Miscellaneous 3 Unit Exercises

(1) (a) Find the values of x for which $\dfrac{3-x}{x+2} > 4$.

(b) (i) Write the equation $(x^2 - x + 1)y = 2x$ as a quadratic equation in x.
(ii) Given that x has real values, use (i) to find the range of values for y.

(2) (a) Solve the inequality

$(5x-7)(x-3) > 16x$.

(b) For what range of values of x is there no real solution to the equation $y^2 = x(1-x)$.

(3) The logarithmic function $y = \ln x$ may be defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad \text{for } x > 0.$$

(a) Use this definition and the substitution $t = \dfrac{1}{u}$ to show that $\ln x = -\ln\left(\dfrac{1}{x}\right)$.

(b) Use the definition of $\ln x$ to write down a definition for $\ln(x^n)$ as an integral. By using the substitution $t = u^n$ in this integral, prove that $\ln(x^n) = n \cdot \ln x$.

(4) A body is placed in a freezer to cool, and the rate of cooling is assumed to be proportional to the difference between its temperature T° and the constant temperature, -5°, of the freezer. This can be expressed by the differential equation $\dfrac{dT}{dt} = k(T+5)$, where t is the time in hours and k is a constant.

(a) Show that $T = Be^{kt} - 5$, where B is a constant, is a solution of the differential equation.

(b) The body cools from 30° to 20° in half an hour. Find its temperature after a further half hour, to the nearest degree.

(5) The population N of a country town increases at a rate proportional to the excess of the population above 1000. This can be expressed by the differential equation $\dfrac{dN}{dt} = k(N - 1000)$, where t is the time in years, and k is a constant.

(a) Show that $N = 1000 + Be^{kt}$, where B is a constant, is a solution of the differential equation.

(b) The population of the town increases from 2000 to 4000 in two years. Show that $k = \frac{1}{2}(\ln 3)$ and find out how many are in the town after four years.

(c) How long does it take for the population to reach 8000?

(6) Prove each of the following results by mathematical induction, given that n represents a positive integer:

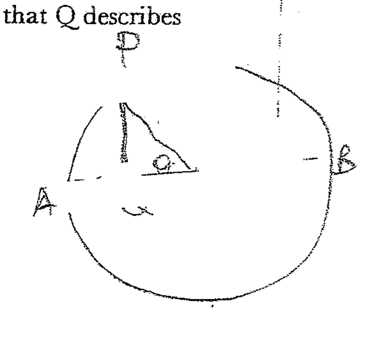
(a) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2}{4}(n+1)^2$

(b) $\dfrac{1}{1.2.3} + \dfrac{1}{2.3.4} + \dfrac{1}{3.4.5} + \cdots + \dfrac{1}{n(n+1)(n+2)} = \dfrac{1}{4} - \dfrac{1}{2(n+1)(n+2)}$

(7) (a) Prove that $n^5 - n$ is divisible by 30 for all positive integers n.

(b) Prove by induction that $6^n < n!$ for all integers $n >$ some n_0, and find the least value of n_0.

(8) P moves around a circle with diameter AB with constant angular speed $\omega = \dfrac{d\theta}{dt}$. Q is the projection of P on AB, i.e., for each position of P, Q lies on AB and PQ is perpendicular to AB. Show that Q describes Simple Harmonic Motion.



A D C B

(9) A particle P moves in a straight line ADCB, where AD = DC, and AC = CB. The speed of P as it passes through C is $10\,\text{m.s}^{-1}$, and it is instantaneously at rest at A and B. P performs four oscillations per second. $T = \frac{2\pi}{n}$
(a) Find the length of the interval AB.
(b) Find the speed of P as it passes through D.

(10) The velocity v of a body is inversely proportional to its displacement x. Initially $x = 6$, and after one second, $x = 7$. Find the equation which relates x to time t.

(11) The manufacturers of 'Wizzo' dog biscuits carefully monitor the number of people n who buy their product at any time t . . . They discover that in January the number of customers is decreasing at a rate proportional to the square root of n, and mount a large advertising campaign. In February, the number of people buying 'Wizzo' remains constant, and in March the number of customers increases at a rate proportional to n. Write down differential equations which express the progress of sales for each of January, February and March.

(12) A curve has a gradient which is inversely proportional to x^2. The curve also passes through the points (2, 4), (4, 5). Find the equation of the curve.

(13) (a) Draw a sketch of the curve $y^2 = \dfrac{1}{x^4}$ by first drawing the curves $y = x^2, y = \dfrac{1}{x^2}$.
(b) Use implicit differentiation to show that
$$\frac{dy}{dx} = \frac{-2}{x^5 y}.$$
(c) (i) In which quadrant(s) must the tangents to the curve have negative gradient?
(ii) Find the coordinates of the point(s) on the curve for which the tangent has a gradient of -2.

(14) (a) Draw a sketch of the curve $y = \sec^2 x$ for $0 \leqslant x \leqslant 2\pi$ by first drawing the curves $y = \cos x$, $y = \cos^2 x$ in this domain.
(b) What is the area contained between the curve, the x axis and the ordinates $x = 0$, $x = \dfrac{\pi}{4}$?
(c) What is the value of x_0 such that the area contained between the curve, the x axis and the ordinates $x = 0$, $x = x_0$ is $\dfrac{1}{\sqrt{3}}$?
(d) If in (c) the area contained is changed to $\dfrac{2}{\sqrt{3}}$, is the corresponding position of the ordinate changed to $2x_0$?

(15) P is any point $(2at, at^2)$ on the parabola $x^2 = 4ay$. A line parallel to the x axis passes through P and intersects the line $x = -2a$ at Q. Find the equation of the locus of M, the midpoint of PQ.

(16) Prove that if PQ is any focal chord of the parabola $x^2 = 4ay$ then the tangents at P, Q intersect on the directrix and at right angles.

(17) A variable chord PQ of the parabola $x^2 = 4ay$ is drawn such that the chord has a fixed gradient m, i.e. the equation of PQ is $y = mx + c$, where m is constant. Find the locus of M, the midpoint of PQ.

(18) OA, OB, OC are three intervals at right angles to each other. OA = h cm, $\widehat{OAB} = 45°$, $\widehat{OAC} = 60°$. Show that $\cos \widehat{BAC} = \dfrac{1}{2\sqrt{2}}$, and BC = $2h$ cm. What is the exact value of $\cos \widehat{ACB}$?

(19) A right pyramid ABCDE has a square base ABCD of side $2x$. The face BCE makes an angle α with the base, and the edge BE makes an angle β with the base. Show that $\cot^2\beta = 2\cot^2\alpha$.

(20) A man observes two towers, one due north of him, and the other tower due east of the first. The angle of elevation β of both towers is the same, but the height of one is twice the height of the other. The angle of elevation of the top of one tower from the top of the other is α. Show that $\cot^2\alpha = 3\cot^2\beta$, and find the bearing of the second tower from the man.

(21) The situation described in (20) applies again, except that the second tower is no longer due east of the first, but on a bearing of θ°T from the man. Show that $\cot^2\alpha = \cot^2\beta\,(5 - 4\cos\theta)$.

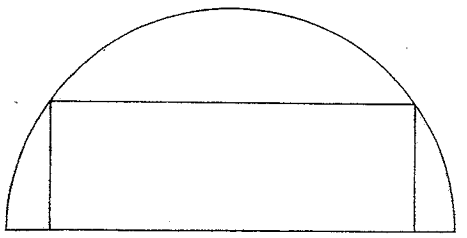
(22) Show that $y = e^x \sin x$ has turning points at intervals of π in x. Distinguish between maxima and minima, and show that successive maximum values are in the ratio $e^{2\pi}$.

(23) A right circular cylinder of height h is inscribed in a sphere of radius a.
(a) Show that the volume of the cylinder is
$$\frac{\pi h}{4}(4a^2 - h^2).$$
(b) The volume of the cylinder is to be made as large as possible. Find the height of the cylinder in terms of a.

(24) A right circular cylinder of height h is inscribed in a sphere of radius a.
(a) Show that the curved surface area of the cylinder is $\pi h\sqrt{4a^2 - h^2}$.
(b) The curved surface area is to be as large as possible. Show that $h = \sqrt{2}a$, and that the height of the cylinder must be equal to its diameter.

(25) A right circular cone has its vertex at the centre of a given sphere, and its base is bounded by a circle which lies on the surface of the sphere. Show that the greatest possible volume of the cone is $\frac{\sqrt{3}}{18}V$, where V is the volume of the sphere.

(26)



A rectangle is inscribed in a semi-circle of radius a, with one of its sides along the diameter of the semi-circle as shown. What are the largest and smallest perimeters of the rectangle?

(27) (a) An urn contains five blue marbles and three red marbles. Marbles are drawn one by one from the urn without replacement. What is the probability of first drawing a blue marble, then a red, then a blue, etc. until there are only blue marbles left?
(b) Suppose the urn contains b blue marbles, and r red marbles, where $b > r$. What is the probability now?

$\frac{5\times4\times3\times3!}{{}^8P_6} = \frac{360}{{}^8P_6} = \frac{1}{56}$

(28) (a) Two urns each contain two black and one white balls. A ball is drawn at random from the first urn and placed in the second; a ball is then drawn from the second and placed in the first. What is the probability that each urn still contains two black and one white balls?
(b) Would it make any difference if the two urns each contained one black and two white balls?

(29) Three urns each contain one white and two black balls. A ball is drawn from the first and placed in the second; then a ball is drawn from the second and placed in the third; then a ball is drawn from the third and placed in the first. What is the probability now that each urn still contains one white and two black balls?

(30) A touring cricket side of 15 players contains five regular bowlers.
(a) How many different elevens can be picked which contain exactly three of the five regular bowlers?
(b) What is the probability that if an eleven is picked at random it will contain only one regular bowler?
(c) What is the probability that if an eleven is picked at random it will contain at least three of the regular bowlers?

(31) Find the greatest coefficient in the expansion of $(2x + 3)^{12}$.

(32) Find the greatest term(s) in the expansion of $(2x + 3y)^{12}$, where $x = 1, y = 3$.

(33) Prove that for

$$1 \leqslant r \leqslant n, \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

(34) Use the identity
$(1 + x)^{3n} = (1 + x)(1 + x)^{2n}$ to prove that

$$\binom{n}{0}\binom{2n}{0} + \binom{n}{1}\binom{2n}{1} + \cdots + \binom{n}{n}\binom{2n}{n} = \binom{3n}{n}.$$

(35) (a) Using the substitution $u = 1 + x$, evaluate $\int_0^1 x(1 + x)^n dx$.
(b) Use the binomial expansion of $(1 + x)^n$ to write down an expansion for $x(1 + x)^n$.
(c) By integrating your expression in (b) evaluate

$$\frac{1}{2}\binom{n}{0} + \frac{1}{3}\binom{n}{1} + \frac{1}{4}\binom{n}{2} + \cdots + \frac{1}{n+2}\binom{n}{n}.$$

(36) In a certain variety of flower, the chance that a seed produces pink flowers is known to be $\frac{1}{5}$. Show that if the probability of obtaining at least one pink flower is to be at least 99%, then the number of seeds I should sow is

$$\frac{2 \ln 10}{\ln 5 - 2 \ln 2}.$$

(37) A large box contains red and green sweets mixed together in the radio 3 : 2. A handful of twenty sweets is taken at random from the box.
(a) Find the probability that there will be precisely 15 red sweets in the handful.
(b) Find the probability that the handful of sweets is such that it can be shared between five children so that each child receives the same number of red sweets, this number being at least two.

Chapter 9

(1) (a) $-2 < x < -1$

(b) (i) $x^2y - x(y+2) + y = 0$

(ii) $-\frac{2}{3} \leqslant y \leqslant 2$

(2) (a) $x > 7$ or $x < \frac{3}{5}$

(b) $x < 0$; $x > 1$

(3) $\int_1^{x^n} \frac{1}{t}\,dt$

(4) $13°$

(5) (b) 10 000

(c) approx. 3.5 years

(7) $n_0 = 7$

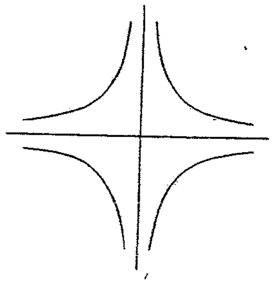
(9) (a) $\dfrac{5}{2\pi}$ (b) $5\sqrt{3}$

(10) $x^2 = 13t + 36$

(11) $\dfrac{dn}{dt} = -k_1\sqrt{n}$, $\dfrac{dn}{dt} = 0$; $\dfrac{dn}{dt} = k_2 n$,
where $k_1, k_2 > 0$, const.

(12) $y = \dfrac{-4}{x} + 6$

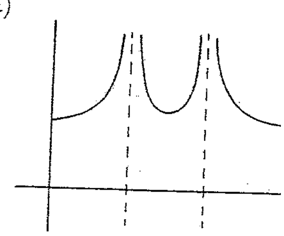
(13) (a)



(c) (i) 1st, 3rd

(ii) $(\pm 1, \pm 1)$; $(\pm 1, \mp 1)$

(14) (a)



(b) 1 sq. unit

(c) $\dfrac{\pi}{6}$

(d) No

(15) $(x+a)^2 = ay$

(17) $x = 2am$

(18) $\frac{3}{4}$

(20) $60°$ T

(23) (b) $h = \dfrac{8a}{3}$

(26) $\dfrac{10a}{\sqrt{5}}$; $2a$

(27) (a) $\dfrac{1}{56}$ (b) $\dfrac{b!\ r!}{(b+r)!}$

(28) (a) $\frac{2}{3}$ (b) No

(29) $\frac{11}{24}$

(30) (a) 450 (c) $\frac{12}{13}$

(b) $\frac{1}{273}$

(31) $792 \cdot 2^5 \cdot 3^7$

(32) $11 \cdot 2^3 \cdot 3^{21}$

(35) (a) $\dfrac{2^{n+2}-1}{n+2} - \dfrac{2^{n+1}-1}{n+1}$

(b) $\binom{n}{0}x + \binom{n}{1}x^2 + \cdots + \binom{n}{n}x^{n+1}$

(c) as for (a)

(37) (a) $\binom{20}{5} \cdot \dfrac{3^{15} \cdot 2^5}{5^{20}}$

(b) $\dfrac{3^{20} + \binom{20}{5}3^{15} \cdot 2^5 + \binom{20}{10}3^{10} \cdot 2^{10}}{5^{20}}$

(38) (a) 0.216

(b) 0.784