Student Number ________

MORIAH COLLEGE

Year 12 – Task 2 - Pre-Trial 2015

MATHEMATICS

Extension 1

Time Allowed: **2 hours**

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OUTCOMES ADDRESSED: P3,P5,H2,H4,H5,H6,H7,H8

General Instructions

- Reading time: 5 minutes
- Working time: 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in each question
- There are 5 sections in this examination paper.
- There are 10 multiple choice questions for 1 mark each
- There are 4 long questions for 15 marks each
- Total marks: 70

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section 1 Multiple choice (10 marks – 1 mark each)

Use the multiple choice answer sheet provided

1. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 5x}{3x}$

   $A.\ 0$ $\quad B.\ 1$ $\quad C.\ \dfrac{5}{3}$ $\quad D.\ \dfrac{3}{5}$

2. What is the exact value of $tan\left\{sin^{-1}\left(\frac{-15}{17}\right)\right\}$?

   $A.\ \dfrac{8}{15}$ $\quad B. \dfrac{-4-8}{15}$ $\quad C. \dfrac{-15}{8}$ $\quad D. \dfrac{15}{8}$

3. Which one of the following is equivalent to $log_{27}729$?

   $A.\ 2$ $\quad B.\ 3$ $\quad C.\ 27$ $\quad D.\ 702$

4. If $x^3 + ax + b$ is divisible by $(x + 3)$ and $(x - 4)$, what is the value of b?

   $A.\ b = 12$ $\quad B.\ b = -12$ $\quad C.\ b = 13$ $\quad D.\ b = -13$

5. What is the exact value of $\cos 105°$?

   A. $\dfrac{-\sqrt{3}-1}{2\sqrt{2}}$

   B. $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$

   C. $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$

   D. $\dfrac{-\sqrt{3}+1}{2\sqrt{2}}$

6. Which one of the following is the correct expression for $\int \frac{dx}{\sqrt{9-25x^2}}$?

   A. $\frac{1}{5}sin^{-1}\left(\frac{5x}{3}\right)$

   B. $\frac{1}{5}sin^{-1}\left(\frac{3x}{5}\right)$

   C. $\frac{1}{3}sin^{-1}\left(\frac{5x}{3}\right)$

   D. $\frac{1}{3}sin^{-1}\left(\frac{3x}{5}\right)$

7. For the series, $log_a 3 + log_a 6 + log_a 12 + log_a 24 \ldots$, which one of the following statements is true?

   A. $r = log_a 3$

   B. $r = log_a 2$

   C. $d = log_a 3$

   D. $d = log_a 2$

8. Solve for x: $x - 8 \leq \frac{20}{x}$

   A. $x \leq -2,\ 0 < x \leq 10$

   B. $x \leq -2,\ 0 \leq x \leq 10$

   C. $x \geq 10, -2 \leq x < 0$

   D. $x \geq 10, -2 \leq x \leq 0$

9. A function is given by the rule $f(x) = \frac{x+3}{x-2}$
Which function is the inverse function of $f(x)$?

A. $f^{-1}(x) = \frac{x-2}{x+3}$

B. $f^{-1}(x) = \frac{2x+3}{x-1}$

C. $f^{-1}(x) = \frac{2x+3}{x+1}$

D. $f^{-1}(x) = \frac{x+2}{x-3}$

10. If $\int_1^k xe^{x^2}\,dx = \frac{1}{2}(e^4 - e)$, where $k > 0$, what is the value of k?

A. 2   B. -2 C. 4   D. -4

Question 11 (15 marks)   Start this question in a new booklet

a) Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$ 3

b) Find the size of the acute angle between the lines $4x - 3y + 1 = 0$ and $x + 4y + 1 = 0$. Give your answer to the nearest minute.   3

c) Prove that $\frac{\cos 2\theta}{\cos\theta} + \frac{\sin 2\theta}{\sin\theta} = \frac{4\cos^2\theta - 1}{\cos\theta}$ 3

d) Using the substitution $t = \tan\frac{\theta}{2}$, prove that $\frac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta} \equiv \cot\theta$.   4

e) If $\sin\theta = \frac{12}{13}$ and $0 \le \theta \le \frac{\pi}{2}$, find the exact value of $\tan 2\theta$.   2

Question 12 (15 marks) Start this question in a new booklet

a) If α, β and γ are the roots of the equation $x^3 - 2x^2 + 3x + 7 = 0$ find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

b) Consider the function $f(x) = \frac{e^x}{e^x+1}$

(i) Prove that $f^{-1}(x)$ exists. **2**

(ii) Find the inverse function $f^{-1}(x)$ **2**

(iii) Find the domain of $f^{-1}(x)$. **2**

c) (i) Prove that the equation $\frac{\log_e x}{x} + 2 = 0$ has a solution between $x = 0.4$ and $x = 0.5$ **1**

(ii) Use one application of Newton's method to find a closer approximation to the solution $x = 0.4$, correct to three decimal places **3**

d) Find $\int \frac{\sec^2 x}{\sqrt{1 - 9\tan^2 x}}\,dx$ using the substitution $u = 3\tan x$. **3**

Question 13 (15 marks) Start this question in a new booklet

a) Prove by mathematical induction that, for all integers $n \geq 1$ **4**

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = 1 + (n-1)2^n$$

b) If α, β and γ are the roots of the equation $x^3 - 12x^2 + Zx - 28 = 0$, find the value of Z if the roots are in arithmetic progression. **2**

c) (i) Show that $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$ **2**

(ii) Hence, find the exact value of $\int_0^{\frac{\pi}{6}} \sin 5x \cos 3x \, dx$ **3**

d) Consider the function $f(x) = 3\tan^{-1}x$.

(i) State the range of the function $y = f(x)$. **1**

(ii) Sketch the graph $y = f(x)$. **1**

(iii) Find the gradient of the tangent to the curve $y = f(x)$ at $x = \frac{1}{\sqrt{3}}$. **2**

Question 14 (15 marks) Start this question in a new booklet

a) (i) Find A and α, to the nearest degree, if $5cosx + 12sinx = Acos(x-\alpha)$, where $A > 0$ and $0° < \alpha < 90°$. Give your answers correct to the nearest minute **2**

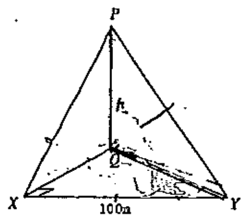
(ii) Hence solve the equation $5cosx + 12sinx = 6.5$ for $0° < x < 360°$. **2**

b) The point $P(2t, t^2)$ is on the parabola with equation $x^2 = 4y$, with focus at F. The point M divides the interval FP externally in the ratio 3:1.

(i) Show that the co-ordinates of M are $x = 3t$ and $y = \frac{1}{2}(3t^2 - 1)$ **2**

(ii) Hence prove that the locus of M is also a parabola and determine the focal length of the locus of this parabola. **3**

c) The angle of elevation to the top of a tower, PQ, from a point X due south of the tower is $45°$. From another point Y, 100 metres due east of X, the angle of elevation is $30°$ to the top of the tower.



(i) Copy this diagram in your examination booklet completing the given information.

(ii) Find the length of XQ and YQ in terms of h **2**

(iii) Show that $h = 50\sqrt{2}$ **1**

(iv) Calculate the bearing of Y from the base of the tower (nearest degree) **3**

END OF TEST

MATHEMATICS EXTENSION 1
PRE-TRIAL 2015 SOLUTIONS

QUESTION 11

(a) $\displaystyle\int_{\pi/4}^{\pi/2} (\cos 2\theta + 1)\, d\theta = \left[\tfrac{1}{2}\sin 2\theta + \theta\right]_{\pi/4}^{\pi/2} = \left(\tfrac{\pi}{2}\right) - \left(\tfrac{1}{2} + \tfrac{\pi}{4}\right) = \dfrac{\pi - 2}{4}$ ✓ 3

(b) $l_1: y = \frac{4}{3}x + \frac{1}{3} \quad \therefore m_1 = \frac{4}{3}$ ✓

$l_2: y = -\frac{1}{4}x - \frac{1}{4} \quad \therefore m_2 = -\frac{1}{4}$

$\therefore \tan\theta = \left|\dfrac{\frac{4}{3} + \frac{1}{4}}{1 - \frac{4}{3}\cdot\frac{1}{4}}\right|$ ✓ $= \dfrac{\frac{4}{3} + \frac{1}{4}}{\frac{2}{3}} = \dfrac{16+3}{8} = \dfrac{19}{8}$

$\therefore \theta = 67°10'$ ✓ 3

(c) LHS: $\dfrac{2\cos^2\theta - 1}{\cos\theta} + \dfrac{2\sin\theta\cos\theta}{\sin\theta}$ ✓ $= \dfrac{2\cos^2\theta - 1}{\cos\theta} + \dfrac{2\cos^2\theta}{\cos\theta}$ ✓ $= \dfrac{4\cos^2\theta - 1}{\cos\theta}$ ✓ $=$ RHS 3

(d) If $t = \tan\frac{\theta}{2}$ then $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$. ✓

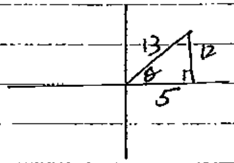
$\therefore \text{LHS} = \dfrac{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}$ ✓

$= \dfrac{1+t^2+2t+1-t^2}{1+t^2+2t-1+t^2}$

$= \dfrac{2(1+t)}{2t(1+t)}$

$= \dfrac{1}{t}$ ✓ $= \cot\dfrac{\theta}{2}$ ✓ $=$ RHS 4

(e) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$



$= \dfrac{2\left(\frac{12}{5}\right)}{1 - \frac{144}{25}}$ ✓

$= \dfrac{120}{25-144}$

$= -\dfrac{120}{119}$ ✓ 2

MULTIPLE CHOICE 1 C 2 C 3 A 4 B 5 D
6 A 7 D 8 A 9 B 10 A

QUESTION 12

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ ✓ $= \dfrac{3}{-7}$ ✓ 2

(b) (i) $f'(x) = \dfrac{(e^x+1)e^x - e^x(e^x)}{(e^x+1)^2} = \dfrac{e^{2x} + e^x - e^{2x}}{(e^x+1)^2} = \dfrac{e^x}{(e^x+1)^2} > 0$ ✓ for all x.

$\therefore$ inverse exists. 1

(ii) $x = \dfrac{e^y}{e^y+1}$ ✓ $\therefore xe^y + x = e^y \quad \therefore xe^y - e^y = -x$

$\therefore e^y(x-1) = -x \quad \therefore e^y = \dfrac{x}{1-x}$

$\therefore y = \ln\dfrac{x}{1-x}$ ✓ 2

(iii) $\dfrac{x}{1-x} > 0$ ✓

$\therefore 0 < x < 1$ ✓ 2

(c) (i) $f(0.4) = -0.29\ldots < 0 \qquad f(0.5) = 0.613\ldots > 0$

Since f is continuous, this means there is a root between 0.4 and 0.5. ✓ 1

(ii) $x_1 = 0.4 - \dfrac{f(0.4)}{f'(0.4)}$ ✓ $\qquad f'(x) = \dfrac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$ ✓

$= 0.424$ ✓ 3

(d) $\dfrac{1}{3}\displaystyle\int \dfrac{3\sec^2 x\, dx}{\sqrt{1 - 9\tan^2 x}}$ $\qquad u = 3\tan x$, $\therefore du = 3\sec^2 x\, dx$ ✓

$= \dfrac{1}{3}\displaystyle\int \dfrac{du}{\sqrt{1-u^2}}$ ✓

$= \dfrac{1}{3}\sin^{-1} u + c$

$= \dfrac{1}{3}\sin^{-1}(3\tan x) + C$ ✓ 3

Question (13)

a) test $n=1$

$LHS = 1\times 2^0 = 1\times 1 = 1$ $\quad RHS = 1+(1-1)\times 2^1 = 1$ $\therefore$ true $n=1$.

Assume true $n=k$.

i.e. Assume $1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \dots + k\times 2^{k-1} = 1+(k-1).2^k$

To Prove true for $n=k+1$.

i.e. We would like to show:-

$$\underbrace{1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \dots + k\times 2^{k-1}} + (k+1)\times 2^k = 1+k\times 2^{k+1}$$

$$LHS = 1+(k-1)\times 2^k + (k+1)\times 2^k \quad \text{(by the above assumption)}$$

$$= 1+2^k(k-1+k+1)$$

$$= 1+2k.2^k$$

$$= 1+k.2^{k+1} = RHS.$$

$\therefore$ Result is true by induction.

or any other valid setting out

b) Roots in A.P. $\Longrightarrow$ roots could be $a-d, a, a+d$

Sum: $3a=12 \Rightarrow a=4$ is a root. OR SIMILAR

$\therefore (4)^3 - 12(4)^2 + 4z - 28 = 0$

$64 - 192 - 28 + 4z = 0$

$64 - 220 + 4z = 0$

$4z = 156 \quad \underline{z=39}$

c) i) $LHS = \sin A\cos B + \cancel{\sin B\cos A} + \sin A\cos B - \cancel{\sin B\cos A}$

$= 2\sin A\cos B. = RHS.$

ii) $\displaystyle\int_0^{\pi/4} \sin 5x\cos 3x\,dx = \frac{1}{2}\int_0^{\pi/6} \sin 8x + \sin 2x\,dx$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right]$$

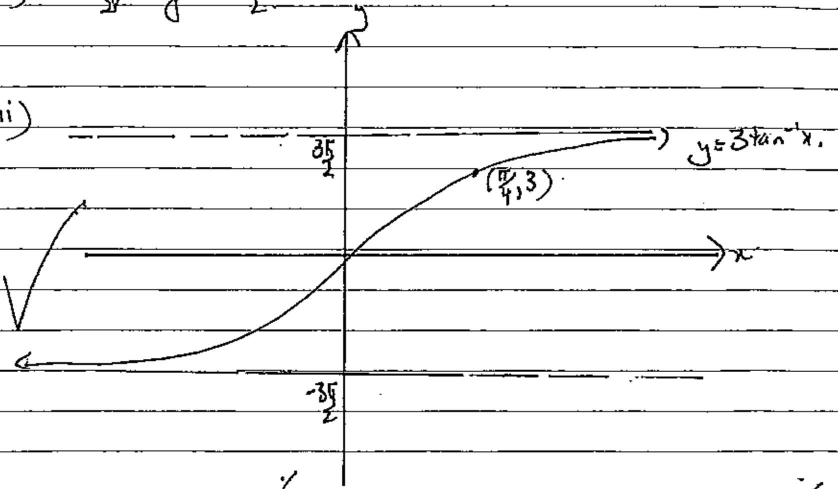
$$= \frac{1}{2}\left[\left(\frac{1}{16}-\frac{1}{4}\right)-\left(-\frac{1}{8}-\frac{1}{2}\right)\right] = \frac{1}{2}\left(\frac{-3}{16}+\frac{10}{16}\right) = \frac{7}{32}$$

Question (13) (cont'd)

d) $f(x) = 3\tan^{-1}x$.

i) $-\frac{3\pi}{2} < y < \frac{3\pi}{2}$

ii)



iii) $m_{tangent} = \dfrac{3}{1+x^2} = \dfrac{3}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \dfrac{3}{4/3} = \dfrac{9}{4} = 2\frac{1}{4}$

PRE TRIAL EXT 1 2015

Question 14

a) (i) $5\cos x + 12\sin x = A\cos(x-\alpha)$ $\qquad A>0,\ 0°<\alpha<90°.$

$5\cos x + 12\sin x \equiv A\cos x\cos\alpha + A\sin x\sin\alpha.$

$A\cos\alpha = 5 \qquad A\sin\alpha = 12$

$\cos\alpha = \frac{5}{A} \qquad \sin\alpha = \frac{12}{A}$

$\left(\frac{5}{A}\right)^2 + \left(\frac{12}{A}\right)^2 = 1$ $\qquad (2)$

$A^2 = 25+144$

$= 169$

$A = 13.\ (A>0)$ ✓ $\qquad \therefore \sin\alpha = \frac{12}{13}$

$\alpha = \sin^{-1}\left(\frac{12}{13}\right)$

$\alpha = 67°$ (nearest deg). ✓

(ii) $5\cos x + 12\sin x = 6\cdot5 \qquad 0°<x<360°$

$\therefore 13\cos(x-67°) = 6\cdot5 \qquad -67°<x-67<253°$

$\cos(x-67) = \frac{6\cdot5}{13}$

$x-67 = \cos^{-1}\left(\frac{6\cdot5}{13}\right)$

$= \cos^{-1}\left(\frac{1}{2}\right)$ ✓

$\therefore x-67° = 60°, 300°, -60°$ $\qquad (2)$

$\therefore x = 60+67, -60+67$

$= 127°, 7°.$ ✓

(b) (i) $P(2t, t^2)$ $\quad x^2 = 4y$ $\quad$ Focus F.

$\therefore a = 1$

$\therefore F(0,1)$ $\qquad M$: externally $3:1$ $\quad FP$

$F(0,1) \quad P(2t, t^2)$

$\therefore M = \left(\frac{(-1\times0)+(3\times2t)}{2}, \frac{-1(1)+(3\times t^2)}{2}\right)$ ✓

$= \frac{6t}{2}, \frac{3t^2-1}{2}$ $\qquad (2)$

$= \left(3t, \frac{1}{2}(3t^2-1)\right)$ ✓

(ii) $x = 3t \qquad y = \frac{1}{2}(3t^2-1)$

$\therefore t = \frac{x}{3} \qquad 2y+1 = 3t^2$ ✓

$2y+1 = 3\left(\frac{x}{3}\right)^2$

$2y+1 = \frac{x^2}{3}$

$x^2 = 6y+3$ ✓ $\qquad (3)$

$x^2 = 6\left(y+\frac{1}{2}\right)$ which is in the form $(x-h)^2 = 4a(y-k)$

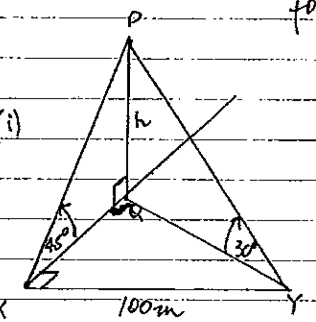
$\therefore$ locus is parabola,

Vertex $\left(0, -\frac{1}{2}\right)$

focal length $= \frac{3}{2}$ ✓

focus $(0,1)$

c) (i)



(ii) $\tan 45 = \frac{h}{XQ}$

$XQ = h.\tan 45$

$\boxed{h = XQ}$ 1

$\tan 30° = \frac{h}{YQ}$

$YQ = \frac{h}{\tan 30}$

$\boxed{YQ = \sqrt{3}h}$. 1 $\qquad (3)$

(iii) $XQ^2 + 100^2 = YQ^2$

$h^2 + 10000 = 3h^2$

$2h^2 = 10000$

$h^2 = 5000$

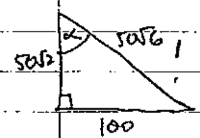
$h = \sqrt{5000}$ (+ive only)

$= \sqrt{2500}\times\sqrt{2}$

$h = 50\sqrt{2}$ $\qquad (1)$

(iv) $\sin\alpha = \frac{100}{50\sqrt{6}}$ 1

$\therefore \alpha = 55°$ (nearest deg)



$\therefore$ Bearing of Y from base of tower $= 180-55$

$= 125°$ 1 $\qquad (3)$