BRIGIDINE COLLEGE RANDWICK

HSC Mathematics Student ______________________

5 December 2006 Teacher ______________________

Time 45 minutes

Show all necessary working.
Neatness may be taken into consideration in the awarding of marks.

There are 6 Questions.

1. Differentiate the following
(leaving answers completely simplified with positive indices)

a. $y = -9(4-x)^3$ *2 marks*

b. $y = x^4 - \dfrac{1}{x^2}$ *2 marks*

c. $y = \sqrt[3]{(2x+1)^2}$ *2 marks*

d. $y = \dfrac{x^2}{3+x^3}$ *2 marks*

2. Given $f(x) = x^3 - 12x + 12$. Find

a. $f(-2)$ *1 marks*

b. $f'(-2)$ *2 marks*

3. Find the value of

a. $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2}$ *1 marks*

b. $\lim_{x\to \infty} \dfrac{x^2 - 3x + 2}{2x^2 - 1}$ *2 marks*

... please turn over

4. The table below is a record of the height of a bush in centimetres, which was measured over a series of months. *2 mark*

month	1	2	3	4	5	6
height	45	65	80	92	100	105

If this table were plotted as a curve, what comments could be made on the first and second derivative of the resulting curve? Justify your answer.

5. Consider the curve given by $y = 3x^2 - x^3$.

a. Determine its x-intercepts. *1 mark*

b. Show there are two stationary values at $x = 0$ and $x = 2$ and determine their nature. *3 mark*

c. Determine any possible points of inflection. *2 mark*

d. Sketch the curve, showing the above features. *2 mark*

e. Find the equation of the tangent to this curve when $x = -1$. *2 mark*

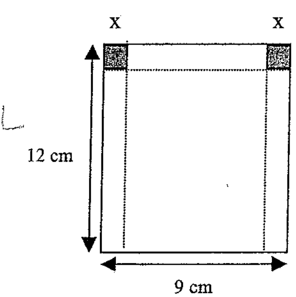
6. A rectangular piece of cardboard measures 9 cm by 12 cm.

From two corners, squares of side x cm are removed,

as shown to the right.

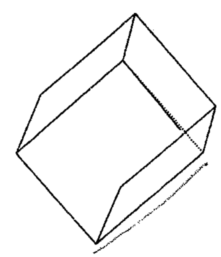
The remainder is folded along the dotted lines

to form a tray.



a. Show that the volume "V" of this tray may be given by

$V = 2x^3 - 33x^2 + 108x$. *1 mark*



b. Find the maximum possible volume of this tray.

3 mark

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HSC-07 MATHEMATICS ASSESSMENT TASK 01: Solutions and Marking Scheme 5/12/06

QUESTION 1

(a)
$$y = -9(4-x)^3$$
$$\frac{dy}{dx} = -9\times 3(4-x)^{3-1}\times\frac{d(4-x)}{dx}$$
$$= -27(4-x)^2\times(-1)$$
$$= 27(4-x)^2$$

(b)
$$y = x^4 - \frac{1}{x^2}$$
$$= x^4 - x^{-2}$$
$$\frac{dy}{dx} = 4x^{4-1} - (-2)x^{-2-1}$$
$$= 4x^3 + 2x^{-3}$$
$$= 4x^3 + \frac{2}{x^3}$$

(c)
$$y = \sqrt[3]{(2x+1)^2}$$
$$= \left[(2x+1)^2\right]^{\frac{1}{3}}$$
$$= (2x+1)^{\frac{2}{3}}$$
$$\frac{dy}{dx} = \frac{2}{3}(2x+1)^{\frac{2}{3}-1}\times\frac{d(2x+1)}{dx}$$
$$= \frac{2}{3}(2x+1)^{-\frac{1}{3}}\times 2$$
$$= \frac{4}{3}(2x+1)^{-\frac{1}{3}}$$
$$= \frac{4}{3}\times\frac{1}{(2x+1)^{\frac{1}{3}}}$$
$$= \frac{4}{3(2x+1)^{\frac{1}{3}}}$$
$$= \frac{4}{3\sqrt[3]{2x+1}}$$

(d)
$$y = \frac{x^2}{3+x^3}$$
$$= \frac{u}{v}$$
$$u = x^2 \qquad v = 3+x^3$$
$$u' = 2x \qquad v' = 3x^2$$
$$\frac{dy}{dx} = \frac{vu'-uv'}{v^2}$$
$$= \frac{(3+x^3)\times 2x - x^2\times 3x^2}{(3+x^3)^2}$$
$$= \frac{6x+2x^4-3x^4}{(3+x^3)^2}$$
$$= \frac{6x-x^4}{(3+x^3)^2}$$
$$= \frac{x(6-x^3)}{(3+x^3)^2}$$

QUESTION 2

$$f(x) = x^3-12x+12$$

(a)
$$f(x) = x^3-12x+12$$
$$f(-2) = (-2)^3-12\times(-2)+12$$
$$= -8+24+12$$
$$= 28$$

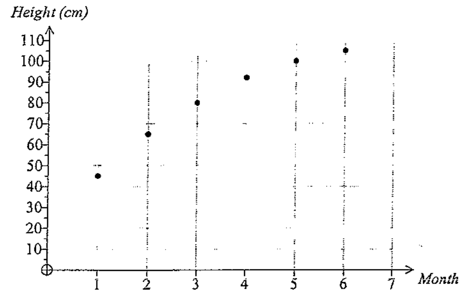
(b)
$$f(x) = x^3-12x+12$$
$$f'(x) = 3x^2-12$$
$$f'(-2) = 3(-2)^2-12$$
$$= 12-12$$
$$= 0$$

QUESTION 3

(a)
$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}$$
$$= \lim_{x\to 2}(x+2)$$
$$= 2+2$$
$$= 4$$

(b)
$$\lim_{x\to\infty}\frac{x^2-3x+2}{2x^2-1}$$
$$= \lim_{x\to\infty}\frac{x^2}{2x^2-1} - \lim_{x\to\infty}\frac{3x}{2x^2-1} + \lim_{x\to\infty}\frac{2}{2x^2-1} \quad \text{OR} \quad \lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{2}{x^2}}{\frac{2x^2}{x^2}-\frac{1}{x^2}}$$
$$= \lim_{x\to\infty}\frac{\frac{x^2}{x^2}}{\frac{2x^2}{x^2}-\frac{1}{x^2}} - 0 + 0 \qquad \lim_{x\to\infty}\frac{1-\frac{3}{x^2}+\frac{2}{x^2}}{2-\frac{1}{x^2}}$$
$$= \lim_{x\to\infty}\frac{1}{2-\frac{1}{x^2}} \qquad \frac{1-0+0}{2-0}$$
$$= \frac{1}{2} \qquad \frac{1}{2}$$

QUESTION 4



- Height is increasing with time
- Rate of height increase is decreasing with increasing time or the curve is concave down.

$\therefore$ $H' > 0$ and $H'' < 0$

QUESTION 5

$$y = 3x^2 - x^3$$

(a) For x – intercepts $y = 0$

i.e.
$$3x^2 - x^3 = 0$$
$$x^2(3-x) = 0$$
$$x = 0 \text{ or } 3$$

The x-intercepts are 0 and 3

(b)
$$y = 3x^2 - x^3$$
$$\frac{dy}{dx} = 6x - 3x^2$$
$$= 3x(2-x)$$

For stationary points
$$\frac{dy}{dx} = 0$$
$$3x(2-x) = 0$$
$$x = 0 \text{ or } 2$$

Nature of stationary points

Method 1:

At $x = 0$

x	0^-	0	0^+
Sign of $\frac{dy}{dx}$	–	0	+
Tangent	\	—	/

$\Rightarrow$ a minimum turning point at $x = 0$

At $x = 0$

x	2^-	2	2^+
Sign of $\frac{dy}{dx}$	+	0	–
Tangent	/	—	\

$\Rightarrow$ a maximum turning point at $x = 2$

At

$x = 0$ there is a minimum turning point;

$x = 2$ there is a maximum turning point.

Method 2:
$$\frac{d^2y}{dx^2} = 6-6x$$
$$x = 0 \quad \frac{d^2y}{dx^2} = 6-6\times 0$$
$$= 6 > 0 \Rightarrow \text{Min t.p.}$$
$$x = 2 \quad \frac{d^2y}{dx^2} = 6-6\times 2$$
$$= -6 < 0 \Rightarrow \text{Max t.p}$$

At

$x = 0$ there is a minimum turning point;

$x = 2$ there is a maximum turning point.

Comment:

Method 1 works for all functions. However, Method 2 will not work for all functions e.g. $y = x^4$

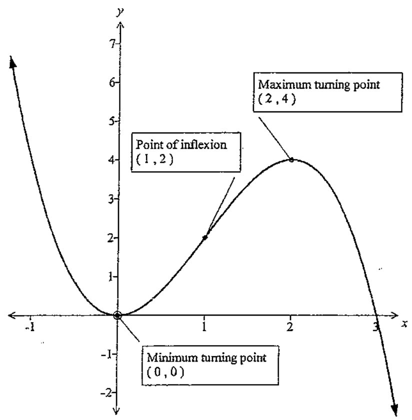
Be aware of this when choosing a method for determining the nature of stationary points.

(c) At a point of inflexion
$$\frac{d^2y}{dx^2} = 0$$
$$6-6x = 0$$
$$6(1-x) = 0$$
$$x = 1$$
$$y(1) = 3\times 1^2 - 1^3$$
$$= 2$$

(1, 2) is a possible point of inflexion.

Note: At a point of inflexion the second derivative is zero and changes sign. It is necessary to analyse the sign of the second derivative in the immediate neighbourhood to confirm a point of inflexion.

(*d*)

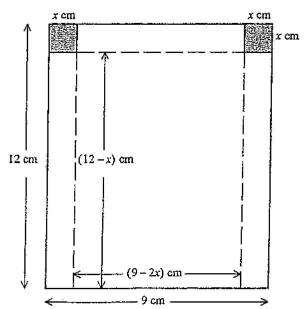


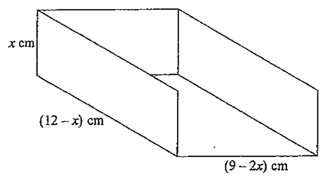
(*e*)
$$y = 3x^2 - x^3$$
$$\frac{dy}{dx} = 6x - 3x^2$$

At $x = -1$
$$y = 3\times(-1)^2 - (-1)^3$$
$$= 4$$
$$\frac{dy}{dx} = 6\times(-1) - 3\times(-1)^2$$
$$= -9$$

At (1, 4) the tangent has gradient -9 and its equation is given by
$$y - y_1 = m(x - x_1)$$
$$y - 4 = -9(x - -1)$$
$$y - 4 = -9(x+1)$$
$$y - 4 = -9x - 9$$
$$y = -9x - 5$$
$$9x + y + 5 = 0$$

QUESTION 6





(*a*) From the diagrams, V, the volume of the tray is given by
$$V = (9-2x)\times(12-x)\times x$$
$$= (108 - 24x - 9x + 2x^2)x$$
$$= (2x^2 - 33x + 108)x$$
$$= 2x^3 - 33x^2 + 108x;\ \ 0 < x < 4\frac{1}{2}$$

(*b*)
$$\frac{dV}{dx} = 6x^2 - 66x + 108$$
$$= 6(x^2 - 11x + 18)$$
$$= 6(x-2)(x-9)$$
$$= 0 \text{ when } x = 2 \text{ or } 9$$
$$\frac{d^2V}{dx^2} = 12x - 66$$

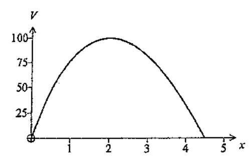
At $x = 2$

x	2^-	2	2^+
Sign of $\frac{dV}{dx}$	+	0	–
Tangent	/	—	\

$\Rightarrow$ a maximum turning point at $x = 2$.

For $0 < x < 2$ $\frac{dV}{dx}$ is positive, thus V is increasing.

For $2 < x < 4½$ $\frac{dV}{dx}$ is negative, thus V is decreasing.



$\therefore$ The maximum value of V occurs when $x = 2$.

$$V_{max} = (12-2)(9-4)2 \text{ cm}^3$$
$$= 100 \text{ cm}^3$$

Comment: *In problems of this type it is necessary to establish the absolute max (or min) within the domain of definition. Here the domain of definition is 0<x<4 5 and the max tp is the absolute maximum.*