Miscellaneous 4 Unit Exercises

(1) The curve in the Argand diagram for which $|z-4|+|z+6|=14$ is an ellipse. Find the coordinates of the centre, the lengths of the major and minor axes of the ellipse, and write down its equation.

(2) The curve in the Argand diagram for which $|z-6|-|z+6|=10$ is a hyperbola. Find the coordinates of the centre, the lengths of the transverse and conjugate axes, and write down its equation. What are the equations of the asymptotes?

(3) $\bar{z}$ denotes the conjugate of z, i.e. if $z=x+iy$, then $\bar{z}=x-iy$. Prove that $z=\bar{z}$ if and only if z is real.

(4) Find the coordinates of the point which represents $\dfrac{3+4i}{1-2i}$ in the Argand plane, and the complex number represented by the reflection of the point in the line $y=x$.

(5) A represents $6i$ in the Argand diagram, B represents 3, and P represents the complex number z. If P moves such that PA = 2 PB, show that

$$z\bar{z}=(4+2i)z+(4-2i)\bar{z}.$$

Show that P moves on a circle, and find its centre and radius.

(6) Determine z such that $|z+3i|=|z+5-2i|$ and $|\bar{z}-4i|=|z+2i|$.

(7) Prove that for any two complex numbers z_1, z_2,

$$|z_1-z_2|^2+|z_1+z_2|^2=2|z_1|^2+2|z_2|^2,$$

and give a geometric interpretation of this result.

(8) if $|z_1-z_2|=|z_1+z_2|$, show that

$$\arg z_1-\arg z_2=\frac{\pi}{2}.$$

(9) Find the values of the constants A, B, C such that

$$x^3+1\equiv x(x-1)(x-2)+Ax(x-1)+Bx+C.$$

(10) (a) Factorise x^4-9 into irreducible factors over the real numbers.
(b) Hence show that

$$\frac{1}{x^4-9}=\frac{\sqrt{3}}{36(x-\sqrt{3})}-\frac{\sqrt{3}}{36(x+\sqrt{3})}-\frac{1}{6(x^2+3)}.$$

(11) Find the quotient and remainder when

$8x^4-4x^2+7x$ is divided by $4x+3$.

(12) Reduce the following polynomials to irreducible factors over the real field.
(a) $2x^4+x^2-3$
(b) x^4+2x^3-x-2
(c) x^6-64

(13) Reduce the following polynomials to irreducible factors over the complex numbers:
(a) x^4+2x^2-3
(b) x^6-8
(c) $(x^2+2x)^2-9$

(14) The polynomial equation $x^3-px+5=0$ has roots α, β, γ. Find, in terms of p

(a) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$

(b) $\alpha^2+\beta^2+\gamma^2$

(c) $\dfrac{\alpha}{\beta\gamma}+\dfrac{\beta}{\alpha\gamma}+\dfrac{\gamma}{\alpha\beta}$

(d) $\alpha^3+\beta^3+\gamma^3$

(15) Express $\dfrac{2x+6}{(x+1)(x^2+3)}$ in partial fractions.

(16) Find constants a, b, c, such that

$$x^2 - 5x + 2 \equiv a(x+1)^2 + b(x+1) + c.$$

(17) (a) A circle has equation $x^2 + y^2 + 2gx + 2fy + c = 0$. Write down the coordinates of its centre C and the length of its radius r.

(b) P(x_1, y_1) is a point outside the circle. Write down expressions for the length PC and t, the length of the tangent from P to the circle, and hence show that

$$t^2 = x_1{}^2 + y_1{}^2 + 2gx_1 + 2fy_1 + c.$$

(c) Find the length of the tangent:

(i) from $(0, 0)$ to $x^2 + y^2 - 6x + 4y + 2 = 0$.

(ii) from (a, b) to $x^2 + y^2 - ax - by + c = 0$.

(18) Find the equations of each of the following

(a) parabola, focus $(-2, -1)$, vertex $(-1, -1)$.

(b) ellipse, major axis $= 6$, minor axis $= 4$, major axis $y = 0$, minor axis $x = 0$.

(c) ellipse, extremities of major axis $(\pm 2, 0)$ and eccentricity $\dfrac{\sqrt{2}}{2}$.

(d) hyperbola, vertices $(\pm 2, 0)$, and eccentricity $\sqrt{2}$.

(e) rectangular hyperbola, coordinates axes as asymptotes, passing through the point $(75, -12)$.

(19) P$(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci S, S′. p, p' are the distances from S, S′ respectively to the tangent at P. Show that $pp' = b^2$.

(20) PQ is a chord of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, with fixed gradient m, i.e. PQ has equation $y = mx + c$ when m is a constant. M is the midpoint of PQ. Show that M always lies on the line $y = \dfrac{b^2 x}{a^2 m}$.

(21) Prove that the tangents at the extremities of a focal chord of the rectangular hyperbola $xy = c^2$ meet on a fixed line. Find the equation of the line.

(22) If a sequence $\{u_n\}$ is defined by $u_{n+1} = 3u_n - 2u_{n-1}$, where $n \geqslant 2$, and it is also given that $u_1 = 0$, $u_2 = 2$, prove that $u_n = 2^n - 2$ using mathematical induction.

(23) A sequence $\{u_n\}$ is defined by $u_1 = 4$, and then $u_n = u_1 + u_2 + \cdots + u_{n-1}$, where $n \geqslant 2$; prove that $u_n = 2^n$ for $n \geqslant 2$.

(24) What happens in (23),

(a) if $u_1 = 0$?

(b) if $u_1 = 2$?

(25) A sequence $\{u_n\}$, the Fibonacci numbers, is defined by

$$u_1 = 1,\ u_2 = 1,\ u_{n+1} = u_n + u_{n-1}, \text{ for } n \geqslant 2.$$

$a = \dfrac{1+\sqrt{5}}{2}$, $b = \dfrac{1-\sqrt{5}}{2}$ are the roots of the quadratic $x^2 - x - 1 = 0$. Prove by induction that $u_n = \dfrac{1}{\sqrt{5}}(a^n - b^n)$.

(26) Find each of the following integrals:

(a) $\displaystyle\int \frac{(x-2)\,dx}{x(x^2-1)}$

(b) $\displaystyle\int \frac{dx}{x^2\sqrt{9+x^2}}$

(c) $\displaystyle\int \sin x \tan x\, dx$

(d) $\displaystyle\int_0^1 \frac{x^2\,dx}{x^6+1}$

(27) Find $\displaystyle\int \sin x \cos x\, dx$

(a) by using the substitution $u = \sin x$

(b) by using the substitution $u = \cos x$

(c) by using the double angle result.

(d) Which trigonometrical formula do these three results illustrate?

(28) (a) Use integration by parts to find $\int x^2 dx$.

(b) Use integration by parts to find $\int x^n dx$, where n is a positive integer > 2.

(29) Show that $\displaystyle\int_0^{\pi/6} \frac{\cos x + \sin x}{\cos x - \sin x}\,dx = \ln(\sqrt{3}+1)$.

(30) Given that $0 < a < b$, sketch the graph of $y = |x - a|$, for $-b \leqslant x \leqslant b$. Hence, or otherwise, evaluate

$$\int_{-b}^{b} |x - a|\, dx.$$

(31) Given that $\displaystyle\int_0^{\pi} x\,\mathrm{f}(\sin x)\,dx = \pi \cdot \int_0^{\pi/2} \mathrm{f}(\sin x)\,dx$

use this result to evaluate $\displaystyle\int_0^{\pi} x \sin^2 x\, dx$.

(32) (a) Sketch the area bounded by the x axis, the curve $y = \ln x$ and the ordinate $x = n + 1$, where n is a positive integer.

(b) This area is divided into n parts by ordinates $x = 1, x = 2, \ldots, x = n + 1$. Show that the sum of lower rectangles $= \ln(n!)$. Show that the sum of upper rectangles $= \ln[(n+1)!]$.

(c) Hence show that

$$\ln(n!) < \int_1^{n+1} \ln x \, dx < \ln[(n+1)!].$$

(d) By evaluating $\int_1^{n+1} \ln x \, dx$ directly, show that

$$\ln(n+1) < \frac{\ln(n!) + n}{n}.$$

(33) First sketch the graph of $y = \ln x$, and then sketch each of the following:
(a) $y = \ln(x-1)$
(b) $y = -\ln(x-1)$
(c) $y = 1 - \ln(x-1)$
(d) $y = x\ln(x-1)$
(e) $y = \ln x(x-1)$

(34) A particle moves in a straight line subject to the condition that $\ddot{x} = t\sin t$. Initially, the particle is at the origin and has initial velocity zero. Express the displacement x as a function of t, and find the displacement of the particle after $\frac{\pi}{4}$ seconds.

(35) The area bounded by the rectangular hyperbola $x^2 - y^2 = a^2$ and the lines $y = \pm b$ is rotated about the y axis. Find the volume of the solid of revolution.

(36) The area bounded by the curve $\sqrt{x} + \sqrt{y} = C$, where C is a constant, the x axis and the y axis is rotated about the x axis. Find the volume of the solid of revolution.

(37) (a) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about the y axis. Find the volume of the solid of revolution using cylindrical shells.
(b) The ellipse is now rotated about the x axis. Find the volume of the solid of revolution.

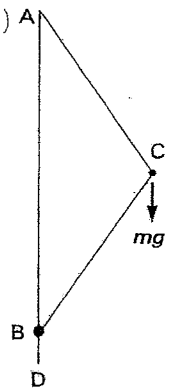
(38) Write down the general solution of the equation $\frac{d^2x}{dt^2} = -n^2x$, where n is a constant. A particle P moves in the x-y plane; its coordinates (x, y) satisfy the equations:

$\frac{d^2x}{dt^2} = -n^2x$, $\frac{d^2y}{dt^2} = -n^2y$, and when $t = 0$, $x = a, y = 0, \frac{dx}{dt} = 0, \frac{dy}{dt} = bn$, where n, a, b are positive constants. Prove that the equation of the path of P is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(39) A race called the Matrices live on an isolated island called Vector. The birth rate is proportional to the population at any time, and the rate of deaths is proportional to the square of the population. If the population at time t is p, show that $\frac{dp}{dt} = ap - bp^2$, where a, b are positive constants. Solve the differential equation for p in terms of t, given that when $t = 0$, $p = \frac{2a}{3b}$. Show that the limiting size of the population is $\frac{a}{b}$.

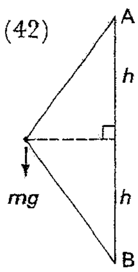
(40) A smooth particle is placed at the highest point of a smooth sphere. It is slightly disturbed. Prove that it leaves the sphere when its height above the horizontal plane through the centre of the sphere is reduced by one-third.

(41)



A particle C of mass m is attached by light inextensible strings AC, BC, each of length l, to a fixed point A, and, at B, to a ring of mass M which slides without friction on a vertical rod AD, as shown in the diagram. The particle C rotates about AD with constant angular velocity ω, and the tensions in the strings AC, BC are T_1, T_2 respectively. Given that each string makes an angle θ with the vertical, write down three equations connecting θ, T_1, T_2 with m, M, g, l and ω, and hence prove that $\omega^2 > \frac{(2M+m)g}{ml}$.

(42)



A, B are two fixed points with A at a distance $2h$ above B. A string of length $2a$, where $a > h$, is attached to A, B and a particle of mass m is fixed to the midpoint of the string. The system rotates about AB with constant angular velocity ω, and both parts of the string are taut.
(a) Find expressions for the tensions in the upper and lower halves of the string and deduce that $\omega^2 < \frac{g}{h}$.
(b) If instead of having a particle at its midpoint, the string carries a smooth ring of mass m which can slide freely on the string, and if this new system is rotated about AB with constant angular velocity Ω such that the lower part of the string is horizontal, prove that $\Omega^2 = \frac{ga^2}{h(a^2 - h^2)}$.

Chapter 10

(1) $(-1, 0)$, 14, $4\sqrt{6}$; $\dfrac{(x+1)^2}{49} + \dfrac{y^2}{24} = 1$

(2) $(0, 0)$, 10, $2\sqrt{11}$; $\dfrac{x^2}{25} - \dfrac{y^2}{11} = 1$;

$y = \pm\dfrac{\sqrt{11}}{5}x$

(4) $-1 + 2i,\ 2 - i$

(5) $4 - 2i,\ 2\sqrt{5}$

(6) $-1 + i$

(7) Sum of squares of diag. of parallelogram

(8) Property of square

(9) $A = 3,\ B = 1,\ C = 1$

(10) (a) $(x + \sqrt{3})(x - \sqrt{3})(x^2 + 3)$

(11) $2x^3 + \frac{3}{2}x^2 + \frac{1}{8}x + \frac{53}{32};\ -\frac{159}{32}$

(12) (a) $(2x^2 + 3)(x + 1)(x - 1)$

(b) $(x - 1)(x^2 + x + 1)(x + 2)$

(c) $(x+2)(x-2)(x^2-2x+4)(x^2+2x+4)$

(13) (a) $(x + 1)(x - 1)(x + \sqrt{3}i)(x - \sqrt{3}i)$

(b) $(x - \sqrt{2})(x + \sqrt{2})(x + 1 - \sqrt{3}i) \times$
$(x + 1 + \sqrt{3}i)$

(c) $(x+3)(x-1)(x+1+\sqrt{2}i)(x+1-\sqrt{2}i)$

(14) (a) $\dfrac{p}{5}$ (c) $-\frac{2}{5}p$

(b) $2p$ (d) -15

(15) $\dfrac{1}{x+1} - \dfrac{x-3}{x^2+3}$

(16) $a = 1,\ b = -7,\ c = 8$

(17) (a) $C = (-g, -f)$; $r = \sqrt{g^2 + f^2 - c}$

(c) (i) $\sqrt{2}$ (ii) $\sqrt{c}$

(18) (a) $y^2 + 4x + 2y + 5 = 0$

(b) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

(c) $\dfrac{x^2}{4} + \dfrac{y^2}{2} = 1$

(d) $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$

(e) $xy = -900$

(21) $x + y = \sqrt{2c}$

(24) (a) $u_n = 0$, all n

(b) $u_n = 2^{n-1}$

(26) (a) $2\ln x - \frac{1}{2}\ln(x - 1) - \frac{3}{2}\ln(x + 1) + c$

(b) $-\dfrac{1}{9}\dfrac{\sqrt{9 + x^2}}{x} + c$

(c) $\ln(\sec x + \tan x) - \sin x + c$

(d) $\dfrac{\pi}{12}$

(27) (a) $\frac{1}{2}\sin^2 x + c_1$

(b) $-\frac{1}{2}\cos^2 x + c_2$

(c) $-\cos 2x + c_3$

(d) $\cos 2x = \cos^2 x - \sin^2 x$

(30) $b^2 + a^2$

(31) $\dfrac{\pi^2}{4}$

(34) $x = 2 - 2\cos t - t\sin t$; $\dfrac{8\sqrt{2} - 8 - \pi}{4\sqrt{2}}$

(35) $\dfrac{2\pi b}{3}(3a^2 + b^2)$

(36) $\dfrac{\pi c^6}{15}$

(37) (a) $\dfrac{4\pi a^2 b}{3}$ (b) $\dfrac{4\pi ab^2}{3}$

(39) $p = \dfrac{2a}{b(2 + e^{-at})}$