



MORIAH COLLEGE

Year 11

MATHEMATICS ENRICHMENT

Class Test: Polynomials, Partial Fractions and Inequalities

Date: Tuesday 7th March, 2006

Time Allowed: 50 min + 5 minutes reading time

Examiners: J. Taylor

General Instructions

- Calculators may be used
- Show all necessary working.
- Start **Each Question** on a new page.
- Questions are not of equal value

Question 1 (8 marks)


a) Use long division to perform each of the following divisions. Express each result in the form $P(x) = D(x)Q(x) + R(x)$

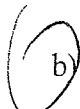
i) $(x^3 + 4x^2 - 5x + 3) \div (x + 1)$ 3


ii) $(2x^3 + 5) \div (x^2 + x + 1)$ 2

b) If $x^4 - 4x^3 - 14x^2 + ax + b$ is exactly divisible by $x^2 - 6x + 3$, find a and b 3

Question 2 (11 marks)

 Express as a sum of partial fractions $\frac{6}{2x^2 - 5x + 2}$ 3

 b) Express as a sum of partial fractions $\frac{x^2 + x + 2}{x(x + 1)}$ 4

 Express as a sum of a partial fractions $\frac{3x^2 - 3x + 2}{(2x - 1)(x^2 + 1)}$ 4

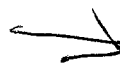
Question 3 (12 marks)

a) If n and r are positive integers such that $n > r > 1$, show that 2

$$\frac{n+1-r}{n-r} > \frac{n+1}{n}$$


b) i) Prove that, for any real numbers a, b , $a^2 + b^2 \geq 2ab$. 2

Hence or otherwise prove that:

 ii) $bc(b + c) + ca(c + a) + ab(a + b) \geq 6abc$ 3

iii) $a^3 + b^3 \geq ab(a + b)$ 2

iv) $2(a^3 + b^3 + c^3) \geq ab(a + b) + bc(b + c) + ca(c + a)$ 1

 v) $\frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} \geq \frac{3}{xyz}$ for any real x, y, z . 2

Question 4 (11 marks)

a) i) Given m, n are positive integers, prove that $\frac{m+2n}{m+n} + \frac{m}{n} = \frac{m+n}{n} + \frac{n}{m+n}$. 2

ii) Deduce that $\frac{m+2n}{m+n} + \frac{m}{n} > 2$. 2

b) Prove that if $\frac{m}{n} < \sqrt{2}$ then

i) $\frac{m+n}{n} < \sqrt{2} + 1$

ii) $\frac{n}{m+n} > \sqrt{2} - 1$

iii) $\frac{m+2n}{m+n} > \sqrt{2}$

$$\frac{m+2n}{m+n} > 2 - \frac{m}{n}$$

$$1 + \frac{n}{m+n} < 2 - \sqrt{2}$$

$$-1 + \sqrt{2} < \frac{n}{m+n}$$

$$\sqrt{2} + 1 > \sqrt{2} - 1$$

1

2

2

c) Explain how a) above and b)iii) lead to the conclusion that

$$2 < \frac{m+2n}{m+n} + \frac{m}{n} < 2\sqrt{2}$$

2

Enrichment 2006 March.

Q1 a) i)

$$\frac{x^2 + 3x - 8}{x+1} \div \frac{x^2 + 4x^2 - 5x + 3}{x^2 + x^2}$$

$$\begin{array}{r} x^2 + x^2 \\ 3x^2 - 5x + 3 \\ \underline{3x^2 + 3x} \\ -8x + 3 \\ \underline{-8x + 8} \\ 11 \end{array}$$

$$\therefore x^3 + 4x^2 - 5x + 3 = (x+1)(x^2 + 3x - 8) + 11$$

ii) $\frac{2x - 2}{x^2 + x + 1} \div \frac{2x^3 + 10x^2 + 0x + 5}{2x^3 + 2x^2 + 2x}$

$$\begin{array}{r} 2x^3 + 2x^2 + 2x \\ \underline{-2x^3 - 2x + 5} \\ -2x^2 - 2x - 2 \end{array}$$

$$\therefore 2x^3 + 5 = (x^2 + x + 1)(2x - 2) + 7$$

b)

$$\frac{x^2 + 2x - 5}{x^2 - 6x + 3} \div \frac{x^4 - 4x^2 - 14x + 6}{x^4 - 6x^2 + 3x^2}$$

$$\begin{array}{r} x^4 - 6x^2 + 3x^2 \\ 2x^3 - 17x^2 + ax + 6 \\ \underline{2x^3 - 12x^2 + 6x} \\ -5x^2 + (a-6)x + 6 \\ \underline{-5x^2 + 30x - 15} \end{array}$$

Remainder = $(a-36)x + 6 + 15$

Since the remainder must be 0, it follows that

$$a = 36 \quad \text{and} \quad b = -15$$

$$(a-36)x + b + 15$$

Q2 a)

$$\frac{6}{2x^2 - 5x + 2} = \frac{6}{(2x-1)(x-2)} = \frac{a}{2x-1} + \frac{b}{x-2}$$

$$\therefore 6 = a(x-2) + b(2x-1)$$

Sub $x = \frac{1}{2}$: $6 = a(-\frac{3}{2})$

$$\therefore a = -4$$

Sub $x = 2$: $6 = 3b$

$$\therefore b = 2$$

$$\therefore \frac{6}{2x^2 - 5x + 2} = \frac{-4}{2x-1} + \frac{2}{x-2}$$

b) $\frac{x^2 + x + 2}{x(x+1)} = \frac{x^2 + x + 2}{x^2 + x}$

$$= 1 + \frac{2}{x^2 + x}$$

Consider $\frac{2}{x(x+1)} = \frac{a}{x} + \frac{b}{x+1}$

$$2 = a(x+1) + bx$$

Sub $x = 0$: $2 = a$

Sub $x = -1$: $b = -2$

$$\therefore \frac{x^2 + x + 2}{x(x+1)} = 1 + \frac{2}{x} - \frac{2}{x+1}$$

c) Let $\frac{3x^2 - 3x + 2}{(2x-1)(x^2+1)} = \frac{a}{2x-1} + \frac{bx+c}{x^2+1}$

$$\therefore 3x^2 - 3x + 2 = a(x^2+1) + (2x-1)(bx+c)$$

Sub $x = \frac{1}{2}$: $\frac{5}{4} = \frac{5}{4}a$

$$\therefore a = 1$$

Coeff x^2 : $3 = a + 2b$

$$\therefore b = 1$$

Sub $x = 0$: $2 = a - 1(0+c)$

$$2 = a - c$$

$$\therefore c = -1$$

$$\therefore \frac{3x^2 - 3x + 2}{(2x-1)(x^2+1)} = \frac{1}{2x-1} + \frac{x-1}{x^2+1}$$

Question 3

a) Consider $\frac{n+1-r}{n-r} - \frac{n+1}{n} = \frac{n^2 + n - nr - n^2 + nr - n + r}{n(n-r)}$ ✓
 $= \frac{r}{n(n-r)}$
 > 0 since $n > 0, n > r$ ✓
 $\therefore \frac{n+1-r}{n-r} > \frac{n+1}{n}$

b) i) Consider $a^2 + b^2 - 2ab = (a-b)^2$ ✓
 ≥ 0 ✓
 $\therefore a^2 + b^2 \geq 2ab$

ii) LHS $= b^2c + bc^2 + c^2a + ca^2 + a^2b + ab^2$ ✓
 $= (b^2 + a^2)c + (a^2 + c^2)b + (b^2 + c^2)a$ ✓
 $\geq 2abc + 2acb + 2bca$ ✓
 $= 6abc$

$\therefore bc(b+c) + ca(c+a) + ab(a+b) \geq 6abc$

iii) $a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$ ✓
 $\geq (a+b)(2ab - ab)$ ✓
 $= (a+b)ab$
 $\therefore a^3 + b^3 \geq ab(a+b)$

iv) Adding similar results:

$a^3 + b^3 + b^3 + c^3 + c^3 + a^3 \geq ab(a+b) + bc(b+c) + ca(c+a)$ ✓
 $\therefore 2(a^3 + b^3 + c^3) \geq ab(a+b) + bc(b+c) + ca(c+a)$

v) Combining (iv) and (iii) gives $a^3 + b^3 + c^3 \geq 3abc$ ✓
 Replace $a \rightarrow \frac{1}{x}$, $b \rightarrow \frac{1}{y}$, $c \rightarrow \frac{1}{z}$: ✓

$\frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} \geq \frac{3}{xyz}$

Question 4

a) i) LHS - RHS $= \frac{m+2n-n}{m+n} + \frac{m-(m+n)}{n}$ ✓
 $= \frac{m+n}{m+n} + -\frac{n}{n}$
 $= 1 - 1$
 $= 0$

$\therefore \frac{m+2n}{m+n} + \frac{m}{n} = \frac{m+n}{n} + \frac{n}{m+n}$

ii) For a a positive integer, $a + \frac{1}{a} \geq 2$ ✓

$\therefore \frac{m+n}{n} + \frac{n}{m+n} \geq 2$ if $m \neq n$ ✓

b) i) $\frac{m+n}{n} = \frac{m}{n} + \frac{1}{n}$ ✓ $\therefore \frac{m+2n}{m+n} + \frac{1}{n} > 2$
 $\leq \sqrt{2} + 1$

ii) $\frac{m}{m+n} > \frac{1}{\sqrt{2}+1}$ ✓ iii) $\frac{m+2n}{m+n} = \frac{m+n+n}{m+n}$
 $= \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{m+n}{m+n} + \frac{n}{m+n}$ ✓
 $= \frac{\sqrt{2}-1}{1} > 1 + \sqrt{2}-1$
 $\therefore \frac{m}{m+n} > \sqrt{2}-1$ $= \sqrt{2}$ ✓

$\therefore \frac{m+2n}{m+n} > \sqrt{2}$ ✓

c) Combining $\frac{m}{n} > \sqrt{2}$ (given) with (ii) we have

$\frac{m+2n}{m+n} + \frac{m}{n} > \sqrt{2} + \sqrt{2}$
 $= 2\sqrt{2}$

Using (i) as well:

$2 < \frac{m+2n}{m+n} + \frac{m}{n} < 2\sqrt{2}$ ✓