MORIAH COLLEGE

Year 11

MATHEMATICS ENRICHMENT

Class Test: Polynomials, Partial Fractions and Inequalities

Date: **Tuesday 7th March, 2006**

Time Allowed: 50 min + 5 minutes reading time

Examiners: J. Taylor

General Instructions

- Calculators may be used
- Show all necessary working.
- Start **Each Question** on a new page.
- Questions are not of equal value

Question 1 (8 marks)

a) Use long division to perform each of the following divisions. Express each result in the form $P(x) = D(x)Q(x) + R(x)$

i) $(x^3 + 4x^2 - 5x + 3) \div (x + 1)$ **3**

ii) $(2x^3 + 5) \div (x^2 + x + 1)$ **2**

b) If $x^4 - 4x^3 - 14x^2 + ax + b$ is exactly divisible by $x^2 - 6x + 3$, find a and b **3**

Question 2 (11 marks)

a) Express as a sum of partial fractions $\dfrac{6}{2x^2 - 5x + 2}$. **3**

b) Express as a sum of partial fractions $\dfrac{x^2 + x + 2}{x(x+1)}$. **4**

c) Express as a sum of a partial fractions $\dfrac{3x^2 - 3x + 2}{(2x-1)(x^2+1)}$. **4**

Question 3 (12 marks)

a) If n and r are positive integers such that $n > r > 1$, show that **2**

$$\frac{n+1-r}{n-r} > \frac{n+1}{n}$$

b) i) Prove that, for any real numbers a, b, $a^2 + b^2 \geq 2ab$. **2**

Hence or otherwise prove that:

ii) $bc(b+c) + ca(c+a) + ab(a+b) \geq 6abc$ **3**

iii) $a^3 + b^3 \geq ab(a+b)$ **2**

iv) $2(a^3 + b^3 + c^3) \geq ab(a+b) + bc(b+c) + ca(c+a)$ **1**

v) $\dfrac{1}{x^3} + \dfrac{1}{y^3} + \dfrac{1}{z^3} \geq \dfrac{3}{xyz}$ for any real x, y, z. **2**

Question 4 (11 marks)

a) i) Given m, n are positive integers, prove that $\dfrac{m+2n}{m+n}+\dfrac{m}{n}=\dfrac{m+n}{n}+\dfrac{n}{m+n}$. **2**

ii) Deduce that $\dfrac{m+2n}{m+n}+\dfrac{m}{n}>2$. **2**

b) Prove that if $\dfrac{m}{n}<\sqrt{2}$ then

$\dfrac{m+2n}{m+n} > 2-\dfrac{m}{n}$

i) $\dfrac{m+n}{n}<\sqrt{2}+1$ **1**

$1+\dfrac{n}{m+n} < 2-\sqrt{}$

ii) $\dfrac{n}{m+n}>\sqrt{2}-1$ **2**

$-1+\sqrt{2} <$ n

iii) $\dfrac{m+2n}{m+n}>\sqrt{2}$ **2**

$\sqrt{2} \ \sqrt{2}-1$

$\sqrt{2}$.

c) Explain how a) above and b)iii) lead to the conclusion that **2**

$$2<\frac{m+2n}{m+n}+\frac{m}{n}<2\sqrt{2}$$

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Q1 a) i)

$$\begin{array}{r|l} & x^2+3x-8 \\ x+1 & x^3+4x^2-5x+3 \\ & x^3+x^2 \\ & \quad 3x^2-5x+3 \\ & \quad 3x^2+3x \\ & \qquad -8x+3 \\ & \qquad -8x-8 \\ & \qquad\quad 11 \end{array}$$ ✓ ✓

$\therefore x^3+4x^2-5x+3 = (x+1)(x^2+3x-8) +$ ✓

ii)

$$\begin{array}{r|l} & 2x-2 \\ x^2+x+1 & 2x^3+0x^2+0x+5 \\ & 2x^3+2x^2+2x \\ & \quad -2x^2-2x+5 \\ & \quad -2x^2-2x-2 \\ & \qquad 7 \end{array}$$ ✓

$\therefore 2x^3+5 = (x^2+x+1)(2x-2) + 7$ ✓

b)

$$\begin{array}{r|l} & x^2+2x-5 \\ x^2-6x+3 & x^4-4x^3-14x^2+ax+b \\ & x^4-6x^3+3x^2 \\ & \quad 2x^3-17x^2+ax+b \\ & \quad 2x^3-12x^2+6x \\ & \qquad -5x^2+(a-6)x+b \\ & \qquad -5x^2+30x-15 \\ & \qquad (a-36)x+b+15 \end{array}$$ ✓ ✓

Remainder $= (a-36)x + b+15$

Since the remainder must be 0, it follows that

$a = 36$ and $b = -15$ ✓ ✓

Q2 a) $\dfrac{6}{2x^2-5x+2} = \dfrac{6}{(2x-1)(x-2)} \equiv \dfrac{a}{2x-1} + \dfrac{b}{x-2}$

$\therefore 6 = a(x-2) + b(2x-1)$ ✓

Sub $x=\frac{1}{2}$: $6 = a \times -1\frac{1}{2}$

$\therefore a = -4$ ✓

Sub $x=2$: $6 = 3b$

$\therefore b = 2$ ✓

$\therefore \dfrac{6}{2x^2-5x+2} = \dfrac{-4}{2x-1} + \dfrac{2}{x-2}$ ✓

b) $\dfrac{x^2+x+2}{x(x+1)} = \dfrac{x^2+x+2}{x^2+x}$

$= 1 + \dfrac{2}{x^2+x}$ ✓

Consider $\dfrac{2}{x(x+1)} \equiv \dfrac{a}{x} + \dfrac{b}{x+1}$

$2 = a(x+1) + bx$ ✓

Sub $x=0$: $a = 2$ ✓

Sub $x=-1$: $b = -2$ ✓

$\therefore \dfrac{x^2+x+2}{x(x+1)} = 1 + \dfrac{2}{x} - \dfrac{2}{x+1}$ ✓

c) Let $\dfrac{3x^2-3x+2}{(2x-1)(x^2+1)} \equiv \dfrac{a}{2x-1} + \dfrac{bx+c}{x^2+1}$

$\therefore 3x^2-3x+2 = a(x^2+1) + (2x-1)(bx+c)$ ✓

Sub $x=\frac{1}{2}$: $\frac{5}{4} = \frac{5}{4}a$

$\therefore a = 1$ ✓

Coeff x^2: $3 = a + 2b$

$\therefore b = 1$ ✓

Sub $x=0$ $2 = a - 1(0+c)$

$2 = a - c$

$\therefore c = -1$ ✓

$\therefore \dfrac{3x^2-3x+2}{(2x-1)(x^2+1)} = \dfrac{1}{2x-1} + \dfrac{x-1}{x^2+1}$ ✓

Question 3

a) Consider $\frac{n+1-r}{n-r} - \frac{n+1}{n} = \frac{n^2+n-nr-n^2+nr-n+r}{n(n-r)}$ ✓

$= \frac{r}{n(n-r)}$

> 0 since $r>0,\ n>r$ ✓

$\therefore \frac{n+1-r}{n-r} > \frac{n+1}{n}$

b) i) Consider $a^2+b^2-2ab = (a-b)^2$ ✓

≥ 0 ✓

$\therefore a^2+b^2 \geq 2ab$

ii) LHS $= b^2c+bc^2+c^2a+ca^2+a^2b+ab^2$ ✓

$= (b^2+a^2)c + (a^2+c^2)b + (b^2+c^2)a$ ✓

$\geq 2abc + 2acb + 2bca$ ✓

$= 6abc$

$\therefore bc(b+c) + ca(c+a) + ab(a+b) \geq 6abc$

iii) $a^3+b^3 = (a+b)(a^2+b^2-ab)$ ✓

$\geq (a+b)(2ab-ab)$ ✓

$= (a+b)ab$

$\therefore a^3+b^3 \geq ab(a+b)$

iv) Adding similar results:

$a^3+b^3+b^3+c^3+c^3+a^3 \geq ab(a+b)+bc(b+c)+ca(a+c)$ ✓

$\therefore 2(a^3+b^3+c^3) \geq ab(a+b)+bc(b+c)+ca(a+c)$

v) Combining (iv) and (ii) gives $a^3+b^3+c^3 \geq 3abc$ ✓

Replace $a \to \frac{1}{x}$ $b \to \frac{1}{y}$ $c \to \frac{1}{z}$: ✓

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3} \geq \frac{3}{xyz}$$

Question 4

a) i) LHS − RHS $= \frac{m+2n-n}{m+n} + \frac{m-(m+n)}{n}$ |

$= \frac{m+n}{m+n} + -\frac{n}{n}$

$= 1-1$

$= 0$ |

$\therefore \frac{m+2n}{m+n} + \frac{m}{n} = \frac{m+n}{n} + \frac{n}{m+n}$

ii) For a a positive integer, $a + \frac{1}{a} \geq 2$ |

$\therefore \frac{m+n}{n} + \frac{n}{m+n} > 2$ if $m \neq n$ |

b) i) $\frac{m+n}{n} = \frac{m}{n} + \frac{n}{n}$ ✓ $\therefore \frac{m+2n}{m+n} + \frac{m}{n} > 2$

$< \sqrt{2}+1$

ii) $\frac{n}{m+n} > \frac{1}{\sqrt{2}+1}$ ✓

$= \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}$

$= \frac{\sqrt{2}-1}{1}$ ✓

$\therefore \frac{n}{m+n} > \sqrt{2}-1$

iii) $\frac{m+2n}{m+n} = \frac{m+n+n}{m+n}$

$= \frac{m+n}{m+n} + \frac{n}{m+n}$ ✓

$> 1 + \sqrt{2}-1$

$= \sqrt{2}$

$\therefore \frac{m+2n}{m+n} > \sqrt{2}$ ✓

c) Combining $\frac{m}{n} > \sqrt{2}$ (given) with iii) we have

$\frac{m+2n}{m+n} + \frac{m}{n} > \sqrt{2}+\sqrt{2}$ |

$= 2\sqrt{2}$

Using a) ii) as well:

$2 < \frac{m+2n}{m+n} + \frac{m}{n} < 2\sqrt{2}$ |