MORIAH COLLEGE

Year 12

MATHEMATICS EXTENSION 2

PRE TRIAL

26TH MARCH 2015

Time Allowed: 3 hours plus 5 minutes reading time

Examiner: E. Apfelbaum

General Instructions

- Write using blue or black pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- A Standard Integrals sheet is attached on last page

Total marks 100

- Attempt Questions 1–16
- Answer Questions 1- 10 on the answer sheet provided.
- Start each of the questions 11-16 in a new booklet

STUDENT NUMBER:..

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

SECTION A: Answer these questions on the multiple choice answer sheet provided

1. The equation $3x^2 + 5y^2 - 24 = 0$ represents
   A. A hyperbola.
   B. A parabola.
   C. A circle.
   D. An ellipse.

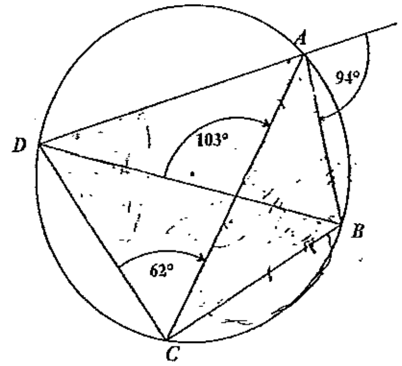
2. Find the distance between the foci of the hyperbola: $\frac{x^2}{4} + \frac{(y-1)^2}{25} = 1$

   A. $2\sqrt{21}$
   B. $4\sqrt{21}$
   C. $\frac{2\sqrt{21}}{5}$
   D. $\frac{4\sqrt{21}}{5}$

3. Find $\sqrt{-8-6i}$

   A. $3i - 1$
   B. $i - 3$
   C. $1 + 3i$
   D. $-1 - 3i$

4.



The size of $< ABC$ is:

A. 94°
B. 84°
C. 107°
D. 106°

5. Simplify: $cos3x - cos7x$.

   A. $2sin5xsin2x$

   B. $2sin7xsin3x$

   C. $2cos7xcos3x$

   D. $2cos5xcos2x$

6. w is a complex cube root of 1. Find the value of $(w+1)^3$.

   A. 5
   B. -2
   C. 8
   D. -1

7. $z = (1-i)^3\left(-\sqrt{3}+i\right)^2$. Find the value of $arg z$.

A. $\frac{\pi}{6}$

B. $\frac{11\pi}{12}$

C. $\frac{-5\pi}{12}$

D. $\frac{7\pi}{12}$

8. $\int_0^1 \frac{dx}{1+x^2} < \int_0^1 \frac{dx}{2+x^2}$..................I

$\int_0^1 \frac{dx}{1+x^2} < \int_0^1 \frac{dx}{1+x^3}$..................II

Without evaluating the above integrals, which of the above statements must be true?

A. I only
B. II only
C. I and II
D. Neither I nor II

9. The cubic polynomial $2x^3 - 5x^2 - 4x + 12 = 0$ has a double root. This root must be:

A. $\frac{3}{2}$
B. 3
C. 2
D. -2

10. Which expression is equal to $\int \frac{dx}{\sqrt{4x^2+1}}$?

A. $sin^{-1}2x + C$
B. $log_e\left(2x + \sqrt{4x^2+1}\right) + C$
C. $\frac{1}{2}log_e\left(x + \sqrt{x^2 + \frac{1}{4}}\right) + C$
D. $\frac{1}{4x}\sqrt{4x^2+1} + C$

SECTION B

Answer these questions on the writing paper provided. Start each question in a new booklet.

Question 11(Start this question in a new booklet.)

a) $\int_0^4 t\sqrt{t^2+9}\,dt$ **2**

b) $\int \frac{dx}{x^2 - 4x + 12}$ **2**

c) $\int \frac{dx}{x^2 - 4x - 12}$ **3**

d) $\int x\ln x\,dx$ **2**

e) $\int sec^4x tan^2x\,dx$ **2**

f) $\int \frac{x^3dx}{\sqrt{x^2+4}}$ using the substitution $x = 2tan\theta$ **4**

Question 12(Start this question in a new booklet.)

a) i) Express the complex number $z = 3 + 3\sqrt{3}i$ in mod/arg form **4**

ii) Hence express each of the following in mod/arg form.

$\alpha)\ \overline{z}$

$\beta)\ z^2$

$\gamma)\ \dfrac{1}{z}$

b) The point P represents a complex number z in the Argand diagram. P moves under the condition that: **5**

$$arg(z-4) - argz = \frac{\pi}{2}$$

i) Draw a diagram representing this situation.
ii) Explain why the locus of P is a semicircle.
iii) Explain why the origin is not included in the locus.
iv) If w is a complex number that satisfies the above condition, explain why $arg(w-2) = 2argw$

c) Given that $u_n = \int_0^1 \frac{dx}{(1+x^2)^n}$,

i) Use integration by parts to show that: **3**

$2nu_{n+1} - (2n-1)u_n = \left(\frac{1}{2}\right)^n$

ii) Hence, by letting $n + 1 = 3$, evaluate u_3 **3**

Question 13(Start this question in a new booklet.)

a) The base of a certain solid is the area contained by the positive branch of the of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ and its latus rectum. Cross-sections perpendicular to the X- axis are parabolas with height twice the width of the base.

i) Sketch this branch of the hyperbola, clearly indicating the directrix. **2**

ii) Find the exact volume of this solid. **3**

b) i) Show that the roots of the equation $x^6 + x^3 + 1 = 0$ are among the roots of the equation $x^9 = 1$. **2**

ii) Find the roots of the equation $x^9 = 1$ in mod /arg form and hence express $x^6 + x^3 + 1$ as the product of three quadratic factors with real coefficients **3**

c) i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **1**

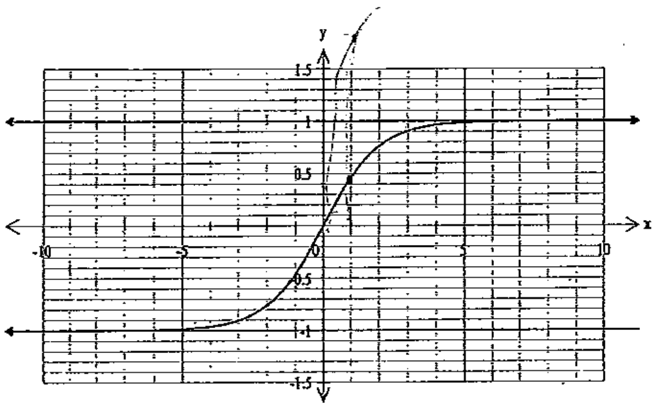
ii) Show $\int_0^{\frac{\pi}{4}} ln(1 + tanx)dx = \int_0^{\frac{\pi}{4}} ln\left(\frac{2}{1+tanx}\right)dx$ **2**

iii) Hence find $\int_0^{\frac{\pi}{4}} ln(1 + tanx)dx$ **2**

Question 14(Start this question in a new booklet.)

a) One of the roots of $x^4+2x^3+6x^2+8x+8=0$ is of the form $x=ki$, where k is a real number. Find all possible values of k. **3**

b) The diagram shows a sketch of the function $y=\frac{e^x-1}{e^x+1}$. (0,0) is a point of inflection.



i) Find the equation of the tangent to $y=\frac{e^x-1}{e^x+1}$ at (0,0). **2**

ii) Find the values of k for which $\frac{e^x-1}{e^x+1}=kx$ has 3 real solutions. **2**

iii) Without using calculus, use the graph of $y=\frac{e^x-1}{e^x+1}$ to sketch on separate diagrams the graphs of:

$\alpha)\ y=\frac{e^x+1}{e^x-1}$ $\qquad\beta)\ y=\left(\frac{e^x-1}{e^x+1}\right)^2$ **4**

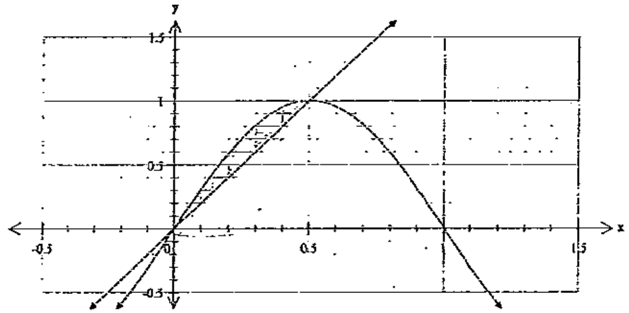
c) Evaluate $\int_0^{\frac{\pi}{2}}\frac{dx}{2-\cos x}$ using the substitution $t=\tan\frac{x}{2}$. **4**

Question 15(Start this question in a new booklet.)

a)

i) Using the method of integration by parts, show that $\int_0^{\frac{1}{2}} x\sin\pi x\,dx=\frac{1}{\pi^2}$ **3**

ii) The diagram shows the region contained by the curves: $y=\sin\pi x$ and $y=2x$. The points of intersection of the two curves are $A(0,0)$ and $B\left(\frac{1}{2},1\right)$. **3**



This region is rotated about the Y- axis. Use cylindrical shells and part (i) to determine the volume of the solid formed. (Leave your answer in simplified exact form.)

b)

i) Show that the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at the point $P(a\cos\theta, b\sin\theta)$ is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **3**

ii) The tangent at P meets the directrix at T. Find the coordinates of T and hence show that $\angle PST=90°$, where S is the nearest focus. **3**

iii) Q is the corresponding point to P on the auxiliary circle of the above ellipse (i.e. Q and P have the same x-coordinate.) **3**

Show that the tangents at P and Q intersect on the major axis of the ellipse.

Question 16(Start this question in a new booklet.)

a) A curve is described by the parametric equations: **3**
$x = t - sint$
$y = 1 - cost$
Find the equation of the normal to this curve at the point where
$t = \frac{\pi}{2}$.

b) i) Draw a sketch of the curve $y = \frac{1}{x}$ and show that $\int_1^{\sqrt{u}} \frac{dx}{x} < \sqrt{u} - 1$, $u > 1$ **1**

ii) Hence show that $0 < lnu < 2(\sqrt{u} - 1)$, for $u > 1$. **1**

iii) Hence show that $\frac{lnu}{u} \to 0$, as $u \to \infty$ **2**

c) Let the points $A_1, A_2, A_3, \ldots\ldots A_{10}$ represent the 10^{th} roots of unity, $w_1, w_2, w_3, \ldots\ldots w_{10}$ in the Argand diagram.

i) Explain why the points $A_1, \ldots A_{10}$ lie on the unit circle. **1**

ii) Prove that $w_1 + w_2 + \cdots + w_{10} = 0$ **1**

iii) P is any other point on the unit circle; i.e. P represents the complex number $z = a + ib$ and $|z| = 1$.
Show that $(PA_i)^2 = (z - w_i)(\bar{z} - \overline{w_i})$, for $i = 1,2, \ldots\ldots,10$ **3**

iv) Prove that $(PA_1)^2 + (PA_2)^2 + \cdots + (PA_{10})^2 = 20$ **3**

Student Number: ....

Section A Multiple Choice Answer Sheet

	A	B	C	D
1.	A ○	B ○	C ○	D ●
2.	A ○	B ○	C ○	D ●
3.	A ●	B ○	C ○	D ○
4.	A ●	B ○	C ○	D ○
5.	A ●	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ●
7.	A ○	B ●	C ○	D ○
8.	A ●	B ○	C ○	D ○
9.	A ○	B ○	C ●	D ○
10.	A ○	B ○	C ●	D ○

7/9

Start here

Question 11

$s + c = 1$

$t + 1 = \sec^2$

a) $\int_0^4 t\sqrt{t^2+9}\,dt$

$= \frac{1}{2}\int_0^4 2t\,(t^2+9)^{\frac{1}{2}}$

$\frac{1}{2}\left[\frac{2(t^2+9)^{\frac{3}{2}}}{3}\right]_0^4$

$= \frac{1}{3}\left[(t^2+9)^{\frac{3}{2}}\right]_0^4 = \frac{1}{3}\left[125 - 27\right]$

$= \frac{98}{3}$

b) $\int \frac{dx}{x^2-4x+12} = \int \frac{dx}{(x-2)^2+8}$

$= \frac{1}{\sqrt{8}}\tan^{-1}\left(\frac{x-2}{\sqrt{8}}\right) + c$

c) $\int \frac{dx}{x^2-4x-12} = \int \frac{dx}{(x-2)^2-16} = \int \frac{dx}{(x-2+4)(x-2-4)}$

$= \int \frac{dx}{(x+2)(x-6)}$

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let $\frac{1}{(x+2)(x+6)} = \frac{A}{x+2} + \frac{B}{x-6}$

$1 = A(x-6) + B(x+2)$

NB $A = -B$

$x = -2$

$1 = -8A$ $A = \frac{-1}{8}$ $B = \frac{1}{8}$

$-\frac{1}{8}\int \frac{1}{x+2}\,dx - \frac{1}{8}\int \frac{1}{x-6}\,dx = -\frac{1}{8}\ln|x+2| - \frac{1}{8}\ln(x-6) + C$

ECT $= -\frac{1}{8}\ln|(x+2)(x-6)| + C$

d) $\int x \ln x \, dx$ $u = \ln x$ $u' = \frac{1}{x}$

$v = \frac{x^2}{2}$ $v' = x$

$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \times \frac{1}{x}\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

e) $\int \sec^4 x \tan^2 x \, dx$

Let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x \, dx$

$\int (1+\tan^2 x)\tan^2 x \cdot \sec^2 x \, dx$

$= \int (1+u^2)\,u^2 \, du$

$= \int u^2 + u^4 \, du$

$= \frac{u^3}{3} + \frac{u^5}{5} + c = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c$

f) $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

$x = 2\tan\theta$

$\frac{dx}{d\theta} = 2\sec^2\theta$

$dx = 2\sec^2\theta\,d\theta$

Integral becomes:

$\int \frac{8\tan^3\theta}{\sqrt{4\tan^2\theta+4}} \times 2\sec^2\theta\,d\theta$

$= \int \frac{8\tan^3\theta}{2\sec\theta} \times 2\sec^2\theta\,d\theta$

$= 8\int \tan^3\theta\sec\theta\,d\theta = 8\int \tan^2\theta \cdot \tan\theta\sec\theta\,d\theta$

~~$= 8\int \tan\theta(\sec^2-1)\sec\theta\,d\theta = 8\int \tan\theta\sec^3\theta - \sec\theta\,d\theta$~~

$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$

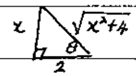
Let $u = \sec\theta \Rightarrow \frac{du}{d\theta} = \tan\theta\sec\theta$

$= 8\int (u^2-1)\,du$

$= 8\left[\frac{u^3}{3} - u\right] + c$

$= 8\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + c$ But $\tan\theta = \frac{x}{2}$

$= 8\left[\frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - \frac{\sqrt{x^2+4}}{2}\right] + c$



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Question 12

a) i) $z = 6\,\text{cis}\left(\frac{\pi}{3}\right)$

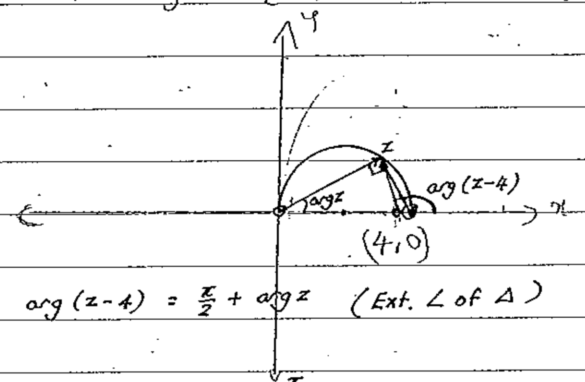
ii) α) $\bar{z} = 6\,\text{cis}\left(-\frac{\pi}{3}\right)$

β) $z^2 = 36\,\text{cis}\left(\frac{2\pi}{3}\right)$

γ) $\frac{1}{z} = \frac{\text{cis}(0)}{6\,\text{cis}\left(\frac{\pi}{3}\right)} = \frac{1}{6}\text{cis}\left(-\frac{\pi}{3}\right)$

b) $\arg(z-4) - \arg z = \frac{\pi}{2}$

i)



(ii) $\arg(z-4) = \frac{\pi}{2} + \arg z$ (Ext. ∠ of Δ)

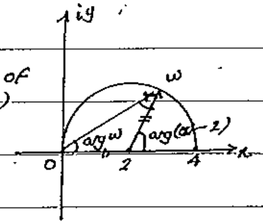
$\therefore \arg(z-4) - \arg z = \frac{\pi}{2}$

(iii) Because $z - 0$ is a vector, and $(0,0)$ is a point.

(iv) Since Δ is isosceles

$\arg(\omega - 2) = \arg\omega + \arg\omega$ (Base ∠s of isos. Δ)

$= 2\arg\omega$



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iii) (c) $\int_0^1 1.(1+x^2)^{-n}\,dx = \left[(1+x^2)^{-n}.x\right]_0^1 - \int_0^1 x.(-n)(1+x^2)^{-n-1}.2x\,dx$

v' u

Let $u_n = 2^{-n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$

$= \left(\frac{1}{2}\right)^n + 2n\int_0^1 \frac{x^2+1-1}{(1+x^2)^{n+1}}\,dx$

iv) $= \left(\frac{1}{2}\right)^n + 2n\left[\underbrace{\int_0^1 \frac{dx}{(1+x^2)^n}}_{u_n} - \underbrace{\int_0^1 \frac{dx}{(1+x^2)^{n+1}}}_{u_{n+1}}\right]$

$u_n = \left(\frac{1}{2}\right)^n + 2n u_n - 2n u_{n+1}$

$\therefore (1-2n)u_n + 2n u_{n+1} = \left(\frac{1}{2}\right)^n$

$\therefore 2n u_{n+1} - (2n-1)u_n = \left(\frac{1}{2}\right)^n$ as required.

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ii) $2\times\cancel{2}\,U_{\cancel{3}} - (\cancel{3})\,U_2 = \frac{1}{\cancel{4}}$

$4U_3 = \frac{1}{4} + 3U_2$

$U_3 = \dfrac{\frac{1}{4} + 3U_2}{4}$

$2\times 1\, U_2 - (1)\,U_1 = \frac{1}{2}$

$U_2 = \dfrac{\frac{1}{2} + U_1}{2}$

$U_1 = \left[\tan^{-1} x\right]_0^1$

$= \frac{\pi}{4}$

$U_2 = \dfrac{\frac{1}{2} + \frac{\pi}{4}}{2} = \frac{\pi + 2}{8}$

$U_3 = \dfrac{\frac{1}{4} + \frac{3(\pi+2)}{8}}{4} = \dfrac{3\pi + 6 + 2}{32}$

$= \dfrac{3\pi + 8}{32}$

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Start here **Question 13**

$a^2 = b^2(e^2 - 1)$

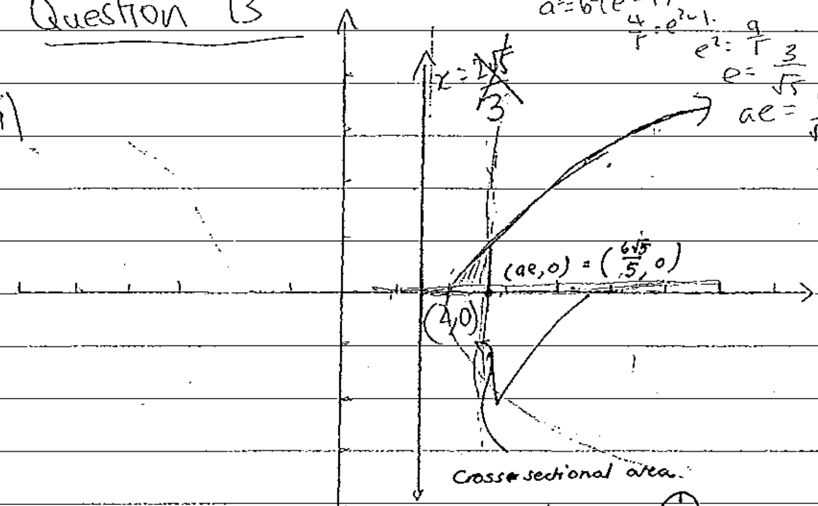
$\frac{4}{5} = e^2 - 1$

$e^2 = \frac{9}{5}$

$e = \frac{3}{\sqrt{5}}$

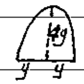
$ae = \frac{6}{\sqrt{5}}$

a) i)



Cross sectional area.

$= A(x)$



Using trapezoidal rule

$A(x) = \frac{y}{2}\left[0 + (4y)\times 2 + 0\right]$

$= 4y^2 = 5x^2 - 20$

directrix $= \pm\frac{a}{e} = \pm\left(\frac{\cancel{2}\times\sqrt{5}}{3}\right) = \pm\frac{2\sqrt{5}}{3}$

ii)

$\therefore$ Volume $= \lim_{\delta x \to 0} \sum_{x=2}^{\frac{6}{\sqrt{5}}} A(x).\,\delta x$

$= \int_2^{\frac{6}{\sqrt{5}}} 5x^2 - 20\, dx$

$= \left[\frac{5x^3}{3} - 20x\right]_2^{\frac{6}{\sqrt{5}}}$

$= \left(\frac{5}{3}\left(\frac{36}{5}\cdot\frac{6}{\sqrt{5}}\right) - 20\left(\frac{6}{\sqrt{5}}\right)\right) - \left(\frac{40}{3} - 60\right)$

$= \frac{72\sqrt{5}}{5} - 24\sqrt{5} + \frac{140}{3}$

$= \frac{140}{3} - \frac{48\sqrt{5}}{5}$

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6) i) $x^6+x^3+1=0$ roots among roots of $x^9=1$

$x^9-1=0$

$(x^3-1)(x^6+x^3+1)=0$

$\therefore$ the roots of $x^6+x^3+1=0$ are among the roots of $x^9=1$

ii) $x^9=cis(0+2k\pi)$

$x=(cis(0+2k\pi))^{\frac{1}{9}}=cis\left(\frac{0+2k\pi}{9}\right)$

$k=0 \quad z_0=cis(0)=1 \qquad z_4=cis\frac{8\pi}{9}$

$k=1 \quad z_1=cis\left(\frac{2\pi}{9}\right) \qquad z_5=cis\left(\frac{10\pi}{9}\right)=cis\left(-\frac{8\pi}{9}\right)=\bar{z}_4$

$z_2=cis\left(\frac{4\pi}{9}\right) \qquad z_6=cis\left(\frac{12\pi}{9}\right)=cis\left(\frac{-6\pi}{9}\right)=\bar{z}_3$

$z_3=cis\left(\frac{6\pi}{9}\right) \qquad z_7=cis\left(\frac{14\pi}{9}\right)=cis\left(\frac{-4\pi}{9}\right)=\bar{z}_2$

$z_8=cis\left(\frac{16\pi}{9}\right)=cis\left(\frac{-2\pi}{9}\right)=\bar{z}_1$

$x^9-1=(x-1)\left(x-cis\frac{2\pi}{9}\right)\left(x-cis\frac{4\pi}{9}\right)\left(x-cis\frac{6\pi}{9}\right)\left(x-cis\frac{8\pi}{9}\right)\left(x-cis\left(\frac{-8\pi}{9}\right)\right)\left(x-cis\frac{-6\pi}{9}\right)$
$\left(x-cis\frac{-4\pi}{9}\right)\left(x-cis\frac{-2\pi}{9}\right)$

$\therefore x^6+x^3+1=(x-z_1)(x-\bar{z}_1)(x-z_2)(x-\bar{z}_2)(x-z_3)(x-\bar{z}_3)(x-z_4)(x-\bar{z}_4)$

$=\left(x^2-\left(\cos\frac{2\pi}{9}\right)x+1\right)\left(x^2-\left(\cos\frac{4\pi}{9}\right)+1\right)\left(x^2-\left(\cos\frac{6\pi}{9}\right)x+1\right)\left(x^2-\left(\cos\frac{8\pi}{9}\right)x+1\right)$

c) let $u=a-x$

$\frac{du}{dx}=-1 \qquad x=0 \quad u=a$

$du=-dx \qquad x=a \quad u=0$

Integral becomes:

RHS $=\int_a^0 f(u)\cdot -du=\int_0^a f(u)\,du$

$=\int_0^a f(x)\,dx=LHS$

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$\therefore \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

ii) show $\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan x}\right)dx$

LHS $=\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}}\ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$=\int_0^{\frac{\pi}{4}}\ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$

$=\int_0^{\frac{\pi}{4}}\ln\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$

$=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan x}\right)dx = RHS.$

(iii) let $u=\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}}\ln 2-\underbrace{\ln(1+\tan x)\,dx}_{u}$

$\therefore 2u=\int_0^{\frac{\pi}{4}}\ln 2\,dx$

$=\left[x.\ln 2\right]_0^{\frac{\pi}{4}}$

$2u=\frac{\pi\ln 2}{4}$

$\therefore u=\frac{\pi\ln 2}{8}$

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Start here Question 14

a) $x^4+2x^3+6x^2+8x+8=0$ one root is $x=ki$

if one root is ki than another is $-ki$

$\therefore \cancel{k^4} - 2ik^3 - \cancel{6k^2} + 8ik + \cancel{8} = \cancel{k^4} + 2k^3 i - \cancel{6k^2} - 8ik + \cancel{8}$

$-4ik^3 + 16ik = 0$

$-4ik(k^2 - 4) = 0$

$k = 0$ but $k \neq 0$ $\quad k^2 = 4$

$k = \pm 2$

$k = +2, -2$

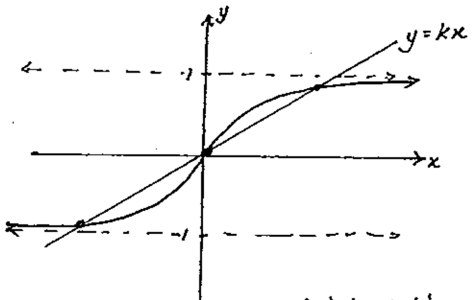
b) i) $\dfrac{dy}{dx} = \dfrac{e^x(e^x+1) - e^x(e^x-1)}{(e^x+1)^2}$ at $(0,0)$

$m = \dfrac{e^0(e^0+1) - e^0(e^0-1)}{(e^0+1)^2} = \dfrac{2}{4} = \dfrac{1}{2}$

$y - 0 = \frac{1}{2}(x - 0)$

$y = \dfrac{x}{2}$

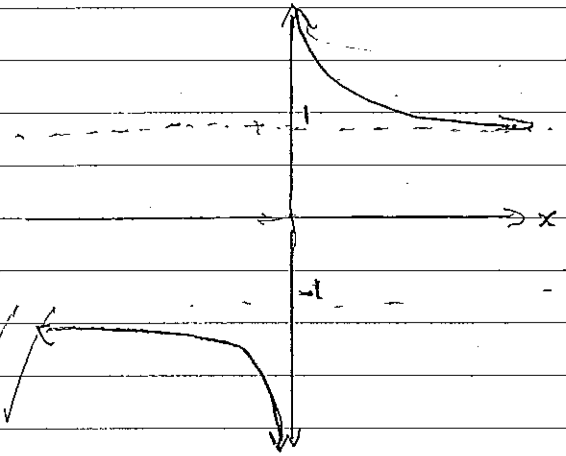
(ii) Using the above diagram



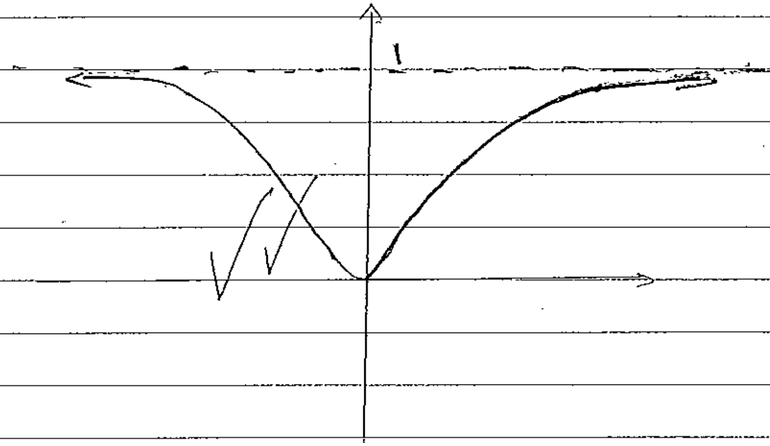
There are 3 points of intersection between $y = \dfrac{e^x - 1}{e^x + 1}$ and $y = kx$ means 3 real solutions.

~~k=0~~

iii) α)



β)



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c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2-\cos x}$

let $t = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1}(t)$

$\frac{dx}{dt} = 2\left[\frac{1}{1+t^2}\right]$ $\quad dx = \frac{2dt}{1+t^2}$

$x = 0 \quad t = 0$

$x = \frac{\pi}{2} \quad t = 1$

Integral becomes:

$\displaystyle\int_0^1 \frac{1}{2-\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$

$= \displaystyle\int_0^1 \frac{2}{1+3t^2}\,dt$

$= \frac{2}{3}\displaystyle\int_0^1 \frac{1}{t^2+(\frac{1}{\sqrt{3}})^2}\,dt$

$= \frac{2}{3}\left[\sqrt{3}\tan^{-1}(\sqrt{3}t)\right]_0^1$

$= \frac{2\sqrt{3}}{3}\cdot\frac{\pi}{3}$

$= \frac{2\pi\sqrt{3}}{9}$

space.

Start here Question 15

a) i) $\displaystyle\int_0^{\frac{1}{2}} x\sin\pi x\,dx$ $\quad u = x \quad v = \frac{1}{\pi}x - \cos\pi x$

$u' = 1 \quad v' = \sin\pi x$

$= \left[-\frac{x}{\pi}\cos\pi x\right]_0^{\frac{1}{2}} + \frac{1}{\pi}\displaystyle\int_0^{\frac{1}{2}}\cos\pi x\,dx$

$= 0 + \frac{1}{\pi^2}\left[\sin\pi x\right]_0^{\frac{1}{2}}$

$= \frac{1}{\pi^2}\left[1-0\right] = \frac{1}{\pi^2}$

ii) $2\pi\displaystyle\int_0^{\frac{1}{2}} rh\,dr$ $\quad r = x$

$h = \sin\pi x - 2x$

$= 2\pi\displaystyle\int_0^{\frac{1}{2}} x(\sin\pi x - 2x)\,dx = 2\pi\int_0^{\frac{1}{2}} x\sin\pi x - 2x^2\,dx$

$= 2\pi\left[\frac{1}{\pi^2}\right] - 2\pi\displaystyle\int_0^{\frac{1}{2}} -2x^2 = \frac{2}{\pi} + 4\pi\int_0^{\frac{1}{2}} x^2$

$= \frac{2}{\pi} + 4\pi\left[\frac{x^3}{3}\right]_0^{\frac{1}{2}}$

$= \frac{2}{\pi} + 4\pi \times \frac{1}{24} = \frac{2}{\pi} + \frac{\pi}{6}$

$= \frac{2\pi + \pi^2}{6\pi}\ u^3$

b) i) $\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

$\frac{2y}{b^2}\cdot\frac{dy}{dx} = \frac{-2x}{a^2}$

$\frac{dy}{dx} = \frac{-xb^2}{ya^2}$ at $P(a\cos\theta, b\sin\theta)$

$m = \frac{-ab^2\cos\theta}{a^2 b\sin\theta} = \frac{-b\cos\theta}{a\sin\theta}$

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$ay\sin\theta + bx\cos\theta = ab(\sin^2\theta + \cos^2\theta)$

$bx\cos\theta + ay\sin\theta = ab$

$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

ii) if perpendicular $m_{PS} \times m_{ST} = -1$

$S(ae, 0)$ $\quad b^2 = a^2(1-e^2)$

$\frac{b^2}{a^2} - 1 = -e^2$

$e^2 = \frac{a^2-b^2}{a^2}$ $\quad e = \sqrt{\frac{a^2-b^2}{a^2}} = \frac{\sqrt{a^2-b^2}}{a}$

$m_{PS} = \frac{-b\sin\theta}{a\frac{\sqrt{a^2-b^2}}{a} - a\cos\theta}$ $= \frac{-b\sin\theta}{\sqrt{a^2-b^2} - a\cos\theta}$

T is where $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ $\cap$ $x = \frac{a}{e}$

$= \frac{a}{\frac{\sqrt{a^2-b^2}}{a}}$

$= \frac{a^2}{\sqrt{a^2-b^2}}$

$\frac{\frac{a^2}{\sqrt{a^2-b^2}}\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

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$\frac{a\cos\theta}{\sqrt{a^2-b^2}} + \frac{y\sin\theta}{b} = 1$

$\frac{y\sin\theta}{b} = \frac{\sqrt{a^2-b^2} - a\cos\theta}{\sqrt{a^2-b^2}}$

$y = \frac{b\sqrt{a^2-b^2} - ab\cos\theta}{\sqrt{a^2-b^2} \times \sin\theta}$

$T\left(\frac{a^2}{\sqrt{a^2-b^2}}, \frac{b\sqrt{a^2-b^2} - ab\cos\theta}{\sin\theta \cdot \sqrt{a^2-b^2}}\right)$

$M_{ST} = \frac{\frac{b\sqrt{a^2-b^2} - ab\cos\theta}{\sin\theta\sqrt{a^2-b^2}}}{\frac{a^2}{\sqrt{a^2-b^2}} - \sqrt{a^2-b^2}}$

$= \frac{b\sqrt{a^2-b^2} - ab\cos\theta}{\sin\theta\sqrt{a^2-b^2}} \times \frac{\sqrt{a^2-b^2}}{b^2}$

$\frac{\cancel{a^2} - a^2 + b^2}{\sqrt{a^2-b^2}}$

$= \frac{\cancel{b}\sqrt{a^2-b^2} - a\cancel{b}\cos\theta}{b\sin\theta}$

$M_{PS} \times M_{ST} = \frac{-b\sin\theta}{\sqrt{a^2-b^2} - a\cos\theta} \times \frac{\sqrt{a^2-b^2} - a\cos\theta}{b\sin\theta}$

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Start here $\cancel{m_{PS} \times m_{ST} = \text{ } -1}$

$$= \cancel{\frac{-b\sin\theta}{\sqrt{a^2-b^2}-a\cos\theta}} \times \cancel{\frac{\sqrt{a^2-b^2}-a\cos\theta}{b\sin\theta}}$$

$$= -1$$

$\therefore$ PS and ST are perpendicular

$\therefore \angle PST = 90^\circ$

iii) tangent on auxillary circle:-

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{a^2}\frac{dy}{dx} = 0$$

$$\frac{2y}{a^2}\frac{dy}{dx} = \frac{-2x}{a^2}$$

$$\frac{dy}{dx} = \frac{-x}{y} \quad \text{at } (a\cos\theta, a\sin\theta)$$

$$m = \frac{-\cos\theta}{\sin\theta}$$

$$y - a\sin\theta = \frac{-\cos\theta}{\sin\theta}(x - a\cos\theta)$$

$$y\sin\theta - a\sin^2\theta = -x\cos\theta + a\cos^2\theta$$

$$x\cos\theta + y\sin\theta = a$$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{a} = 1$$

$\cancel{\frac{x\sin\theta}{a} + \frac{y\cos\theta}{a} = 1 \cap \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1}$

Circle tangent $\cap$ ellipse tangent

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{a} = \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b}$$

$$\frac{y\sin\theta}{a} = \frac{y\sin\theta}{b}$$

$$y\left(\frac{\sin\theta}{a} - \frac{\sin\theta}{b}\right) = 0$$

$$y = 0$$

$$\cancel{\frac{1}{a} = \frac{1}{b}}$$

$$\cancel{1 = \frac{a}{b}}$$

$$\cancel{a = b}$$

They intercept where a=b and thus intercept on the major axis of the ellipse

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(16) (iii) From (ii) $\frac{\ln u}{u} < \frac{2(\sqrt{u}-1)}{u}$ as $u>1$

$\therefore \lim_{u\to\infty} \frac{\ln u}{u} < \lim_{u\to\infty} 2\left(\frac{1}{\sqrt{u}} - \frac{1}{u}\right)$

$= 2(0-0)$

$= 0$

(c) (i) Because $|A_1| = |A_2| = \ldots = |A_{10}| = 1$

(ii) Sum of roots of $w^{10} - 1 = 0$ is $-\frac{b}{a} = 0$

(iii) Note $|PA_1| = |z - w_1|$

$|PA_1|^2 = (z-w_1)^2$

$= (z-w_1)(\overline{z-w_1})$ since $|z|^2 = z.\bar{z}$.

$\therefore |PA_i|^2 = (z-w_i)(\bar{z} - \bar{w}_i)$

(iv) $\sum_{i=1}^{20} PA_i^2 = \sum_{i=1}^{10} (z-w_i)(\bar{z} - \bar{w}_i)$

$= \sum_{i=1}^{10} \underbrace{z\bar{z}}_{|z|^2 \Rightarrow 1} - \underbrace{z\bar{w}_i}_{=0} - \underbrace{w_i\bar{z}}_{0} + \underbrace{w_i\bar{w}_i}_{|w_i|^2 \Rightarrow 1}$

$= \sum_{i=1}^{10} 2$

$= 2 \times 10$

$= 20$

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Q16 (a) $\frac{dx}{dt} = 1 - \cos t$; $\frac{dy}{dt} = \sin t$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{\sin t}{1 - \cos t}$

At $t = \frac{\pi}{2}$, $m_{Tangent} = \frac{1}{1}$

$\therefore m_{Normal} = -1$

At $(\frac{\pi}{2} - 1, 1)$, the eq^n. of the normal is

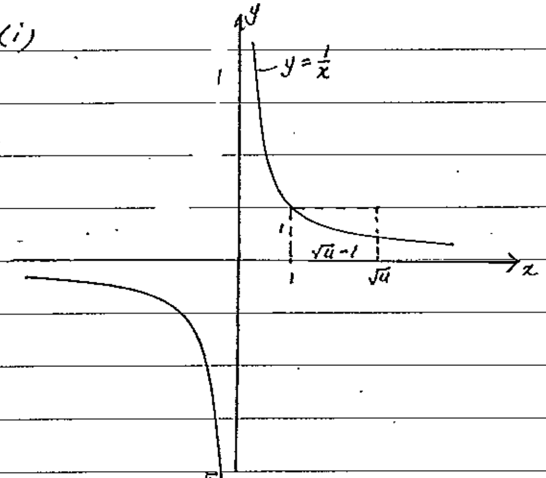
$y - 1 = -1(x - (\frac{\pi}{2} - 1))$

$\therefore y = -x + \frac{\pi}{2} - 1 + 1$

$= -x + \frac{\pi}{2}$

or $x + y - \frac{\pi}{2} = 0$

(b) (i)



Using $\int_1^{\sqrt{u}} \frac{1}{x}\,dx <$ Area of rectangle $(\sqrt{u}-1) \times 1$ as req^d.

(ii) $0 < [\ln x]_1^{\sqrt{u}} < \sqrt{u} - 1$

$0 < \ln\sqrt{u} - \ln 1 < \sqrt{u} - 1$

$0 < \frac{1}{2}\ln u < \sqrt{u} - 1$

$\therefore 0 < \ln u < 2(\sqrt{u} - 1)$ as req^d.

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