J.M.J.

MARCELLIN COLLEGE RANDWICK

EXTENSION 2 (TASK 1)

MATHEMATICS

2013

Weighting: 20% (HSC Assessment Mark)

NAME: ____________________

MARK: / 67

Time Allowed: 90 minutes

Topics: Graphs, Complex Numbers and Polynomials.

Directions:

- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks <u>may not</u> be awarded for careless or badly arranged work

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STANDARD INTEGRALS

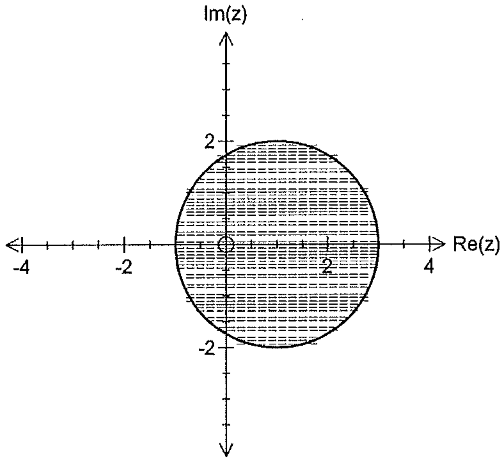
$\int x^n dx$	$= \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$
$\int \frac{1}{x} dx$	$= \ln x,\ x > 0$
$\int e^{ax} dx$	$= \frac{1}{a}e^{ax},\ a \neq 0$
$\int \cos ax\, dx$	$= \frac{1}{a}\sin ax,\ a \neq 0$
$\int \sin ax\, dx$	$= -\frac{1}{a}\cos ax,\ a \neq 0$
$\int \sec^2 ax\, dx$	$= \frac{1}{a}\tan ax,\ a \neq 0$
$\int \sec ax \tan ax\, dx$	$= \frac{1}{a}\sec ax,\ a \neq 0$
$\int \frac{1}{a^2+x^2} dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}} dx$	$= \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}} dx$	$= \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}} dx$	$= \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x,\ x > 0$

Multiple Choice *(10 marks)*

1. The locus below is best described by:

   (A) $|z-1|\le 2$ (B) $|z+1|\le 2$ (C) $|z-1|\le 4$ (D) $|z+1|\le 4$



2. The polynomial $P(z)$ has real coefficients.
   The roots of $P(z) = 0$ include $z = 1 - i$ and $z = 2 + i$. and $z = 3$.
   What is the lowest possible degree of $P(z)$?

   (A) 3 (B) 4 (C) 5 (D) 6

3. Let $z = 3+4i$ and $\omega = 2-2i$. Then $z\overline{\omega}$ would be equal to:

   (A) $x = 6-2i$ (B) $x = 6+2i$ (C) $x = -2-14i$ (D) $x = -2+14i$

4. When $P(x) = x^4 - 1$ is fully factorised over the complex field it may be written as

   (A) $P(x)=(x^2-1)(x^2+1)$ (B) $P(x)=(x-1)(x+1)(x^2+1)$
   (C) $P(x)=(x-1)(x+1)(x+i)(x-i)$ (D) $P(x)=(x^2-1)(x-i)^2$

Multiple Choice continued

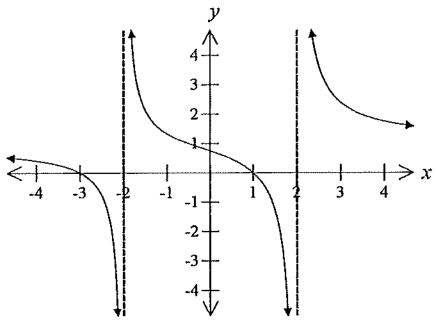
5. The curve $y=\dfrac{x^3+4}{x^2}$ has asymptotes at

   (A) $x = 0$ and $y = 0$ (B) $y = 0$ and $y = x$ (C) $x = 0$ and $y = x$ (D) Only the x - axis

6. If w is a cube root of unity then $2+2\omega+2\omega^2 =$

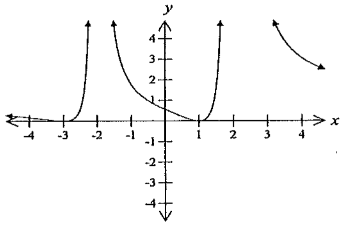
   (A) 2 (B) -2 (C) 1 (D) 0

7. The diagram shows the graph of the function $y = f(x)$.

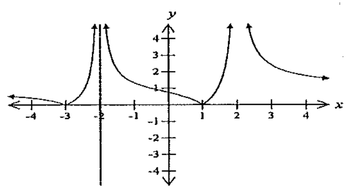


Which of the following is the graph of $y=|f(x)|$?

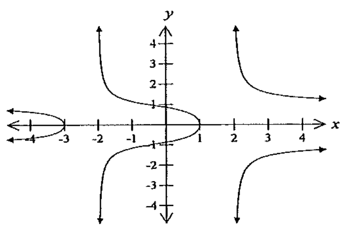
(A)



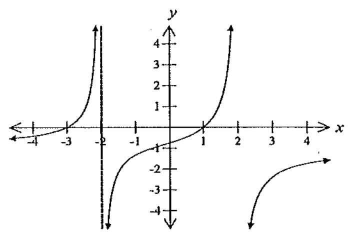
(B)



(C)



(D)



Multiple Choice continued

8. What is $-\sqrt{3}+i$ expressed in modulus-argument form?

(A) $\sqrt{2}(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

(B) $2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

(C) $\sqrt{2}(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$

(D) $2(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$

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9. What are the three roots of $z^3-1=0$ in modulus argument form?

(A) $\text{cis}0,\ \text{cis}\frac{2\pi}{3}$

(B) $\text{cis}0,\ \text{cis}\frac{2\pi}{3},\ \text{cis}-\frac{2\pi}{3}$

(C) $\text{cis}0,\ \text{cis}\frac{\pi}{3}$

(D) $\text{cis}0,\ \text{cis}\frac{\pi}{3},\ \text{cis}-\frac{\pi}{3}$

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10. Let α, β and γ be roots of the equation $x^3+x^2-2x-5=0$. Which of the following polynomial equations have roots $\alpha-2$, $\beta-2$ and $\gamma-2$?

(A) $x^3+7x^2+14x+3=0$

(B) $x^3+7x^2+21x+3=0$

(C) $x^3+x^2-6x+9=0$

(D) $x^3+2x^2-6x+9=0$

Marks

Question 11 (15 marks) **[START A NEW PAGE]**

(a) Express $\dfrac{31-2i}{3+4i}$ in the form $a+bi$ where a and b are real. 2

(b) If $z=3-i$ express $\dfrac{1}{z}$ in the form $x+iy$, where, x and y are real numbers. 1

(c) Find the two square roots of $16-30i$ 2

(d) Let $\omega=1-i\sqrt{3}$

(i) Express ω in modulus-argument form. 2

(ii) Hence find the value of ω^9 2

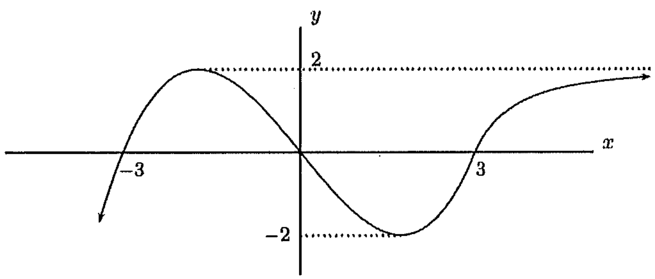
(e) Shade the region in the complex plane where $|z+2|\le 2$ and $-\frac{\pi}{6}\le\arg(z+3)\le\frac{\pi}{3}$ are simultaneously satisfied. 3

(f) Express $\cos 4\theta$ as a polynomial in $\cos\theta$ by expanding $(\cos\theta+i\sin\theta)^4$ and applying De Moivre's theorem. 3

Marks

Question 12 (16 marks) **[START A NEW PAGE]**

(a)



The graph of a certain function $y = f(x)$ is sketched above. Draw neat third-page sketches of the following graphs.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = e^{f(x)}$ **2**

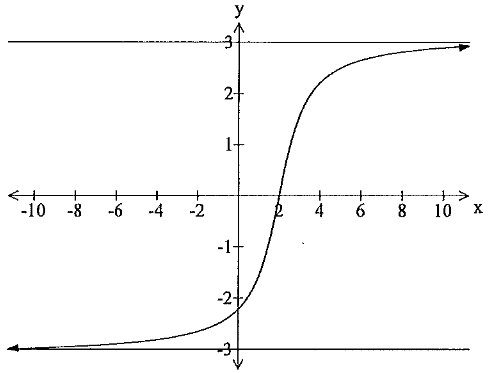
(iii) $y^2 = f(x)$ **2**

(b) (i) Find the five roots of the equation $z^5 = 1$. **1**
Give the roots in modulus-argument form.

(ii) Show that $z^5 - 1$ can be factorised as **2**
$$(z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$$

Question 12 continued Marks

(c) The diagram shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = f(x+2)$ **1**

(ii) $y = |f(x)|$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = f(|x|)$ **2**

Marks

Question 13 (17 marks) **[START A NEW PAGE]**

(a) It is known that $5 + 6i$ is a zero of the polynomial $P(x) = 2x^3 - 19x^2 + 112x + d$, where d is real.

(i) Find the other two zeroes of $P(x)$ **2**

(ii) Find the value of d. **2**

(b) (i) Suppose the polynomial $P(x)$ has a double root at $x = \alpha$. Prove that $P'(x)$ also has a root at $x = \alpha$. **2**

(ii) The polynomial $P(x) = x^4 + ax^2 + bx + 28$ has a double root at $x = 2$. **2**

(iii) Factorise the polynomial $P(x)$ of part (ii) over the real numbers. **2**

(c) Consider the polynomial $P(x) = (x+c)^4 - 32x$ where 'c' is a constant.
If $P(x) = 0$ has a double root at $x = \alpha$

(i) Prove that $\alpha = 2 - c$ **2**

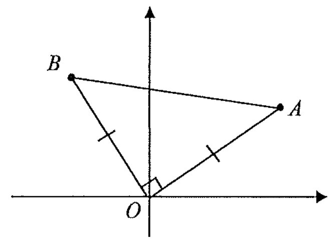
(ii) Find the numerical values of α and c **2**

(c) The polynomial equation $2x^3 - x^2 + 5 = 0$ has roots a, b and g. **3**
Find a polynomial equation with integer coefficients whose roots are α^3, β^3 and γ^3

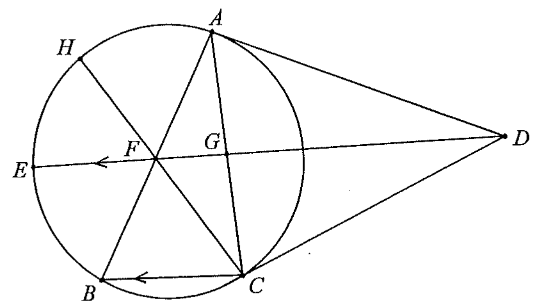
Marks

Question 14 (9 marks) **[START A NEW PAGE]**

(a) The Argand diagram shows the points A and B, which represent the complex numbers z_1 and z_2 respectively. Given that ΔBOA is a right-angled, isosceles triangle, show that $(z_1 + z_2)^2 = 2z_1 z_2$. **3**



(b)



The triangle ABC is inscribed in a circle. From an external point D, tangents are drawn to the circle, touching it at A and C. The chord ED is drawn parallel to BC, meeting AB at F and AC at G.
The line CF is produced to meet the circle at H.

(i) Prove that AFCD is a cyclic quadrilateral. **3**

(ii) Prove that $HF = AF$. **3**

Ext 2 Half-Yearly solutions 2013

Multiple Choice

1. A	4. C	7. B	10. A
2. C	5. C	8. D	
3. D	6. D	9. B	

Question 11

(a) $\dfrac{31-2i}{3+4i} \times \dfrac{3-4i}{3-4i}$

$= \dfrac{93-6i-124i-8}{9+16}$

$= \dfrac{17}{5} - \dfrac{26}{5}i$

(b) $\dfrac{1}{z} = \dfrac{1}{3-i} \times \dfrac{3+i}{3+i}$

$= \dfrac{3}{10} + \dfrac{i}{10}$

(c) $x+iy = \sqrt{16-30i}$

$x^2-y^2+2xyi = 16-30i$

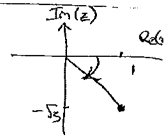
$x^2-y^2=16 \qquad xy=-15$

$x=5 \qquad y=-3$

$x=-5 \qquad y=3$

$\pm(5-3i)$

(d)(i) $w = 1-i\sqrt{3}$



$\sqrt{1^2+(\sqrt{3})^2} = 2$

$\tan\theta = \sqrt{3}$

$\theta = -\dfrac{\pi}{3}$

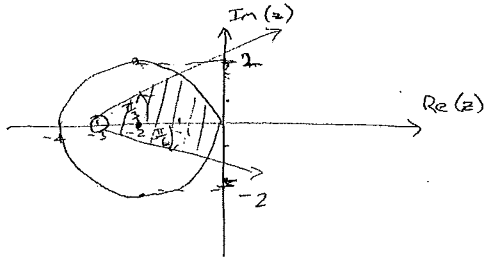
$2\,\text{cis}\left(\dfrac{-\pi}{3}\right)$

(ii) $w^9 = 2^9\left[\text{cis}\left(-\dfrac{9\pi}{3}\right)\right]$

$= 512(-1+i(0))$

$= -512$

(e)



(f) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

$= \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$

$= \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta + 4\cos^3\theta\, i\sin\theta - 4\cos\theta\, i\sin^3\theta$

$\therefore \cos 4\theta = \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta$

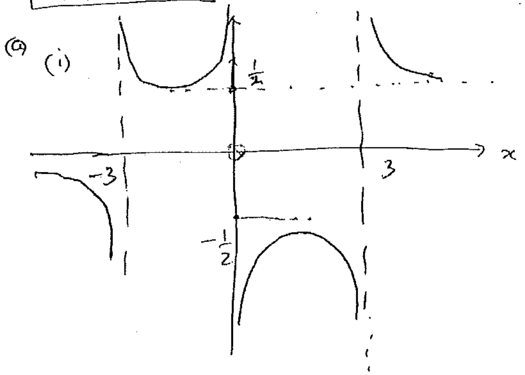
$= \cos^4\theta + \left[1-\cos^2\theta\right]^2 - 6(\cos^2\theta)(1-\cos^2\theta)$

$= \cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

Question 12

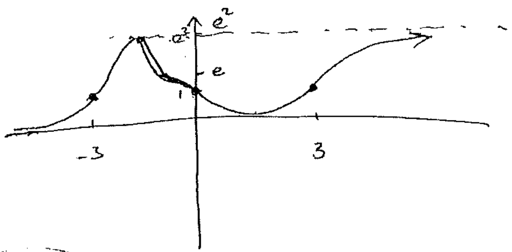
(a) (i)



(2)

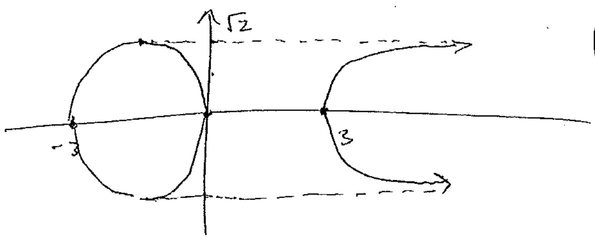
$y = \frac{1}{f(x)}$

(ii) $y = e^{f(x)}$



(2)

(iii) $y^2 = f(x)$



(2)

(b) (i) $z^5 - 1 = (z-1)\left[z^4 + z^3 + z^2 + z + 1\right]$

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$\frac{360}{5} = 72. \quad \frac{2}{5}$

$z_1 = 1$
$= \cos(0) + i\sin(0)$

$z_2 = \cos 72 + i\sin 72$
$= \cos\left(\frac{2\pi}{5}\right) + i\sin\frac{2\pi}{5}$

$z_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$z_4 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$ (1)
$= \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$
$= \overline{z_2}$

$z_5 = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$
$= \overline{z_3}$

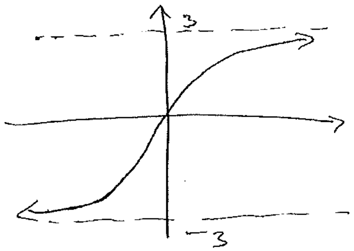
(ii) $(z-1)(z-z_2)(z-\overline{z_2})(z-z_3)(z-\overline{z_3})$ (1)

$= (z-1)\left[z^2 - 2(z_2+\overline{z_2}) + z_2\overline{z_2}\right]\left[z^2 - 2(z_3+\overline{z_3}) + z_3\overline{z_3}\right]$

$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$ (1)

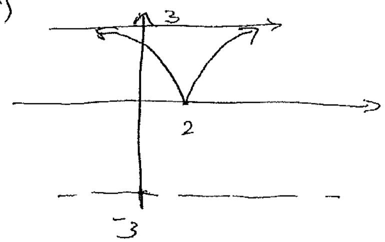
(c)

(i)



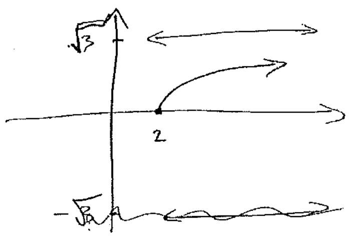
(1)

(ii)



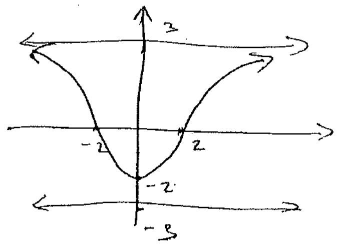
(2)

(iii)



(2)

(iv)



(2)

Question 13

(a) (i) $5+6i$ is a zero

$\therefore$ $5-6i$ is a zero because real coefficients

$5+6i+5-6i+\alpha = \frac{19}{2}$

$10+\alpha = \frac{19}{2}$.

$\therefore \alpha = -\frac{1}{2}$

(ii) $(5+6i)(5-6i)\left(-\frac{1}{2}\right) = -\frac{d}{2}$.

$(25+36)\left(-\frac{1}{2}\right) = -\frac{d}{2}$

$\therefore d = 61$

(b) (i) Double root. $P(x) = (x-\alpha)^2 Q(x)$ (1)

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$

$= (x-\alpha)\left[2Q(x) + (x-\alpha)Q'(x)\right]$

$\therefore P'(x)$ also has $x=\alpha$ as a root (1)

(ii) $P(x) = x^4 + ax^2 + bx + 28$

$P'(x) = 4x^3 + 2ax + b$

at $x=2$

$P(2) = 2^4 + a \times 2^2 + b \times 2 + 28$

$0 = 44 + 4a + 2b$ (1)

$P'(2) = 4 \times 2^3 + 2a(2) + b$

$32 + 4a + b = 0$

① - ②

$12 + b = 0$

$b = -12$ 1

$\therefore a = -5$

(iii)

$$\begin{array}{r} x^2+4x+7 \\ x^2-4x+4 \overline{\smash{)}\, x^4+0x^3-5x^2-12x+28} \\ x^4-4x^3+4x^2 \\ \hline 4x^3-9x^2 \\ 4x^3-16x^2+16x \\ \hline 7x^2-28x \\ 7x^2-28x+28 \\ \hline 0 \end{array}$$

1

$P(x) = (x-2)^2(x^2+4x+7)$ 1

(c)

(i) $P'(x) = 4(x+c)^3 - 32$

$P'(\alpha) = 4(\alpha+c)^3 - 32$ 1

$\therefore 4(\alpha+c)^3 = 32$

$(\alpha+c)^3 = 8$

$\alpha + c = 2$

$\therefore \alpha = 2 - c.$ 1

(ii) $P(\alpha) = (\alpha+c)^4 - 32\alpha$ 1

$0 = (2)^4 - 32\alpha.$

$\therefore \alpha = \frac{16}{32}$

$\alpha = \frac{1}{2}$ $\therefore c = \frac{3}{2}$ 1

(d) $2x^3 - x^2 + 5 = 0$

$2\alpha^3 - \alpha^2 + 5 = 0$

let $y = \alpha^3$

$\therefore \alpha = \sqrt[3]{y}$

$2(\sqrt[3]{y})^3 - (\sqrt[3]{y})^2 + 5 = 0$

$2y - y^{\frac{2}{3}} + 5 = 0$

$2y + 5 = y^{\frac{2}{3}}$

$(2y+5)^3 = y^2$

$8y^3 + 60y^2 + 150y + 125 = y^2$

$8y^3 + 59y^2 + 150y + 125 = 0$

Question 14

(a) $z_2 = iz_1$ since $\angle BOA = \frac{\pi}{2}$

$z_1 + z_2 = z_1 + iz_1$

$= z_1(1+i)$

$(z_1+z_2)^2 = z_1^2(1+i)^2$

$= 2z_1^2 i$

$= 2z_1z_2$

(b) (i) $\angle CAD = \angle ABC$ (alternate segment) 1

$\angle ABC = \angle AFD$ (corresponding angles)

$AD = CD$ (tangents from an external point)

$\angle CAD = \angle ACD$ (isosceles Δ) 1

$\angle AFD = \angle ACD$

$\therefore$ AFCD is a cyclic Quad since $\angle$'s standing on the same arc are equal. 1

(ii) $\angle DAC = \angle DFC$ ($\angle$'s at circumference are equal) 1

$\angle FCB = \angle DFC$ (alternate $\angle$'s)

ΔBCF is isosceles

$\angle AFH = \angle BFC$ (vertically opposite)

$\angle AHF = \angle ABC$ ($\angle$'s at circumference) 1

$\therefore \Delta HFA ||| \Delta BFC$ (equiangular)

$\therefore \Delta HFA$ is Isosceles

$\therefore$ ~~A~~ $HF = AF$ (sides of isos Δ opposite equal corresponding sides) 1