PRELIMINARY EXAMINATION

2000

MATHEMATICS

MATHEMATICS EXTENSION 1

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Write your Student Name / Number on every page of the question paper and your answer sheets .
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- The answers to the six questions are to be handed in separately, clearly marked Question 1, Question 2 etc.
- ***This question paper must not be removed from the examination room.***

STUDENT NUMBER / NAME: ...

Question 1 (Start a new page) **Marks**

a) Find $\lim_{x \to 1} \frac{x^3-1}{x^2-1}$. **2**

b) Find the derivative of : **6**

(i) $\frac{2x-3}{3x+5}$

(ii) $\frac{1}{\sqrt{2x-7}}$

(iii) $\left(3x^2-2\right)^6$

c) For the curve represented by the equation $y = x - x^3$: **4**

(i) Find $\frac{dy}{dx}$

(ii) Find the equation of the tangent to the curve at the point (1,0).

(iii) State the gradient of the normal at this point

Question 2 (Start a new page) **Marks**

(a)

The tenth term of an arithmetic sequence is 29 and the fifteenth term is 44. **3**

(i.) Find the value of the common difference and the value of the first term.

(ii.) Find the sum of the first 75 terms.

(b) A geometric series has second term 6 and the ratio of the seventh term to the sixth term is 3.

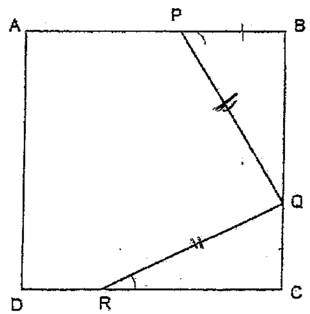
(i.) Find the common ratio r. **3**

(ii.) What is the first term a?

(iii.) Calculate the sum of the first 12 terms.

(c) If $f(x) = 15x^{-2} - 9x^3$, find the value of $f'(-1)$. **2**

(d) In the diagram below, ABCD is a square. P, Q and R are points on sides AB, BC and CD respectively such that PB = QC = RD.



Copy the diagram onto your worksheet.

(i) Prove that $\Delta BPQ \equiv \Delta CQR$. **2**

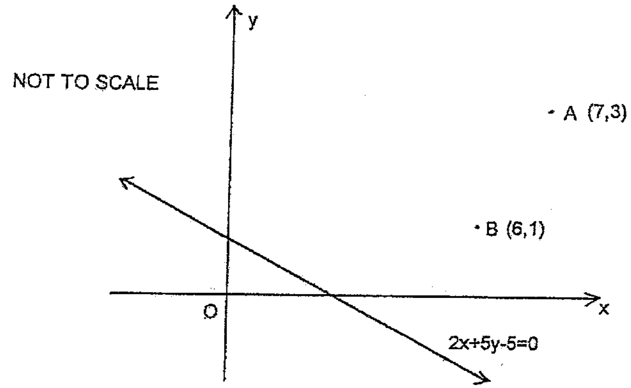
(ii) Hence or otherwise prove that PQ = RQ. **1**

(iii) Deduce that $\angle PQR = 90°$. **1**

Question 3 (Start a new page) **Marks**

(a) Solve for x: $x^4 - 5x^2 - 36 = 0$. **2**

(b)



The diagram above shows the line $2x + 5y - 5 = 0$ and the points A (7,3) and B (6,1).

Copy the diagram onto your worksheet.

(i) Find the equation of the line AB. **2**

(ii) Find the coordinates of the point of intersection, P, of the line $2x + 5y - 5 = 0$ and the line AB. **1**

(iii) Find the shortest distance from P to the line $y + 2 = 0$. **1**

(c) The equation $2x^2 - 7x + 12 = 0$ has roots α and β. Find the value of;

(i) $\alpha + \beta$. **1**

(ii) $\alpha\beta$. **1**

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$. **1**

(d) Find A, B and C such that $A(x-1)^2 + Bx + C \equiv x^2$ **3**

Question 4 (Start a new page) **Marks**

(a) Find the values of m, given that $(3m-1)x^2 - 4mx + 2 = 0$ has equal roots. **2**

(b) Given the parabola $8y = x^2 - 6x - 23$

(i) Write the equation in the form $(x-h)^2 = -4a(y-k)$. **1**

(ii) Find the coordinates of the vertex and focus. **2**

(iii) Find the equation of the axis of symmetry of the parabola. **1**

(iv) Draw a neat sketch of the parabola showing the above information. **2**

(c) If $x^2 - 4px + 3p - 2 = 0$: **4**

(i) Write down an expression, containing p, for the product of the roots of the above equation.

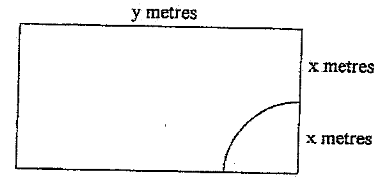
(ii) Find the value of p given that the product of the roots is three times the sum.

Question 5 (Start a new page) **Marks**

(a) Convert $0.3\dot{4}$ to a fraction in its simplest form. **2**

(b) How much will \$6000 accumulate to at the end of five years if it is invested in a fund which pays an interest rate of 4% p.a. compounded quarterly? **2**

(c) **6**



The diagram shows a rectangular paddock which has an area of 1 hectare (10 000 m^2). The paddock requires fencing around the perimeter and also along a circular arc in one corner. The radius of the arc is half the width of the paddock.

i. Show that the total amount of fencing required is given by
$L = 4x + \frac{\pi x}{2} + \frac{10000}{x}$ metres, where x is the radius of the circular arc.

ii. Hence show that when x is approximately 42.4 metres, the paddock will require the least amount of fencing.

(d) Find $\lim_{x \to \infty} \left[\frac{4x^2 - 2x^3 + 1}{7x^3 - \frac{2}{x}} \right]$ **2**

Question 6 (Start a new page) **Marks**

(a) For the function $y = x^3 - 6x^2 + 9x + 1$ find the: 7

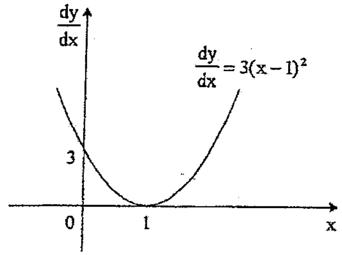
i. stationary points and determine their nature.

ii. co-ordinates of any points of inflection.

iii. values of x for which the curve is increasing.

Hence sketch the curve $y = x^3 - 6x^2 + 9x + 1$.†

(b) The gradient function $\frac{dy}{dx}$ of a curve is illustrated by the graph below: 5

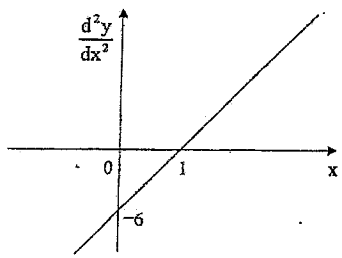


i. A stationary point is located at $x = 1$. Justify this statement by reference to the graph.

ii. α. Comment on the sign of $\frac{dy}{dx}$ for all x, $x \neq 1$.

β. What does this imply about the curve $y = f(x)$?

iii. The graph of $\frac{d^2y}{dx^2}$ is given below.



Copy and complete this table:

x	0	1	2
sign of $\frac{d^2y}{dx^2}$			

iv. What is the nature of the stationary point at $x = 1$?

Question 7 (Start a new page) **Marks**

(a) The sum of the first n terms of a certain arithmetic series is given by $S_n = \frac{n(3n+1)}{2}$. 5

i. Calculate S_1 and S_2.

ii. Find the first three terms of this series.

iii. Find an expression for the n-th term.¤

(b) Over the years the statistics showed the population of a particular town was decreasing in such a way that at the end of each year the population could be determined in the following way:

10% of the population at the beginning of the year moved out and
500 new citizens moved in during the year.

At the beginning of 1990, before the 10% moved out, the population of the town was 10 000.

i. Show that at the end of 1992 the population of the town was:
$10\,000(0.9)^3 + 500(1 + 0.9 + 0.9^2)$.

ii. This trend continued indefinitely. Find the population of the town at the end of the year 2009. (i.e. the end of the 20th year)†

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Mary MacKillop College Prelim 2000 D. Ong

78/84

1.(a). $\lim_{x \to 1} \frac{x^3-1}{x^2-1}$

$= \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)}$ ✓

$\lim_{x\to 1} = 3/2$ ✓

(b)(i) $\frac{2x-3}{3x+5}$

$= \cancel{\ln\left(\frac{2x-3}{3x+5}\right)}$

$= \cancel{\ln(2x-3) - \ln(3x+5)}$

$\cancel{\frac{d}{dx} = \frac{2}{2x-3} - \frac{3}{3x+5}}$ $\Rightarrow$ $\cancel{\frac{6x+10-6x+9}{(2x-3)(3x+5)}}$

$\cancel{= \frac{19}{(2x-3)(3x+5)}}$

Normally

$u = 2x-3$ $v = 3x+5$

$u' = 2$ $v' = 3.$

$\frac{d}{dx} = \frac{vu'-uv'}{v^2}$

$= \frac{2(3x+5) - 3(2x-3)}{(3x+5)^2}$ ✓

$= \frac{6x+10-6x+9}{(3x+5)^2}$

$= \frac{19}{(3x+5)^2}$ ✓

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(ii). $\frac{1}{\sqrt{2x-7}}$ $= \frac{1}{2}\ln(2x-7)$

$= (2x-7)^{-1/2}$ $= \frac{-1}{2} \times \frac{2}{2x-7}$

$\frac{d}{dx} = -\frac{1}{2} \times 2\,(2x-7)^{-3/2}$ ✓ $= \frac{-1}{2x-7}$

$= \frac{-1}{(2x-7)\sqrt{2x-7}}$ ✓

(iii). $(3x^2-2)^6$

$\frac{d}{dx} = 6 \times 6x\ (3x^2-2)^5$ ✓

$= 36x\ (3x^2-2)^5.$ ✓

(c)(i) $y = x - x^3$

$\frac{dy}{dx} = 1 - 3x^2$ ✓

(ii) At $x = 1$

$m = 1-3$

$= -2.$ ✓

$\therefore y - 0 = -2(x-1)$ ✓

$y = -2x + 2$

(iii) $m_2 = 1/2.$ ✓

2 (a)(i) $T_{10} = 29 \Rightarrow a + 9d = 29 \Rightarrow a = 29 - 9d.$

$T_{15} = 44$ $a + 14d = 44.$

$29 - 9d + 14d = 44$

$5d = 15$ ✓

$d = 3.$ ✓ $\therefore a = 2.$ ✓

(ii). $S_{75} = \frac{n}{2}(2a + (n-1)d)$ ✓

$= \frac{75}{2}(4 + 74 \times 3)$ ✓

$= 8475$ ✓

(b).(i) $T_n = ar^{n-1} = 6$

$\frac{T_7}{T_6} = \frac{ar^6}{ar^5} = 3$

$\therefore r = 3$ ✓

(ii) $a \times 3 = 6.$

$\therefore a = 2$ ✓

12

(iii). $S_{12} = \frac{a(r^n-1)}{r-1}$

$= \frac{2(3^{12}-1)}{2}.$ ✓

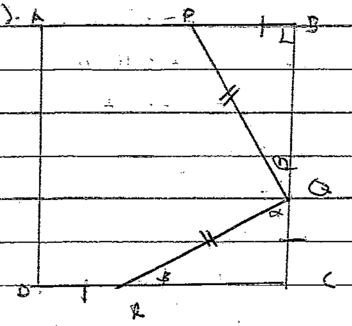
$= 531440.$ ✓

(c). $f(x) = 15x^{-2} - 9x^3$

$f'(x) = -30x^{-3} - 27x^2$ ✓

$f'(1) = 30 - 27$ ✓

$= +3$

(a). 

(i) In Δ's BPQ & Δ CQR

PB = QC (given)

RC = DC − DR = BQ = BC − QC (square)

∴ ~~QR = PQ (Pythagoras theorem)~~ (ii) ∴ $\angle PBQ = \angle RCQ = 90°$ (Property of square)

∴ $\Delta BPQ \equiv \Delta CQR$ (S.A.S)

(ii) By Pythagoras, $PQ^2 = PB^2 + BQ^2$

$RQ^2 = QC^2 + RC^2$

But $PB = QC$ ∩ $BQ = RC$

∴ $PQ^2 = RQ^2$

$PQ = RQ$

(iii). Let $RQC = \alpha$

∴ $QRC = \beta$

$\alpha + \beta = \pi/2$ ⟹ $\alpha = \pi/2 - \beta$

In Line CQB

$RQR = \pi - 2\alpha\ (\alpha+\beta)$ (supplementary)

$= \cancel{\pi - 2(\pi/2 - \beta)}$ $\pi - \pi/2$

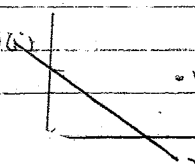
$= \cancel{2\beta}$ $= \pi/2$

Q3.(a) $x^4 - 5x^2 - 36 = 0$.

$(x^2 - 9)(x^2 + 4) = 0$.

$\therefore x^2 = 9$

$x = \pm 3$

(b)(i)  A (7,3)

B (6,1)

(i) $m_{AB} = \frac{3-1}{7-6}$

$= 2$

$y - 1 = 2(x - 6)$

$y = 2x - 11$

(ii) $y = 2x - 11$

$y + 11 = 2x$ — ①.

$2x = 5 - 5y$ — ②.

∴ $y + 11 = 5 - 5y$

$6y = -6$

$y = -1$

$-1 = 2x - 11$

$10 = 2x$

$x = 5$.

∴ P(−1, 5)

(.). $\frac{|(a)x_1 + (b)y_1 + 2|}{\sqrt{a^2+b^2}}$

$= \frac{|5+2|}{\sqrt{1}}$

$= 7$ units.

(c). $2x^2 - 7x + 12 = 0$.

(i) $\alpha + \beta = \frac{-b}{a}$

$= 7/2$

(ii). $\alpha\beta = \frac{c}{a}$

$= 6$.

(iii) $\frac{1}{\alpha} + \frac{1}{\beta}$

$\frac{\alpha+\beta}{\alpha\beta}$

$= \frac{7}{12}$

(d). $A(x-1)^2 + Bx + C \equiv x^2$

$Ax^2 - 2Ax + A + Bx + C$

$A = 1$ $B - 2A = 0$. $A + C = 0$.

$B = 2$ $C = -1$

4.(a). $(3m-1)x^2 - 4mx + 2 = 0$.

∴ $P'(x) = 0$.

$2x(3m-1) - 4m = 0$.

~~$12m - 2x - 4m = 0$.~~

$2m = 2$.

$m = 1$

3unit Method.

$\frac{4m \pm \sqrt{16m^2 - 8(3m-1)}}{2(3m-1)} = 0$.

$\frac{4m \pm \sqrt{16m^2 - 24m + 8}}{2(3m-1)}$

~~$4m \pm 2\sqrt{4m^2 - 6m + 2}$~~

$4m \pm \sqrt{16m^2-24m+8} = 0.$ $\qquad \Delta = 0$

$\cancel{\sqrt{16m^2-24m+8} = 16m^2}$ $\qquad 16m^2-24m+8=0$

$\cancel{16m^2-24m+8 = 16m^2}$ $\qquad 2m^2-3m+1=0$

$\cancel{24m = 8}$ $\qquad (2m-1)(m-1)=0$

$\cancel{m=3}$. $\qquad m=\frac{1}{2}$ or 1.

4unit Method.

$P(x) = (3m-1)x^2 - 4mx + 2$

$P'(x) = 0.$

$2x(3m-1) - 4m = 0.$

$x = \frac{4m}{6m-2}$

$x = \frac{2m}{3m-1}$

Sub back $\cancel{(3m-1)}\frac{4m}{(3m-1)^2} - \frac{8m^2}{3m-1} + 2 = 0.$

$\frac{4m-8m^2}{3m-1} = -2$

$8m^2-4m = 2(3m-1)$

$8m^2-4m = 6m-2$

$8m^2-10m+2=0.$

$m = 1, 1/4.$

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b). $8y = x^2-6x-23$

$8y+9 = x^2-6x+9-23.$

$8y+32 = x^2-6x+9$

$\cancel{8(y+7/4) = (x-3)^2}$ $\qquad 8(y+4) = (x-3)^2$

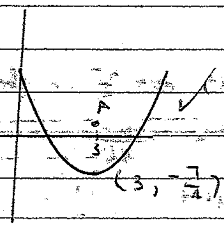
(i) $\cancel{V:(3,-\frac{7}{4})}$ $\qquad V(3,-4)$

Focus, $a=2$. ✓

$\therefore$ $\cancel{(3,\frac{7}{4})}$ $\qquad$ Focus $(3,-1)$

(ii). $x=3$ ✓

(iv)



(c) (i) $x^2-4px+3p-2=0.$

$\alpha\beta = \frac{c}{a}$

$= (3p-2)$ ✓

(ii) $\sum\alpha = 4p$

$12p = 3p-2$

$9p = -2$

$p = \frac{-2}{9}$ ✓

5(a). $0{\cdot}3\dot{4}$

Let $x = 0{\cdot}3\dot{4}$

$\cancel{100x = 34{\cdot}\dot{4}}$ $\qquad 10x = 3{\cdot}\dot{4}\dot{4}$

$\cancel{\therefore 90x = 33}$ $\qquad 9x = 3{\cdot}1$

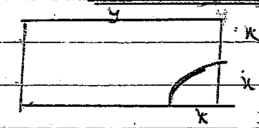
$x = \frac{31}{90}.$ ✓✓

(b). $A_n = P(1+\frac{r}{100})^n$

$= 6000(1{\cdot}04)^{20}$

$= \cancel{13\,146{\cdot}74}$ $\$7321{\cdot}14$

(c).



$A = 10000$

$2xy = 10000$

$y = \frac{10000}{2x}$ ✓

Circle, Perimeter $= \frac{2\pi r}{4}.$

$= \frac{\pi x}{2}$

$\therefore$ Total Perimeter $= y+2x+(y-x) + \frac{\pi x}{2} + 2x$

$\cancel{= 4x + \frac{10000}{2x} + \frac{10000}{2x} - x + \frac{\pi x}{2}}$

$= 3x + 2y - x + \frac{\pi x}{2} + 2x$

$L = 4x + \frac{10000}{x} + \frac{\pi x}{2}.$

(ii). $\frac{dL}{dx} = 4 + -\frac{10000}{x^2} + \pi/2 = 0.$

$4 + \pi/2 = \frac{10000}{x^2}$ ✓

$x^2 = \frac{10000}{4+\pi/2}$

$x^2 \doteqdot 1795{\cdot}07$

$x = 42{\cdot}36$

Test $\quad 41 \quad 42{\cdot}36 \quad 43$

$\quad \backslash \quad — \quad /$

10

(d) $\lim_{x\to\infty}\left[\frac{4x^2-2x^3+1}{7x^3-2/x}\right]$

$= \frac{x(4x^2-2x^3+1)}{7x^4-2}$ ✓

$= \frac{4x^3-2x^4+x}{7x^4-2}$

$\lim_{x\to\infty} \frac{\frac{4}{x}-2+\frac{1}{x^3}}{7-\frac{2}{x^4}}$

$= \frac{-2}{7}$ ✓

6.(a)(i) x^3-6x^2+9x+1

$y' = 3x^2-12x+9 = 0$

$x^2-4x+3=0$ ✓

$(x-3)(x-1)=0$

$x=3,1$

Test 0 1 2 3 4
y' 3 0 -1 0 3
/ — \ — /

$\therefore (1,5)$ Max
$(3,1)$ Min

(ii) $y'' = 2x-4=0$

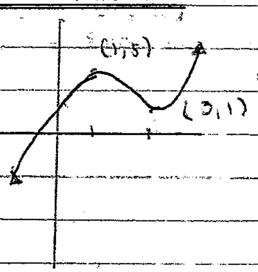
$x=2$

Test $x=$ 1 2 3
$y''=$ -2 0 2
− 0 + ✓

Change in sign $(2,3)$ P.o.I

(iii) $y'>0$ ✓

$x<1 \cap x>3$



(b)(i) A point of inflection is at $x=1$ ✓

(ii)(α) All positive i.e. $f(x)$ increasing ≥ 0

(β) Increasing ✓

12

(iii) 0 1 2
$\frac{d^2y}{dx^2}$ − 0 + ✓
\ — /

(iv) P.o.I ✓

7.(a)(i) $S_n = \frac{n(3n+1)}{2}$

$S_1 = 4/2$ ✓
$= 2$

$S_2 = 7$ ✓

(ii) $T_1 = 2$, $T_2 = 5$, $T_3 = 8$

(iii) $T_n = a+(n-1)d$ ✓
$= 2+3(n-1)$

(b)(i) $P_1 = 10000\times 0.9 + 500$

$P_2 = (P_1)0.9 + 500 \Rightarrow 10000\times 0.9^2 + 500\times 0.9 + 500$

$P_3 = (P_2)\times 0.9 + 500$
$= 10000\times 0.9^3 + 500\times 0.9^2 + 500\times 0.9 + 500$ ✓
$= 10000\times(0.9)^3 + 500(1+0.9+0.9^2)$

12

(ii) $P_n = PR^n + \frac{M(R^n-1)}{R-1}$

$P_{20} = 10000\times 0.9^{20} + \frac{500(0.9^{20}-1)}{-0.1}$ ✓

$= 5607.88$

$= 5607$ people ✓