic. ✓

(iv). TP = TQ (shown above).

(b)(i) $(1+z)^n = c_0 + c_1 z + \ldots c_n z$

Let $n = 0$.

$(1+z)^0 = c_0$

$\underline{1 = c_0.}$ ✓

Let $z = 1 \; : \; 2^n = c_0 + c_1 z + \ldots c_n z^n \ldots (i)$

(ii). Let $z = -1$

LHS: 0 RHS: $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots (-1)\binom{n}{n}$

∴ $\underline{0 = c_0 - c_1 + c_2 - c_3 + \ldots (-1)^n c_n.}$ ✓

(iii). $c_0 + c_2 + c_4 + \ldots c_{n-1} = c_1 + c_3 + c_5 + \ldots c_n$

Add $c_1 + c_3 + c_5 + \ldots c_n$ to both sides

$c_0 + c_1 + c_2 + c_3 + \ldots c_{n-1} + c_n = 2[c_1 + c_3 + c_5 + \ldots c_n]$

$2^n = 2[c_1 + c_3 + c_5 + \ldots c_n]$

$\frac{2^n}{2} = [c_1 + c_3 + c_5 + \ldots c_n]$

$\underline{2^{n-1} = c_1 + c_3 + \ldots c_{n-2} + c_n}$

Alternatively

From (i)

$c_0 + c_1 + c_2 + \ldots + c_n = 2^n \ldots$ (

From (ii) for n odd

$c_0 - c_1 + c_2 - \ldots + c_n = 0 \ldots$ (

(I) - (II)

$2c_1 + 2c_3 + \ldots + 2c_n = 2^n$

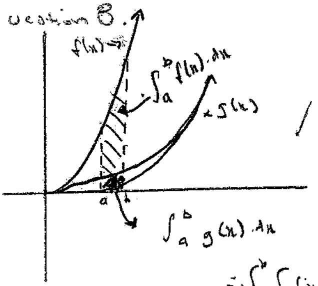
$2(c_1 + c_3 + \ldots + c_n) = 2^n$

∴ $c_1 + c_3 + \ldots + c_n = 2^{n-1}$ as req$\underline{d}$.

(iv)

Question 8.

(a)



✓

$\therefore \underline{\int_a^b f(x) > \int_a^b g(x).dx}$ ✓

(b). $U_n = \int_0^1 (1-t^2)^{\frac{n-1}{2}} . dt$

$u = (1-t^2)^{\frac{n-1}{2}} \qquad v = t$

$u' = -t(n-1)\left[1-t^2\right]^{\frac{n-3}{2}} \qquad v' = 1$

$I = \left[t(1-t^2)^{\frac{n-1}{2}} \right]_0^1 + (n-1) \int_0^1 t^2 (1-t^2)^{\frac{n-3}{2}} . dt$

$U_n = 0 + (n-1) \int_0^1 t^2 (1-t^2)^{\frac{n-3}{2}} . dt + \int_0^1 (1-t^2)^{\frac{n-3}{2}} . dt.$

$= (n-1) \int_0^1 -(1-t^2)(1-t^2)^{\frac{n-3}{2}} dt + \int_0^1 (1-t^2)^{\frac{n-3}{2}} . dt$

$= (n-1) - \int_0^1 (1-t^2)^{\frac{n-1}{2}} . dt$ ✓ $+ U_{n-2}$

$U_n = (n-1)[-U_n + U_{n-2}]$

$(n-1)U_n + U_n = (n-1)U_{n-2}$

$(n-1+1)U_n = (n-1)_{n-2}$ ✓

(iv). $V_n = nU_nU_{n-1}$ $nU_n = (n-1)U_{n-2}$

RTP: $V_n = \frac{1}{2}\pi$ $U_n = \frac{(n-1)}{n} U_{n-2}$ $\Rightarrow U_{n-1} = \frac{(n-2)}{n-1} U_{n-3}$

$V_n = \frac{(n-1)}{n} U_{n-2} \times \frac{(n-2)}{n-1} U_{n-3}$

But $U_{n-2} = \frac{n-3}{n-2} U_{n-4}$.

$U_2 = \frac{1}{2} U_0$.

$U_0 = \int_0^1 (1-t^2)^{-1/2} \, dt$

$= \int_0^1 \frac{1}{\sqrt{1-t^2}} \, dt$ ✓

$= \left[\sin^{-1}(t)\right]_0^1$.

$= \frac{\pi}{2}$.

$\therefore V_n = \frac{(\cancel{n-1})}{n} \times \frac{(\cancel{n-2})}{(\cancel{n-1})} \times \frac{(\cancel{n-3})}{(\cancel{n-2})} \times \dots \frac{\pi}{2}$ ✓

$= \underline{\underline{\frac{\pi}{2}}}$. ✓

(iii) ~~$\int_a^b f(x)\,dx \; \int_a^b g(x)\,dx$~~

~~$f(x) > g(x)$~~ (from (e))

~~$U_n > U_{n-1}$~~

~~$\therefore 0 < U_n < U_{n-1}$~~

~~$U_n < U_{n-1}$~~

~~Let $n = n+1$~~

~~$U_{n+1} < U_{n+1-1}$~~

~~$U_{n+1} < U_n$~~

~~$\therefore U_{n+1} < U_n < U_{n-1}$~~

~~$U_n < \frac{V_n}{nU_{n-1}}$~~

~~$< \frac{\pi}{2(n-1)} \times \frac{1}{U_{n-2}} < \dots < U_1$~~

~~$U_1 = \int_0^1 (1-t^2)^0 \, dt$~~

~~$= 1$~~

~~$\therefore \sqrt{\frac{n}{2n+2}} \; U_n < \sqrt{\frac{n}{2n}}$~~

If $f(x) > g(x) > 0$

i.e. $u_{n-1} > u_n > 0$

$\therefore u_n < u_{n-1} = \frac{V_n}{n u_n}$

$= \frac{\pi}{2n u_n}$ since $V_n = \frac{\pi}{2}$ for all n

$\therefore u_n^2 < \frac{\pi}{2n}$

$u_n < \sqrt{\frac{\pi}{2n}}$

Now sub $n = n+1$

$\therefore u_{(n+1)-1} > u_{n+1}$

$\therefore u_n > u_{n+1}$

$u_n > u_{n+1} = \frac{V_{n+1}}{(n+1)u_n}$

$= \frac{\pi}{2(n+1)u_n}$

$\therefore u_n^2 > \frac{\pi}{2(n+1)}$

$\therefore u_n > \sqrt{\frac{\pi}{2(n+1)}}$

$\therefore \sqrt{\frac{\pi}{2(n+1)}} < u_n < \sqrt{\frac{\pi}{2n}}$ as req^{d}.