Meriden School

Year 12

Mathematics Half-Yearly Examination

March, 2002

Total Marks: 100
Assessment Weighting: 20%

Time Allowed: 3 hours (plus 5 minutes reading time)

Instructions:

- Attempt ALL questions
- All questions are NOT of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Hand up in **FOUR** separate sections:

  Section A: Questions 1, 2 and 3 (29 marks)
  Section B: Question 4 (18 marks)
  Section C: Questions 5, 6 and 7 (29 marks)
  Section D: Questions 8 and 9 (24 marks)

MATHEMATICS OUTCOMES

A student:

P1 - demonstrates confidence in using mathematics to obtain realistic solutions to problems

P2 - provides reasoning to support conclusions which are appropriate to the context

P3 - performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities

P4 - chooses and applies appropriate arithmetic, algebraic, graphical, and trigonometric and geometric techniques

P5 - understands the concept of a function and the relationship between a function and its graph

P6 - relates the derivative of the function to the slope of its derivative

P7 - determines the derivative of a function through routine application of the rules of differentiation

P8 - understands and uses the language and notation of calculus

H1 - seeks to apply mathematical techniques to problems in a wide range of practical contexts

H2 - constructs arguments to prove and justify results

H4 - expresses practical problems in mathematical terms based on simple given models

H5 - applies appropriate techniques from the study of calculus, geometry, trigonometry and series to solve problems

H6 - uses the derivative to determine the features of the graph of a function

H7 - uses the features of a graph to deduce information about the derivative

H8 - uses techniques of integration to calculate areas and volumes

H9 - communicates using mathematical language, notation, diagrams and graphs

SECTION A (29 marks):

START A NEW PAGE:
Question 1 (10 marks):

(a) Factorise: $12x^2 - 3y^2$ **2 marks**

(b) Solve: $(x-3)(2x+1) = 9$ **3 marks**

(c) Find: $\frac{d}{dx}(2-5x)^8$ **1.5 marks**

(d) Find the primitive of $3x^2 + 4x - 2$ **1.5 marks**

(e) If $\phi(x) = x + 2x^2$, solve $\phi'(x) = \phi''(x)$. **2 marks**

START A NEW PAGE:
Question 2 (10 marks):

For the curve $y = x^3 - 6x^2$,

(a) Find the stationary points and determine their nature. **4 marks**

(b) Show there is an inflexion point when $x = 2$. **2 marks**

(c) Find the intercepts on both axes. **1 mark**

(d) Sketch the curve on at least a $\frac{1}{3}$ of a page for the restricted domain of $-1 \le x \le 7$ **3 marks**

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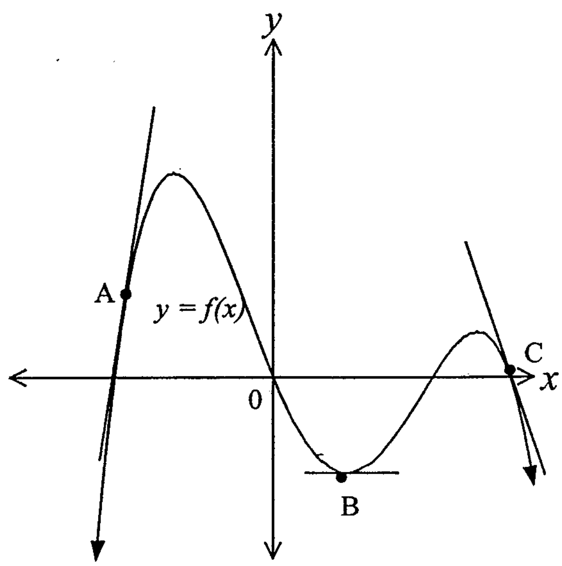
Question 3 (9 marks):

(a)



(i) For the curve above, $y = f(x)$, briefly *describe* (in words) the value/sign of each of the gradients of the tangents at the points A, B and C.

1.5 marks

(ii) Copy or trace the curve onto your page and indicate the normal to the curve at B on your diagram.

1 mark

(iii) On the SAME set of axes clearly sketch the curve that is $y = f'(x)$, showing all essential features.

2.5 marks

(b) Find the equation of the tangent to the curve $y = \dfrac{2x+5}{x-1}$ when x = 2 (answer in general form).

4 marks

SECTION B (18 marks):

START A NEW PAGE:
Question 4 (18 marks):

(a) Consider the series $5 + 11 + 17 + \ldots\ldots.$

(i) *Explain* why the series is either *arithmetic*, *geometric* or *neither*. **1 mark**

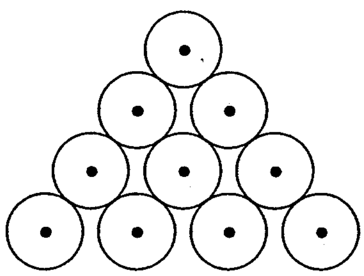
(ii) Find the eighth term. **2 marks**

(iii) Find the sum of the first eight terms. **2 marks**

(b) (i) For the series $20 + 4 + 0.8 + \ldots\ldots$, *justify* why a limiting sum exists. **1 mark**

(ii) Find the limiting sum of this series. **2 marks**

(c) Rolls of material are stacked in a triangular pattern. The cross-section of the stack is shown.



The top row has one roll, the second row has 2 rolls, and the third row has 3 rolls and so on.

(i) Find an algebraic expression that would give the sum of all the rolls in k rows of material. **2 marks**

(ii) If there are 666 rolls of material in a stack, how many rows are there? **3 marks**

Question 4 continued next page

Question 4 (continued)

(d) A woman invests \$2000 into a superannuation fund at the beginning of *each* year. The fund pays interest 4% p.a. compounded annually at the end of the year.

(i) If the first \$2000 is invested on 1st January, 1986, to what value has it grown on 1st January, 1987.

1 mark

(ii) When the second \$2000 is paid into the fund on 1st January, 1987, what is the TOTAL amount to which the two instalments have now grown on 1st January, 1988.

1.5 marks

(iii) What will be the TOTAL amount in her superannuation fund on 1st January, 2010 *before* she makes the payment for that year? (answer to the nearest dollar)

2.5 marks

SECTION C (29 marks):

Question 5 (8 marks):

(a) Given that α and β are the roots of the quadratic equation $2x^2+3x-6=0$,

Without evaluating for α or β:

(i) Find the value for:

1. $\alpha+\beta$ **1 mark**

2. $\alpha\beta$ **1 mark**

3. $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ **1.5 marks**

(ii) ***Show*** that $\alpha^2+\beta^2=8.25$ **1.5 marks**

(b) For $x^2+2x+k=0$, find the values of k for the equation to have real roots.

1.5 marks

(c) For the equation $px^2-4x-7=0$, find the value of p if the roots are reciprocals of each other.

1.5 marks

START A NEW PAGE:
Question 6 (8 marks):

(a) (i) For $y = \dfrac{1}{\sqrt[3]{3x+2}}$ show that $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt[3]{(3x+2)^4}}$. **2 marks**

(ii) Find the equation of the normal to the curve $y = \dfrac{1}{\sqrt[3]{3x+2}}$ at the point (2,0.5) [answer in general form].

3.5 marks

(b) The line $y = mx + b$ is a tangent to the curve $y = x^3 + 2x - 4$ at the point (-1,-7). Find m and b.

2.5 marks

START A NEW PAGE:
Question 7 (13 marks):

(a) A quadrilateral ABCD is to be plotted. The co-ordinates of the points A and B are (0,1) and (4,4) respectively.

Give a possible set of co-ordinates for C and D (give reasons for your answer), if ABCD is to be a square.

3 marks

(b) Consider the three vertices of a ΔABC where A(0,0), B($2q$, $2r$) and C($2p$, 0) where $r > 0$ and $0 < q < p$.

(i) On a set of axes draw a neat sketch to illustrate this ΔABC. **2 marks**

(ii) Show the area of ΔABC is $2pr$ units^2. **1 mark**

(iii) Find the co-ordinates of the points L and N where L is the midpoint of AB and N is the midpoint of BC.

2 marks

(iv) Find the equation of the line LN. **1 mark**

(v) Show the distance of LN is p units. **1 mark**

(vi) ***Describe*** fully, in words, ***two*** ways in which AC and LN are related. ***Justify*** your answer.

3 marks

SECTION D (24 marks):

START A NEW PAGE:
Question 8 (10 marks):

(a) Evaluate: (i) $\int_{0}^{2} (x^2 - 2)^2 \, dx$ **2 marks**

(ii) $\int_{1}^{4} \frac{x^3 + 4x^2 - 1}{x^2} \, dx$ **2 marks**

(b) Use the trapezoidal rule with three function values to find an approximate value for $\int_{1}^{3} (x^2 + 4) dx$. **3 marks**

(c) Use the Simpson's Rule to find an approximate value for the area between the curve and the x-axis from $x = 10$ to $x = 13$ defined by the following points:

x	10	10.5	11	11.5	12	12.5	13
$f(x)$	2.4	2.7	3.6	4.4	3.5	3.2	2.0

3 marks

START A NEW PAGE:
Question 9 (14 marks):

(a) (i) Draw a neat sketch of the parabola $x^2 = 12y$ clearly indicating the ***vertex, focus (S)*** and ***directrix.*** **2 marks**

(ii) Let P(x, y) be a point on this parabola and drop a perpendicular from P to meet the directrix at M (illustrate this information on your diagram).

1 mark

(iii) Explain clearly why $\angle PSM = \angle PMS$.

2 marks

(b) Find the equation of the parabola that has its focus at (1, 2) and directrix $x = 7$.

3 marks

(c) The cost per hour of running a truck at an average speed of v km h^{-1} is given by: $(64 + \frac{v^2}{100})$ dollars per hour.

(i) Show the total cost, C (in dollars), of running a truck from Sydney to Katoomba (a distance of 100 km) is $v + \frac{6400}{v}$.

(NB Speed (v) = $\frac{\text{Distance}}{\text{Time}}$ *)*

3 marks

(ii) What is the minimum cost?

3 marks

[redacted]

Year 12 Mathematics H-Y Examination: page 1.

Section A:

Q1 (a) $12x^2-3y^2$

$= 3(4x^2-y^2)$

$= 3(2x-y)(2x+y)$

b) $(x-3)(2x+1)=9$

$2x^2+x-6x-3=9$

$2x^2-5x-3=9$

$2x^2-5x-12=0$

$(2x+3)(x-4)=0$

$x=4, -\frac{3}{2}$

(c) $\frac{d}{dx}(2-5x)^8 = 8(2-5x)^7 \times -5$

$= -40(2-5x)^7$

(d) $\int (3x^2+4x-2)\,dx$

$= \frac{3x^3}{3} + \frac{4x^2}{2} - 2x + C$

$= x^3 + 2x^2 - 2x + C$

e) $\phi(x) = x + 2x^2$

$\phi'(x) = 1 + 4x$

$\phi''(x) = 4$

let $\phi'(x) = \phi''(x)$

$1+4x = 4$

$4x = 3$

$x = \frac{3}{4}$

Q2 $y = x^3 - 6x^2$

(a) $y' = 3x^2 - 12x$

$y'' = 6x - 12$

For St. Pts let $y'=0$

$3x^2-12x=0$

$x^2-4x=0$

$x(x-4)=0$

$x=0, 4$

let $x=0$: $y = 0^3 - 6(0^2)$ $= 0$ $\therefore (0,0)$

let $x=4$: $y = 4^3 - 6(4)^2$ $= -32$ $\therefore (4,-32)$

Check Max/Min:

let $x=0$: $y'' = 6(0)-12$ $= -12 < 0$ $\therefore$ ∩ MAX

let $x=4$: $y'' = 6(4)-12$ $= 12 > 0$ $\therefore$ ∪ MIN

So $(0,0)$ is maximum & $(4,-32)$ is a minimum.

(b) For Inflexion point let $y''=0$:

$y'' = 6x-12 = 0$

$6x = 12$

$x = 2$

$y = 2^3 - 6(2)^2$

$= 8 - 24$

$= -16$ $\therefore (2,-16)$

check concavity change:

x	2^-	2	2^+
y''	$-$	0	$+$

since the sign of y'' has changed either side of $x=2$ hence $(2,-16)$ is an inflexion point.

page 2

Q2 CONT:

(c) let $x=0$: $y=0$ (already found)

let $y=0$: $x^3-6x^2=0$

$x^2(x-6)=0$

$x=0, 6$

$\therefore$ Intercepts $(0,0)$ & $(6,0)$

(d) Endpoints:

$x=-1$: $y = (-1)^3 - 6(-1)^2$

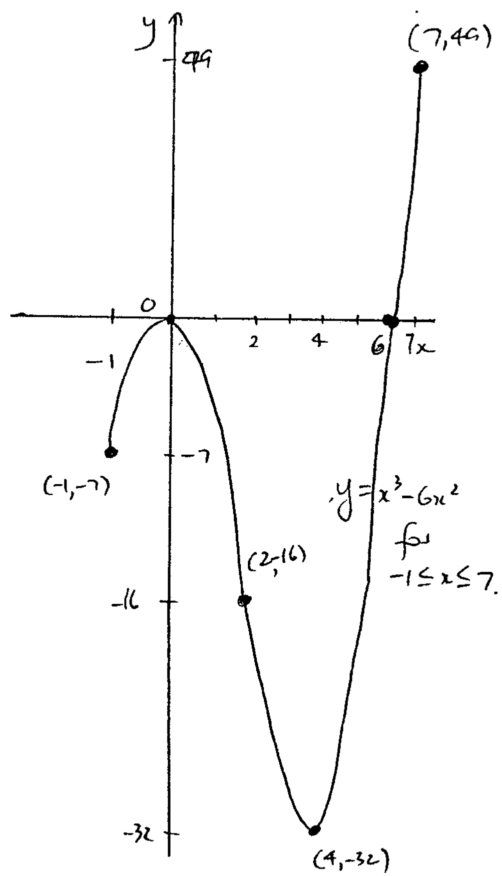
$= -1 - 6\times 1$

$= -1-6$

$= -7$ $\therefore (-1,-7)$

$x=7$: $y = 7^3 - 6(7)^2$

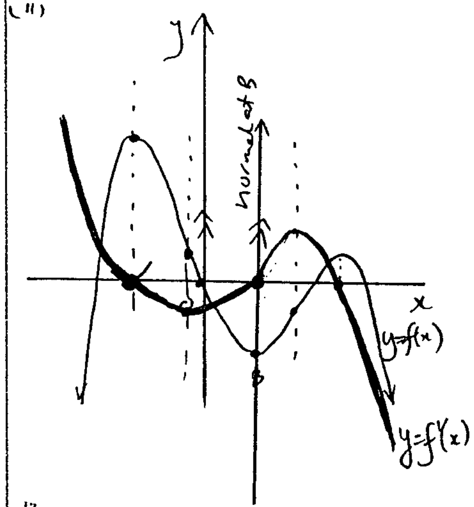
$= 49$ $\therefore (7,49)$



Q3

(a)(i) At point A the gradient of the tangent to the curve is +ve, at B the gradient is zero, at C the gradient is negative.

(ii)



(iii) See above

(b) $y = \frac{2x+5}{x-1}$ $\quad u = 2x+5,\ v = x-1$; $u' = 2,\ v' = 1$

$y' = \frac{vu' - uv'}{v^2}$

$= \frac{2(x-1) - 1(2x+5)}{(x-1)^2}$

$= \frac{2x - 2 - 2x - 5}{(x-1)^2}$

$= \frac{-7}{(x-1)^2}$

[let $x=2$]

$m_T = \frac{-7}{(2-1)^2}$

$= -7$

Find y: $y = \frac{2(2)+5}{2-1}$ $= 9$

Equation:

$y - y_1 = m(x-x_1)$

$y - 9 = -7(x-2)$

$y - 9 = -7x + 14$

$7x + y - 23 = 0$

[redacted]

Question 4:

(a)(i) $T_1 = 5$

$T_2 = 11$

$T_3 = 17$

$T_3 - T_2 = 17 - 11 = 6$
$T_2 - T_1 = 11 - 5 = 6$ } ∴ since there is a common difference hence

the series is ARITHMETIC

N.B.
For Geometric

$\frac{T_2}{T_1} = \frac{11}{5}$, $\frac{T_3}{T_2} = \frac{17}{11}$ } $\frac{11}{5} \neq \frac{17}{11}$ ∴ not Geometric

(ii) $T_n = a + (n-1)d$

$T_8 = 5 + 7 \times 6$

$= 47$

(iii) $S_n = \frac{n}{2}[a + l]$

$S_8 = \frac{8}{2}[5 + 47]$

$= 208$

(b)(i) A limiting sum exists, only if $|r| < 1$ [ie. the common ratio, r, lies between $-1 < r < 1$]

$r = \frac{T_3}{T_2} = \frac{0.8}{4} = 0.2$
or $r = \frac{T_2}{T_1} = \frac{4}{20} = 0.2$ } Common ratio $= 0.2$

since $|r| = |0.2| < 1$

∴ a limiting sum exists.

(ii) $S_\infty = \frac{a}{1-r} = \frac{20}{1-0.2} = 25.$

(c)
(i) $T_1 = 1$
$T_2 = 2$ } $a = 1$, $d = 1$

$T_3 = 3$

⋮

$T_k = k$

$S_n = \frac{n}{2}[a + l]$

$S_k = \frac{k}{2}[1 + k]$

$S_k = \frac{k(k+1)}{2}$

(ii) let $S_k = 666$

∴ $666 = \frac{k(k+1)}{2}$

$1332 = k(k+1)$

$1332 = k^2 + k$

$k^2 + k - 1332 = 0$

$(k+37)(k-36) = 0$

$k = -37, 36$

k must be positive integer

∴ $k = 36$

There are 36 rows of material

(d)(i) $A_n = P(1 + \frac{r}{100})^n$

$A_1 = 2000(1.04)^1$

$= 2080$

It has grown to \$2080 in the first year.

(ii) $A_2 = 2000(1.04)^2$

$A_1 = 2000(1.04)^1$

Total $= A_1 + A_2$

$= 2000(1.04)^1 + 2000(1.04)^2$

$= 2000(1.04)[1 + 1.04]$

$= \$4243.20$

(iii) let A_n be the amount each P has to which has grown in n years.

$A_1 = 2000(1.04)^1$ [sum put in on 1st Jan, 2009]

$A_2 = 2000(1.04)^2$ [. . . . . . . 2008]

⋮

$A_{24} = 2000(1.04)^{24}$ [sum put in on 1st Jan, 1986]

Total $= A_1 + A_2 + \ldots + A_{24}$

$= 2000(1.04) + 2000(1.04)^2 + \ldots + 2000(1.04)^{24}$

$= 2000(1.04)[1 + 1.04^1 + \ldots + (1.04)^{23}]$

$a = 1$, $r = 1.04$, $n = 24$, $S_n = \frac{a(r^n - 1)}{r-1}$

$= \frac{2000(1.04)(1)(1.04^{24} - 1)}{(1.04 - 1)}$

$= 81\ 291.816\ldots$

$= \$81\ 292$

Section C:

Q5(a)(i) (1): $\alpha + \beta$

$= -\frac{b}{a}$

$= \frac{-3}{2}$

(2): $\alpha\beta = \frac{c}{a}$

$= \frac{-6}{2}$

$= -3$

(3): $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$= -\frac{3}{2} \div -3$

$= \frac{1}{2}$

(ii) $\alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2(\alpha\beta)$

$= (-\frac{3}{2})^2 - 2 \times -3$

$= \frac{9}{4} + 6$

$= 2.25 + 6$

$= 8.25$, as required

(b) For Real Roots: $\Delta \geq 0$

∴ $b^2 - 4ac \geq 0$

$(2)^2 - 4 \times 1 \times k \geq 0$

$4 - 4k \geq 0$

$-4k \geq -4$

$k \leq 1$

(c) $px^2 - 4x - 7 = 0$

let α (first root)

let $\frac{1}{\alpha}$ (2nd root)

∴ $\alpha + \beta = -\frac{b}{a} \Rightarrow \alpha + \frac{1}{\alpha} = +\frac{4}{p}$. (1)

$\alpha\beta = \frac{c}{a} \Rightarrow \alpha \times \frac{1}{\alpha} = \frac{-7}{p}$. (2)

Using equation (2)

∴ $1 = \frac{-7}{p}$

$p = -7$

Q6

(a)(i) $y = \dfrac{1}{\sqrt[3]{3x+2}}$

$= (3x+2)^{-\frac{1}{3}}$

$\dfrac{dy}{dx} = -\frac{1}{3}(3x+2)^{-\frac{4}{3}} \times 3$

$= -1\,(3x+2)^{-\frac{4}{3}}$

$= \dfrac{-1}{\sqrt[3]{(3x+2)^4}}$, as required

(ii) Find gradient:

m_T (tangent) $= \dfrac{-1}{(\sqrt[3]{3\times2+2})^4}$

$= \dfrac{-1}{(\sqrt[3]{8})^4}$

$= -\dfrac{1}{2^4}$

$= -\dfrac{1}{16}$

$\therefore m_N$ (normal) $= +16$ (since $m_T \times m_N = -1$ for perp. lines)

Equation:

$y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = 16(x-2)$

$y - \frac{1}{2} = 16x - 32$

$2y - 1 = 32x - 64$

$\therefore 32x - 2y - 63 = 0$ is the equation of the normal.

(b) $y = x^3 + 2x - 4$

$y' = 3x^2 + 2$

[at $x = -1$]

$m_T = 3(-1)^2 + 2$

$= 5$

Eqn of tangent:

$y - y_1 = m(x - x_1)$

$y + 7 = 5(x+1)$

$y + 7 = 5x + 5$

$y = 5x - 2$

$\therefore m = 5,\ b = -2$

Alternatively:

$y = mx + b$

[let $m = 5$, $x = -1$, $y = -7$]

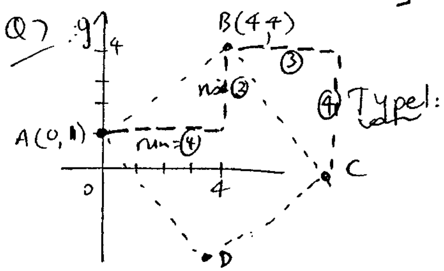
$-7 = 5(-1) + b$

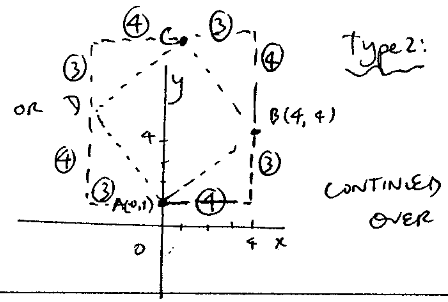
$-7 = -5 + b$

$-2 = b$

$\therefore m = 5,\ b = -2$

Q7





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Q7 Type 1: opposite sides of square have equal gradients (parallel sides)

$m_{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{4-1}{4-0}$

$= \dfrac{3}{4}$

C $(x_2\ y_2)$

B $(4\ 4)$ x_1, y_1

for $m_{BC} = -\dfrac{4}{3}$, so $-\dfrac{4}{3} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$-\dfrac{4}{3} = \dfrac{y_2 - 4}{x_2 - 4}$

$\therefore x_2 - 4 = 3 \Rightarrow x_2 = 7$

$y_2 - 4 = -4 \Rightarrow y_2 = 0$

$\therefore C(7, 0)$

Similarly: $m_{AD} = -\frac{4}{3}$ (BC // AD → opp. sides of square are parallel)

$\therefore -\dfrac{4}{3} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$-\dfrac{4}{3} = \dfrac{y_2 - 1}{x_2 - 0}$

so $y_2 - 1 = -4 \Rightarrow y_2 = -3$

$3 = x_2 - 0 \Rightarrow x_2 = 3$

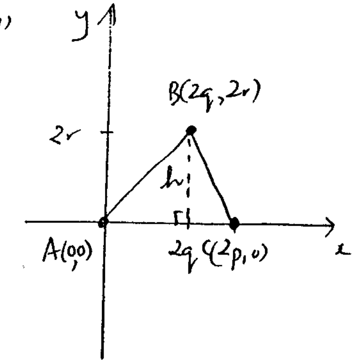
$\therefore D(3, -3)$

Type 2: Similarly:

$C(1, 8)$

& $D(-3, 5)$

(b)(i)



(ii) Area $= \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times AC \times h$

$= \frac{1}{2} \times 2p \times 2r$

$= (2pr)$ units^2

(iii) $L = M_{AB}$

$= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

$= \left(\dfrac{0 + 2q}{2}, \dfrac{0 + 2r}{2}\right)$

$= (q, r)$

$N = M_{BC}$

$= \left(\dfrac{2q + 2p}{2}, \dfrac{2r + 0}{2}\right)$

$= (p + q, r)$

(iv) equation of LN: $y = r$

(v) $LN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(p + q - q)^2 + (r - r)^2}$

$= \sqrt{p^2}$

$= p$ units

(vi) $d_{AC} = 2p$ units, $d_{LN} = p$ units] $d_{AC} = 2d_{LN}$

LN has equation $y = r$ where

AC has equation $y = 0$ (x-axis)

LN is half the distance of AC and LN // AC (since both have zero gradients).

Section D:

Q8

(a)(i) $\int_0^2 (x^2-2)^2\,dx$

$= \int_0^2 (x^4 - 4x^2 + 4)\,dx$

$= \left[\frac{x^5}{5} - \frac{4x^3}{3} + 4x\right]_0^2$

$= \left(\frac{2^5}{5} - \frac{4(2)^3}{3} + 4(2)\right) - \left(\frac{0^5}{5} - \frac{4(0)^3}{3} + 4(0)\right)$

$= \frac{32}{5} - \frac{32}{3} + 8$

$= 3\frac{11}{15}$

(ii) $\int_1^4 \frac{x^3 + 4x^2 - 1}{x^2}\,dx$

$= \int_1^4 \left(\frac{x^3}{x^2} + \frac{4x^2}{x^2} - \frac{1}{x^2}\right)dx$

$= \int_1^4 (x + 4 - x^{-2})\,dx$

$= \left[\frac{x^2}{2} + 4x - \frac{x^{-1}}{-1}\right]_1^4$

$= \left[\frac{x^2}{2} + 4x + \frac{1}{x}\right]_1^4$

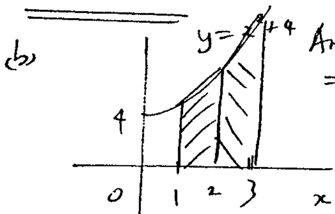
$= \left(\frac{4^2}{2} + 4(4) + \frac{1}{4}\right) - \left(\frac{1^2}{2} + 4(1) + \frac{1}{1}\right)$

$= \left(8 + 16 + \frac14\right) - \left(\frac12 + 4 + 1\right)$

$= 24.25 - 5.5$

$= 18.75$

(b)



Area of trapezium $= \frac12 \times h \times$ (sum of // sides)

$A \doteq \frac12 \times (2-1) \times [f(1) + f(2)] + \frac12 \times (3-2) \times [f(2) + f(3)]$

$= \frac12 \times 1 \times [(1^2+4) + (2^2+4)] + \frac12 \times 1 \times [(2^2+4) + (3^2+4)]$

$= \frac12 \times 1 \times [5+8] + \frac12 \times 1 \times [8+13]$

$= 6.5 + 10.5$

$= 17$ units^2

Alternatively:

$A \doteq \frac{b-a}{2n}[y_0 + 2y_1 + y_2]$ (where $n=2$)

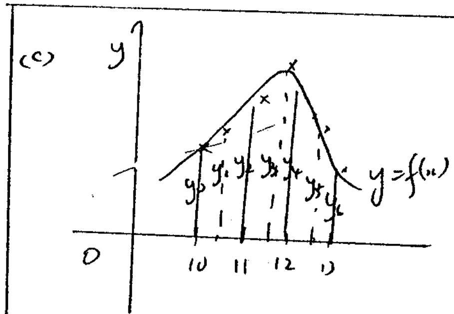
$= \frac{3-1}{2\times2}[(1^2+4) + 2(2^2+4) + (3^2+4)]$

$= \frac{2}{4}[5 + 2\times8 + 13]$

$= \frac12[34]$

$= 17$ units^2

(c)



$A \doteq \int_a^b f(x)$ $h = \frac{b-a}{n}$ {n+1 function values}

$\doteq \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

7 function values (n+1) $\Rightarrow n = 6$; $h = \frac{13-10}{6} = 0.5$

$A = \frac{0.5}{3}[f(10) + f(13) + 4(f(10.5) + f(11.5) + f(12.5)) + 2(f(11) + f(12))]$

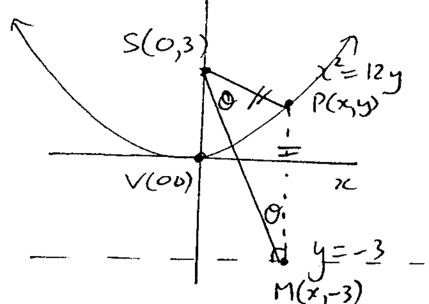
$= \frac{0.5}{3}[2.4 + 2.0 + 4(2.7 + 4.4 + 3.2) + 2(3.6 + 3.5)]$

$= \frac{0.5}{3}[(4.4) + (4 \times 10.3) + 2 \times (7.1)]$

$= 9.9\dot{6}$ units^2

Q9(a)(i) $x^2 = 12y$

$[x^2 = 4ay] \Rightarrow 4a = 12$, $a = 3$



Vertex: (0,0)

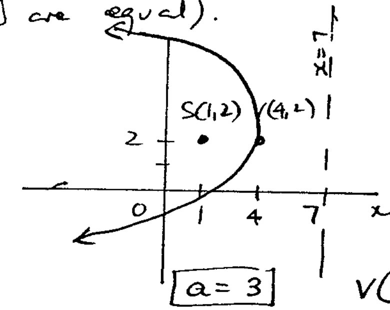
Focus: (0,3)

directrix: $y = -3$

(ii) By definition of a parabola, a point on the curve is equidistant from the point to the focus as from the point to the directrix, ie. PS = PM.

So ΔPSM is isosceles, hence ∠PSM = ∠PMS (angles opp. equal sides in an isosceles Δ are equal).

(b)



h, k
$V(4,2)$

$(y-k)^2 = -4a(x-h)$

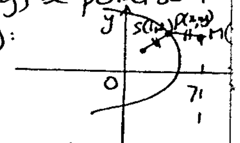
$(y-2)^2 = -4 \times 3 \times (x-4)$

$(y-2)^2 = -12(x-4)$

OR $y^2 - 4y + 4 = -12x + 48$

$12x + y^2 - 4y - 44 = 0$

Alternatively: let $P(x,y)$ be point on curve (parabola):



$PS = PM$

$\Rightarrow PS^2 = PM^2$

$(x-1)^2 + (y-2)^2 = (x-7)^2 + (y-y)^2$

$x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 - 14x + 49$

$12x + y^2 - 4y - 44 = 0$

(c) Cost per hour $= \left(64 + \frac{v^2}{100}\right)$ dollars/hour

(i) Speed = $\frac{\text{Distance}}{\text{Time}} \Rightarrow$ Time = $\frac{\text{Distance}}{\text{Speed}}$

# of hours $= \frac{100}{v}$

$\therefore C = \left(\text{Cost per hour}\right) \times (\text{\# of hours})$

$= \left(64 + \frac{v^2}{100}\right) \times \frac{100}{v}$

$= \frac{64 \times 100}{v} + \frac{v^2}{100} \times \frac{100}{v}$

$= \left(\frac{6400}{v} + v\right)$ dollars, as required.

(ii) $C = v + \frac{6400}{v}$

$= v + 6400v^{-1}$

$\frac{dC}{dv} = 1 - 6400v^{-2}$ or $1 - \frac{6400}{v^2}$

$\frac{d^2C}{dv^2} = +12800v^{-3}$ or $\frac{12800}{v^3}$

For "St Points" let $\frac{dC}{dv} = 0$

$1 - \frac{6400}{v^2} = 0$

$\frac{6400}{v^2} = 1$

$v^2 = 6400$

$v = \sqrt{6400}$

$= 80$ (take +ve only as speed is required)

Check Min:

$\frac{d^2C}{dv^2} = \frac{12800}{80^3} > 0$ ∪ MIN

Cost $= v + \frac{6400}{v}$

$= 80 + \frac{6400}{80}$

$= 80 + 80$

$= 160$

The minimum cost for the trip is $160.