P. 18.

2.50 MIXED INTEGRATION QUESTIONS

1. Find the indefinite integral (primitive function) of:

   (a) $x^2 + 4x - 3$

   (b) $(7x + 1)^5$

   (c) e^{5x}

   (d) $\sin\left(\frac{\pi x}{3} - 1\right)$

   (e) $\frac{x^4 + 3x^2 - x}{x^2}$

   (f) $\sec^2 2x$

   (g) $\frac{1}{x+3}$

   (h) $\sqrt{5 - x}$

   (i) $\cos 3x$

   (j) $\frac{x}{3x^2 + 1}$

2. Evaluate:

   (a) $\int_1^2 x^2 dx$

   (b) $\int_0^1 e^{3y} dy$

   (c) $\int_0^1 (4x - 3)^2 dx$

   (d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2\theta d\theta$

   (e) $\int_{-1}^{1} \frac{dx}{2x + 3}$

   (f) $\int_0^{\frac{\pi}{6}} \cos 2x dx$

3. (a) Evaluate $\int_{-1}^{1} x^5 dx$.
   (b) Find the area bounded by the curve $y = x^5$, the x-axis and the lines $x = -1$ and $x = 1$.

4. Find the area bounded by the curve $y = \ln x$, the x-axis and the lines $x = 1$ and $x = 5$ to three significant figures by using Simpson's rule with five function values.

5. Use the trapezoidal rule with two subintervals to find an approximation to $\int_0^1 \tan x dx$ correct to three significant figures.

6. Find the area bounded by the curve $y = \sin 2x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{4}$.

7. Find the area enclosed between the curve $y = 2x^3$, the x-axis and the line $x = 2$.

8. Find the area bounded by the parabola $y = x^2 - 7x + 10$ and the x-axis.

9. Find the area enclosed between the curve $y = \ln x$, the y-axis and the ordinates $y = 2$ and $y = 5$.

10. Find the exact area bounded by the curve $y = x^2$, the y-axis and the lines $y = 0$ and $y = 2$ in the first quadrant.

11. Find the volume of the solid formed if the curve $y = x^2 + 1$ is rotated about the (a) x-axis from $x = 1$ to $x = 2$; (b) y-axis from $y = 0$ to $y = 5$.

12. Find the volume of the solid formed by rotating the curve $y = e^{2x}$ about the x-axis from $x = 0$ to $x = 1$.

13. The curve $y = \sec x$ is rotated about the x-axis from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$. Find the volume of the solid of revolution formed.

14. The rate of change in volume flow of water in a pipe is given by

$$R = 2t^2 + 20t + 5$$

litres per minute. If there is an initial volume of 10 litres in the pipe, find the volume after 5 minutes.

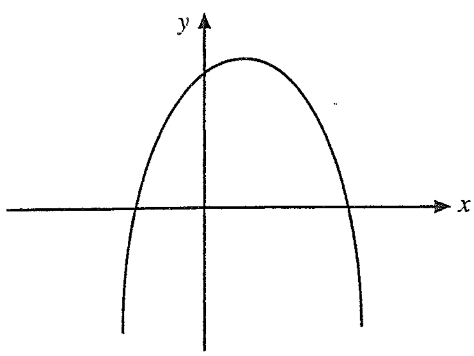
15. The velocity of a particle is given by $v = 6 \sin 3t$ m s^{-1}. If the particle is initially 2 m to the right of the origin, find:

(a) the displacement after $\frac{\pi}{6}$ s
(b) the initial acceleration

16. The acceleration of a particle is $\ddot{x} = 4e^{2t}$ cm s^{-2}. Initially the particle is at rest 1 cm to the right of the origin. Find the equation of the displacement of the particle.

17. The second derivative of a function is given by $\frac{d^2y}{dx^2} = 12x - 6$. If the function has a stationary point at (1, 2), find the equation of the function.

18. The graph below is the tangent (derivative) function of $y = f(x)$. Sketch a graph of $y = f(x)$ that could have this tangent function. Is your graph unique?



v5

<u>xed Integration Questions</u>

1a. let $y' = x^2 + 4x - 3$

$\therefore y = \frac{1}{3}x^3 + 2x^2 - 3x + c$ ✓

b. let $y' = (7x+1)^5$

$\therefore y = \frac{(7x+1)^6}{6 \times 7} = \frac{1}{42}(7x+1)^6 + c$ ✓

c. let $y' = e^{5x}$

$\therefore y = \frac{1}{5}e^{5x} + c$ ✓

d. let $y' = \sin\left(\frac{\pi x}{3} - 1\right)$

$\therefore y = \dfrac{-\cos\left(\frac{\pi x}{3} - 1\right)}{\frac{\pi}{3}}$

$= -\frac{3}{\pi}\cos\left(\frac{\pi x}{3} - 1\right) + c$ ✓

e. let $y' = \frac{x^4 + 3x^2 - x}{x^2} = x^2 + 3 - \frac{1}{x}$

$\therefore y = \frac{1}{3}x^3 + 3x - \ln x + c$ ✓

f. let $y' = \sec^2 2x$

$\therefore y = \frac{1}{2}\tan 2x + c$ ✓

g. let $y' = \frac{1}{x+3}$

$\therefore y = \ln(x+3) + c$ ✓

h. let $y' = \sqrt{5-x} = (5-x)^{\frac{1}{2}}$

$\therefore y = \dfrac{(5-x)^{3/2}}{\frac{3}{2} \times -1} + c = -\frac{2}{3}\left(\sqrt{5-x}\right)^3 + c$ ✓

i. let $y' = \cos 3x$

$\therefore y = \frac{1}{3}\sin 3x + c$ ✓

j. let $y' = \frac{x}{3x^2+1} = \frac{1}{6} \times \frac{6x}{3x^2+1}$

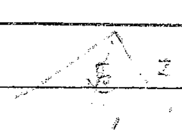
$\therefore y = \frac{1}{6}\ln(3x^2+1) + c$ ✓

2a. $\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$ ✓

b. $\int_0^1 e^{3y}\,dy = \left[\frac{1}{3}e^{3y}\right]_0^1$

$= \frac{1}{3}e^3 - \frac{1}{3}e^0 = \frac{1}{3}e^3 - \frac{1}{3}$ ✓

c. $\int_0^1 (4x-3)^2\,dx$



$= \left[\frac{(4x-3)^3}{3 \times 4}\right]_0^1$

$= \frac{1}{12} + \frac{27}{12} = \frac{28}{12} = \frac{7}{3}$ ✓

d. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2\theta\,d\theta = \left[\tan\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \tan\frac{\pi}{3} - \tan\frac{\pi}{6} = \sqrt{3} - \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{3\sqrt{3} - \sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$ ✓

e. $\int_{-1}^1 \frac{dx}{2x+3} = \frac{1}{2}\int_{-1}^1 \frac{2}{2x+3}\,dx$

$= \frac{1}{2}\left[\ln(2x+3)\right]_{-1}^1 = \frac{1}{2}\ln 5 - \frac{1}{2}\ln 1$

$= \frac{1}{2}\ln 5.$ ✗ $\frac{1}{2}\ln 5$

f. $\int_0^{\frac{\pi}{6}} \cos 2x\,dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\sin\frac{\pi}{3} - \frac{1}{2}\sin 0$

$= \frac{\sqrt{3}}{4} - 0 = \frac{\sqrt{3}}{4}$ ✓

3a. $\int_{-1}^1 x^5\,dx = \left[\frac{1}{6}x^6\right]_{-1}^1$

$= \frac{1}{6} - \frac{1}{6} = 0$ ✓

b. $A = \left|\int_{-1}^0 x^5\,dx\right| + \int_0^1 x^5\,dx$

$= \left|\left[\frac{1}{6}x^6\right]_{-1}^0\right| + \left[\frac{1}{6}x^6\right]_0^1$

$= \left|0 - \frac{1}{6}\right| + \frac{1}{6} - 0 = \frac{1}{3}$ u^2 ✓

$y = \ln x$

4.

x	1	2	3	4	5
y	0	0.6931	1.0986	1.3863	1.6094

$A \doteq \frac{1}{3}\left(0 + 4(0.6931) + 1.0986\right)$

$+ \frac{1}{3}\left(1.0986 + 4(1.3863) + 1.6094\right)$

$= \cancel{5.4169}$ $= 4.04$ (3sf)

$= \cancel{5.42}$ $\cancel{3}$ $1 \; 2^u$ $\cancel{\quad 4.04}$

y	0	0.5463	1.5574

$y = \tan x$

$A \doteqdot \frac{1}{4}\left(0 + 2(0.5463) + 1.5574\right)$

$= 0.663\ u^2\ (3\ sf)$ ✓ ~~0.663~~

6. $A = \int_0^{\frac{\pi}{4}} \sin 2x\, dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\cos 0$

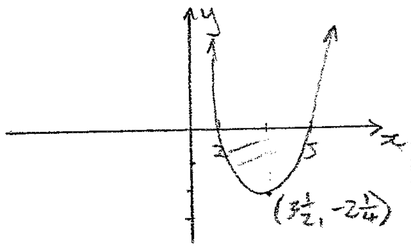
$= \frac{1}{2}\ u^2$ ✓

7. $A = \int_0^2 2x^3\, dx = \left[\frac{1}{2}x^4\right]_0^2 = 16$

8. $y = x^2 - 7x + 10$

$(x-2)(x-5) = 0$

$V\left(3\frac{1}{2}, -2\frac{1}{4}\right)$



$A = \left|\int_2^5 x^2 - 7x + 10\, dx\right|$

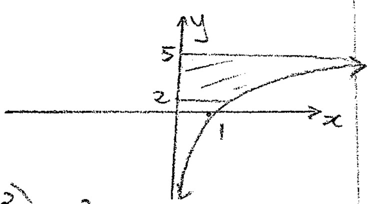
$= \left|\left[\frac{1}{3}x^3 - \frac{7}{2}x^2 + 10x\right]_2^5\right|$

$= \left|\frac{125}{3} - \frac{175}{2} + 50 - \frac{8}{3} + \frac{28}{2} - 20\right|$ ✓

$= 4\frac{1}{2}\ u^2$

9. $y = \ln x$

$x = e^y$



$A = \int_2^5 e^y\, dy$

$= \left[e^y\right]_2^5 = (e^5 - e^2)\ u^2$ ✓

10. $y = x^2$

$x = \sqrt{y}$ ($y>0$ in quad. 1)

$\therefore A = \int_0^2 y^{\frac{1}{2}}\, dy = \left[\frac{2}{3}y^{3/2}\right]_0^2$

$= \frac{2}{3}(\sqrt{2})^3 = \frac{2\times 2\sqrt{2}}{3} = \frac{4\sqrt{2}}{3}\ u^2$ ✓

→ 11a. $V = \pi\int_1^2 x^4 + 2x^2 + 1\, dx$

$= \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_1^2$

$= \pi\left(\frac{32}{5} + \frac{16}{3} + 2 - \frac{1}{5} - \frac{2}{3} - 1\right)$ ✓

$= \frac{178\pi}{15}\ u^3$

b. $V = \pi\int_0^5 y - 1\, dy$

$= \pi\left[\frac{1}{2}y^2 - y\right]_0^5$

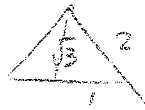
$= \pi\left(\frac{25}{2} - 5\right)$

$= \frac{15\pi}{2}\ u^3$ ✓

12. $V = \pi\int_0^1 e^{4x}\, dx$

$= \pi\left[\frac{1}{4}e^{4x}\right]_0^1 = \pi\left(\frac{1}{4}e^4 - \frac{1}{4}\right)$

$= \frac{\pi}{4}(e^4 - 1)\ u^3$ ✓



13. $V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x\, dx$

$= \pi\left[\tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \pi\left(\tan\frac{\pi}{3} - \tan\frac{\pi}{6}\right)$

$= \pi\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = \pi\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right)$

$= \pi\left(\frac{3\sqrt{3} - \sqrt{3}}{3}\right) = \frac{2\sqrt{3}\pi}{3}\ u^3$ ✓

14. $\frac{dV}{dt} = 2t^2 + 20t + 5$ L/min

$V = \frac{2}{3}t^3 + 10t^2 + 5t + c$

when $t=0$, $V=10$

$\therefore c = 10$

$\therefore V = \frac{2}{3}t^3 + 10t^2 + 5t + 10$

when $t=5$, $V = 368\frac{1}{3}$ L. ✓

15a. $v = 6\sin 3t$ m/s

$x = -2\cos 3t + c$

when $t=0$, $x=2$

$c = 2\cos 0 = 2$

$\therefore x = -2\cos 3t + 2$

when $t = \frac{\pi}{6}$, $x = -2\cos\frac{\pi}{2} + 2$

$= 2$

$\therefore$ particle is 4 m to right of origin.

× 4 m

b. $\ddot{x} = 18\cos 3t$

when $t=0$, $\ddot{x} = 18$ m/s^2 ✓

16. $\ddot{x} = 4e^{2t}$ cm/s^2

when $t=0$, $v=0$, $x=1$

$v = 2e^{2t} + c$

$\therefore v = 2e^{2t} - 2$

$x = e^{2t} - 2t + k$

$\Rightarrow 1 = 1 + k \Rightarrow k = 0$

$\boxed{\therefore x = e^{2t} - 2t}$ ✓

17. $\frac{d^2y}{dx^2} = 12x - 6$

$\frac{dy}{dx} = 6x^2 - 6x + c$

when $x = 1$, $\frac{dy}{dx} = 0$

$\therefore c = -6 + 6 = 0$

$\therefore \frac{dy}{dx} = 6x^2 - 6x$

$\therefore y = 2x^3 - 3x^2 + k$

when $x = 1$, $y = 2$,

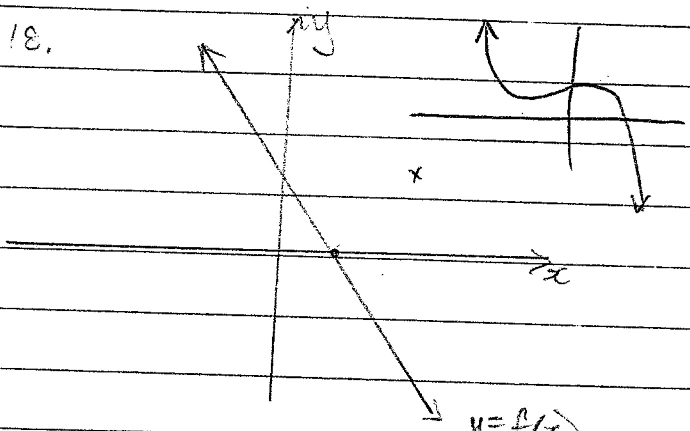
$\Rightarrow 2 = 2 - 3 + k$

$k = 3$

✓

$\therefore y = 2x^3 - 3x^2 + 3$

18.



not unique.

need to know the co-efficients.