J.M.J.

MARCELLIN COLLEGE RANDWICK

EXTENSION I

MATHEMATICS

HSC TASK 2

2012

Weighting: 20% (Assessment Mark)

NAME: ____________________

MARK: / 35

Time Allowed: 45 minutes

Topics: Parametric Equations and Applications of Calculus to the Physical World

Directions:

- There are two questions on this paper
- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks <u>may not</u> be awarded for careless or badly arranged work

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Marks

Question 1 (17 marks) Use a SEPARATE writing booklet.

(a) A piece of hot metal is placed in a room with a surrounding air temperature of 20°C and allowed to cool. It loses heat according to Newton's law of cooling, $\frac{dT}{dt} = -k(T - A)$ where T is the temperature of the metal in degrees Celsius at time t minutes, A is the surrounding air temperature and k is a positive constant. After 6 minutes the temperature of the metal is 80°C, and after a further 2 minutes it is 50°C.

(i) Verify that $T = A + Be^{-kt}$ satisfies the above equation. **1**

(ii) Show that $k = \frac{\log_e 2}{2}$. **3**

(iii) What is the value of B? **1**

(iv) Determine the initial temperature of the metal. **1**

(b) A particle is moving such that its displacement x metres at time t seconds is given by $x = 4\cos(3t - 1)$.

(i) Show that the motion is simple harmonic. **2**

(ii) Find the centre and the period of the motion. **2**

(iii) Find the speed of the particle when $x = 2$, correct to 3 significant figures. **1**

(c) A particle moves in a straight line with acceleration given by $\frac{d^2x}{dt^2} = 9(x - 2)$ where x is the displacement in metres from an origin O after t seconds. Initially, the particle is 4 metres to the right of O, so that $x = 4$, and has velocity $v = -6$.

(i) Show that $v^2 = 9(x - 2)^2$. **2**

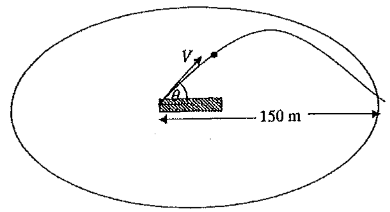
(ii) Find an expression for v and hence find x as a function of t. **2**

(iii) Explain whether the velocity of the particle is ever zero. **2**

Marks

Question 2 (18 marks) Use a SEPARATE writing booklet.

(a) A batsmen stands at the crease which is at the centre of a circular cricket ground of radius $150m$. He hits the ball at angle of elevation of θ with a speed of V *metres / second*. (Take $g = 10m/s^2$).



(i) Assuming the origin is at the point at which the ball is hit, show That the equations of motion are given by: **2**

$$x = Vt\cos\theta \text{ and } y = -\frac{gt^2}{2} + Vt\sin\theta.$$

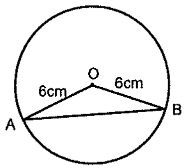
(ii) A batsmen hits the ball at an angle of elevation of 40^0 with a velocity of 36 m/s. What are the maximum height and the horizontal range of the path of this ball? (Answer to two decimal places.) **2**

(iii) A second batsmen hits the ball at an angle of elevation of 60^0. At what speed must the ball be hit in order to clear the boundary of 150 metres. (Answer to two decimal places) **2**

Marks

Question 2 continued

(b) O is the centre of a circle with radius 6cm and $\angle AOB = \theta$ radians.
θ is increasing at a rate of 0.2 radians/second.



(i) Find the rate of change of the area of ΔAOB **3**

(ii) Find the rate of change of the area of the minor segment formed by AB when $\angle AOB = \dfrac{2\pi}{3}$. **3**

(c) $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ are two points on the parabola $x^2 = 4ay$.

(i) If the chord PQ passes through the point $R\left(2a, 3a\right)$, show that $pq = p + q - 3$. **2**

(ii) If M is the midpoint of PQ, show that the coordinates of M are **2**

$$\left[a(pq+3), \frac{a}{2}\left\{(pq+3)^2 - 2pq\right\}\right].$$

(iii) Hence, find the locus of M. **2**

Question 1

(a) (i) $T = A + Be^{-kt}$

$\frac{dT}{dt} = -kBe^{-kt}$

$= -k(T-A)$

(ii) $A = 20$

when $t = 6$ $T = 80$

$80 = 20 + Be^{-6k}$

$\therefore Be^{-6k} = 60$ ①

when $t = 8$ $T = 50$

$50 = 20 + Be^{-8k}$

$\therefore Be^{-8k} = 30$ ②

① ÷ ②

$e^{2k} = 2$

$2k = \ln 2$

$k = \frac{\ln 2}{2}$

(iii) $Be^{-6 \times \frac{\ln 2}{2}} = 60$

$Be^{-3\ln 2} = 60$

$B = 60 \div e^{-3\ln 2}$

$= 480$

(iv) when $t=0$

$T = 20 + 480e^{0}$

$= 500°C$

(b) (i) $x = 4\cos(3t-1)$

$\dot{x} = -12\sin(3t-1)$

$\ddot{x} = -36\cos(3t-1)$

$= -9[4\cos(3t-1)]$

$= -9x$

$= -n^2 x$

(iii) Centre is $x = 0$

period $= \frac{2\pi}{3}$

(iii) $v^2 = n^2(a^2 - x^2)$

$= 9(16-4)$

$v = \sqrt{108}$

$= 10.4\text{ m/s}$

OR

$2 = 4\cos(3t-1)$

$\frac{1}{2} = \cos(3t-1)$

$\frac{\pi}{3} = 3t - 1$

$t = \frac{\pi+3}{9}$

$v = -12\sin\left(\frac{\pi+3}{3} - 1\right)$

$= 10.4\text{ m/s}$

(c) (i) $\frac{d^2x}{dt^2} = 9x - 18$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{9x^2}{2} - 18x + C$

$\frac{1}{2}(36) = \frac{9(16)}{2} - 18(4) + C$

$\therefore C = 18$

$\frac{1}{2}v^2 = \frac{9x^2}{2} - 18x + 18$

$v^2 = 9x^2 - 36x + 36$

$v^2 = 9(x-2)^2$

(ii) $v = -3(x-2)$

since when $x = 4$ $v = -6$

$\frac{dx}{dt} = -3x + 6$

$\frac{dt}{dx} = -\frac{1}{3} \cdot \frac{1}{x-2}$

$t = -\frac{1}{3}\ln(x-2) + C$

$0 = -\frac{1}{3}\ln 2 + C$

$C = \frac{1}{3}\ln 2$

$t = -\frac{1}{3}\ln(x-2) + \frac{1}{3}\ln 2$

$-3t = \ln\left(\frac{x-2}{2}\right)$

$e^{-3t} = \frac{x-2}{2}$

$2e^{-3t} = x - 2$

$x = 2 + 2e^{-3t}$

(iii) $\frac{dx}{dt} = -6e^{-3t}$

when $t = 0$

$\frac{dx}{dt} = -6\text{ m/s}$

as $t \to \infty$

$e^{-3t} \to 0$ but $e^{-3t} \neq 0$

$\therefore v \to 0$ but can never $= $ zero.

Question 2

(a)
(i) $\boxed{\ddot{x} = 0}$ $\boxed{\ddot{y} = -g}$

$\boxed{\dot{x} = V\cos\theta}$ $\boxed{\dot{y} = -gt + V\sin\theta}$ 1

$x = Vt\cos\theta + c$ $\quad y = \frac{-gt^2}{2} + Vt\sin\theta + c$

$t = 0,\ x = 0,\ y = 0$

$\therefore\ c = 0$ 1

$\boxed{x = Vt\cos\theta}$ $\boxed{y = \frac{-gt^2}{2} + Vt\sin\theta}$.

(ii) $\dot{y} = 0$

$0 = -gt + V\sin\theta$

$t = \frac{V\sin\theta}{g}$

$= \frac{36\sin 40}{10}$

$= 2.31$ sec.

$y = -\frac{g(2.31)^2}{2} + 36(2.31)\sin 40$.

$= 26.77$ m. 1

$x = 36 \times 4.628 \times \cos 40$.

$= 127.63$ m 1

(iii) $y = 0$

$0 = -5t^2 + Vt\sin 60$.

$0 = t(-5t + V\sin 60)$

$5t = V\sin 60$

$t = \frac{V\sin 60}{5}$ 1

$150 = V \times \frac{V\sin 60}{5} \times \cos 60$.

$V^2 = \frac{150 \times 20}{\sqrt{3}}$

$V = 41.62$ m/s

$\therefore\ V > 41.62$ m/s 1

(b) (i) $A = \frac{1}{2}(6)^2 \sin\theta$

$A = 18\sin\theta$.

$\frac{dA}{d\theta} = 18\cos\theta$ 1

$\frac{d\theta}{dt} = 0.2$ r/sec

$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$ 1

$= 18\cos\theta \times 0.2$

$= \frac{18}{5}\cos\theta$. 1

(iii) $A = \frac{1}{2}r^2(\theta - \sin\theta)$

$\frac{dA}{d\theta} = \frac{1}{2}r^2(1-\cos\theta)$ 1

$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= 18(1-\cos\theta)\times 0.2$ 1

$= 3.6(1 - -\frac{1}{2})$

$= 5.4\ \text{cm}^2/\text{s}$. 1

(c) (i) $PQ_m = \frac{ap^2 - aq^2}{2ap - 2aq}$.

$= \frac{\cancel{a}(p+q)\cancel{(p-q)}}{2\cancel{a}\cancel{(p-q)}}$

$= \frac{p+q}{2}$

$y - ap^2 = \frac{p+q}{2}[x - 2ap]$. 1

$2y - \cancel{2ap^2} = (p+q)x - \cancel{2ap^2} - 2apq$

$2y = (p+q)x - 2apq$

$R(2a, 3a)$

$2(3a) = (p+q)2a - 2apq$.

$6a = 2a(p+q) - 2apq$. 1

$3 = (p+q) - pq$.

$pq = p+q-3$.

(ii) $M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left[a(p+q), \frac{a}{2}(p^2+q^2)\right]$ 1

$= \left[a(pq+3), \frac{a}{2}\{(p+q)^2 - 2pq\}\right]$ 1

$$= \left[a(pq+3),\ \frac{a}{2}\left\{(pq+3)^2 - 2pq\right\}\right]$$

(iii) $x = a(pq+3) \qquad y = \frac{a}{2}\left\{(pq+3)^2 - 2pq\right\}$

$$pq + 3 = \frac{x}{a}$$

& $pq = \frac{x}{a} - 3$

$$\therefore \quad y = \frac{a}{2}\left[\frac{x^2}{a^2} - 2\left(\frac{x}{a} - 3\right)\right] \qquad 1$$

$$y = \frac{x^2}{2a} - x + 3a.$$

$$2ay - 6a^2 = x^2 - 2ax$$

$$2ay - 6a^2 + a^2 = x^2 - 2ax + a^2$$

$$(x-a)^2 = 2ay - 5a^2$$

$$(x-a)^2 = 2a\left(y - \frac{5a}{2}\right) \qquad 1$$