MORIAH COLLEGE

Year 12

2008 EXTENSION 1 MATHEMATICS

ASSESSSMENT TASK 2 (PRE-TRIAL)

Time Allowed: 2 hours plus 5 minutes reading time.

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Instructions:

- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.

Standard integrals are at the end of the paper.

Question 1

(a) Find $\lim_{x\to 0}\dfrac{\sin 3x}{x}$ [2]

(b) Solve for x: $\dfrac{x}{x+1}\le\dfrac{1}{6}$ [3]

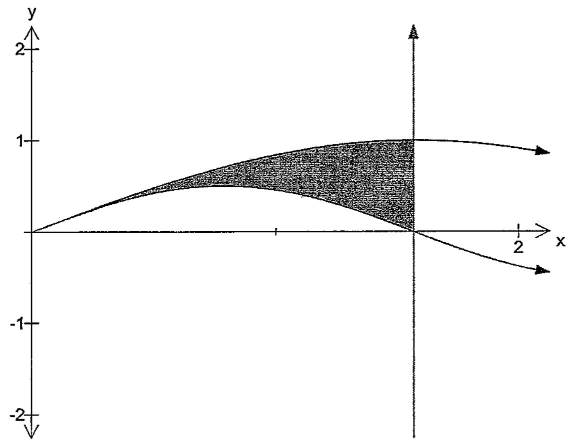
(c) Given A:(1,4) and B: (5,2), find the coordinates of the point P which divides the interval AB externally in the ratio 1:3. [3]

(d) $\displaystyle\int_0^2 \frac{dx}{\sqrt{16-x^2}}$ [2]

(e) Find the acute angle between the lines $2x+y=3$ and $x-y=3$ [2]
Give your answer correct to the nearest minute.

Question 2

(a) The shaded area is bounded by the curve $y = \sin x$ and $y = \frac{1}{2}\sin 2x$ and the line $x=k$



i) Show that $k = \frac{\pi}{2}$ **[1]**

ii) Calculate the shaded area **[3]**

(b) A normal is drawn to the curve $y = \tan x$ at $x = \frac{\pi}{4}$. Find where this normal intercepts the x-axis. Leave your answer in exact form **[3]**

(c) Evaluate:

i) $\int_0^1 \frac{dx}{4+x^2}$ correct to 2 decimal places **[2]**

ii) $\int_0^{\frac{\pi}{4}} \sin^2 dx$ in exact form **[3]**

Question 3

(a) Find $\int \frac{\ln x}{x} dx$ using the substitution $u = \ln x$ **[2]**

(b)

i) Express $\cos x + \sqrt{3}\sin x$ in the form $R\cos(x-\alpha)$ where $r > 0$ and α is in radians **[2]**

ii) Hence find the general solution of the equation $\cos x + \sqrt{3}\sin x = \sqrt{2}$ **[2]**

iii) Give the solution in the interval $-\pi \le x \le 2\pi$ **[2]**

(c) The mass M (measured in kilograms) of a certain species of fish is known to grow at a rate given by $\frac{dM}{dt} = \frac{\ln t}{t}$ where $t \ge 1$ and t is measured in weeks.

i) After 1 week, the mass of a particular fish of this species was 2 kilograms. Using part a) or otherwise, find its mass after 2 weeks(correct to 2 places decimals) **[2]**

ii) Find the maximum value of $\frac{dM}{dt}$, giving your answer in exact form and stating the units of measurement. **[2]**

Question 4

(a)

i) Sketch the curve $y = 3\sin^{-1} 2x$ indicating clearly the domain and range. **[2]**

ii) Show that the equation of the tangent to the curve at $x = \frac{1}{4}$ **[3]**

is given by $y = 4\sqrt{3}x + \frac{\pi}{2} - \sqrt{3}$

iii) Does the tangent found in part ii) meet the curve again. **[2]**
Use appropriate y-values to support your answer.

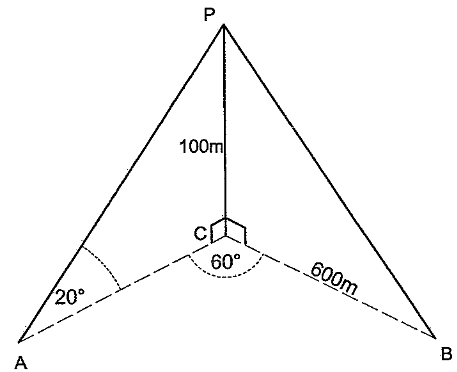
(b)

i) Prove by mathematical induction that for $n \geq 1$,
$1^2 + 3^2 + ... + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$ **[3]**

ii) Hence, calculate the sum: $17^2 + 19^2 + ... + 71^2$ **[2]**

Question 5

(a) Two yachts, A and B, subtend an angle of 60° at the base C of a cliff. From yacht A the angle of elevation of the point P, 100 metres vertically above C is 20°. Yacht B is 600 metres from C.



i) Calculate the length AC, correct to 2 decimal places **[2]**

ii) Calculate the distance between the two yachts correct to the nearest metre. **[2]**

(b) For a triangle, you are given two side lengths 4 and 5.
The angles opposite these sides are α° and $(\alpha + 30)^\circ$ respectively.

i) Show this information on a triangle **[1]**

ii) Show that $\tan\alpha = \frac{10 + 4\sqrt{3}}{13}$ **[3]**

(c) The volume V of a sphere of radius r mm is increasing at a constant rate of 200mm^3 per second.

i) Find $\frac{dr}{dt}$ in terms of r **[2]**

ii) Determine the rate in which the surface area S, of the sphere, is increasing when the radius is 50mm. **[2]**

[You may assume $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$]

Question 6

(a) Find the general solution to the equation: $\sin^2\theta + 2\cos 2\theta = \frac{1}{2}$, where θ is measured in degrees. **[3]**

(b) Consider the parabola $4ay = x^2$ where $a > 0$, and suppose the tangents at $P:(2ap, ap^2)$ and $Q:(2aq, aq^2)$ intersect at the point T.
Let $S:(0, a)$ be the focus of the parabola.

i) Find the coordinates of the point T
[You may assume that the equation of the tangent at P is $y = px - ap^2$] **[2]**

ii) Show that $SP = a(p^2 + 1)$ **[2]**

iii) Suppose that P and Q move on the parabola in such a way that $SP + SQ = 4a$. Show that the locus of T is a parabola. **[2]**

(c) Evaluate: $\int_0^{\frac{1}{2}} \frac{4x}{(1+2x)^4}\,dx$ using the substitution $u = 1 + 2x$ **[3]**

Question 7

Consider the function $f(x) = \frac{e^x}{3 + e^x}$

i) Show that $f(x)$ has no stationary points **[2]**

ii) Find the coordinates of the point of inflexion of $f(x)$ **[3]**

iii) Describe the behaviour of $f(x)$ as $x \to \pm\infty$ **[2]**

iv) Sketch the curve of $f(x)$ **[2]**

v) Explain why $f(x)$ has an inverse function **[1]**

vi) Find $f^{-1}(x)$, the inverse function of $f(x)$, stating its domain **[2]**

Solutions to Pretrial 2008

Question 1

a. as $x \to 0$ $\quad \sin 3x \to 3x$

$$\therefore \lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3x}{x} = 3$$

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b. Note that $x \neq -1$

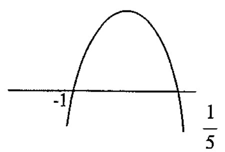
$$\frac{x}{x+1} \le \frac{1}{6} \qquad /6(x+1)^2 \qquad \Rightarrow \qquad 6x(x+1) \le (x+1)^2$$

$$\Rightarrow \quad 0 \le (x+1)^2 - 6x(x+1)$$

$$\Rightarrow \quad 0 \le (x+1)[(x+1) - 6x]$$

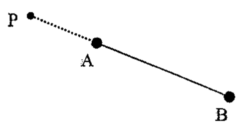
$$\Rightarrow \quad 0 \le (x+1)(1-5x)$$

$$-1 < x \le \frac{1}{5}$$



1:-3

(1,4) (5:2)



$$x_p = \frac{(1\times -3) + (5\times 1)}{1-3} = -2$$

$$y_p = \frac{(4\times -3) + (2\times 1)}{1-3} = 5$$

$$P:(-2,5)$$

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c. $$\int_0^2 \frac{dx}{\sqrt{16-x^2}} = \int_0^2 \frac{dx}{\sqrt{4^2-x^2}} = \left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^2 = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

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d. Find the acute angle between the lines $2x+y=3$ and $x-y=3$

$m_1 = -2 \qquad m_1 = 1$

$$\tan\theta = \left|\frac{-2-1}{1+1\times(-2)}\right| \quad \Rightarrow \quad \tan\theta = 3 \quad \Rightarrow \quad \theta = 77°34'$$

Question 2

a. The shaded area is bounded by the curve $y = \sin x$ and $y = \frac{1}{2}\sin 2x$ and the line $x=k$

i. Solution to $\frac{1}{2}\sin 2x = 0 \quad \Rightarrow \quad \sin 2x = 0 \Rightarrow \quad 2x = \pi$

$$\Rightarrow \quad x = \frac{\pi}{2} \qquad k = \frac{\pi}{2}$$

ii $$A = \int_0^{\frac{\pi}{2}} (\sin x - \frac{1}{2}\sin 2x)dx$$

$$A = \left[-\cos x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}u^2$$

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b. The given point has coordinates: $\left(\frac{\pi}{4}, 1\right)$

$y = \tan x \quad \Rightarrow \quad y' = \sec^2 x$

At $x = \frac{\pi}{4}$ $\; y' = 2 \; \to m_\perp = -\frac{1}{2}$

Equation of normal $\quad y - 1 = -\frac{1}{2}(x - \frac{\pi}{4})$

Intercepts the x-axis when y=0: $\quad -1 = -\frac{1}{2}(x - \frac{\pi}{4})$

$$2 + \frac{\pi}{4} = x \qquad \left(2+\frac{\pi}{4}, 0\right)$$

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c.

i. $$\int_0^1 \frac{dx}{4+x^2} = \int_0^1 \frac{dx}{2^2+x^2} = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^1 = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) = 0.23 \quad \text{(2dp)}$$

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ii. Using the identity: $\sin^2 x = \frac{1}{2}[1 - \cos 2x]$

$$\int_0^{\frac{\pi}{4}} \sin^2 x dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 - \cos 2x)dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) = \frac{\pi}{8} - \frac{1}{4}$$

Question 3

a. $u=\ln x \qquad \frac{du}{dx}=\frac{1}{x}\Rightarrow \qquad du=\frac{dx}{x}$

So, rewrite $\int \ln x\frac{dx}{x}$ and hence: $\int u\,du\ =\frac{u^2}{2}+C$

Therefore: $\int\frac{\ln x}{x}dx=\frac{1}{2}(\ln x)^2+C$

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b.

i. $\cos x+\sqrt{3}\sin x\equiv R\cos(x-\alpha)$

$\cos x+\sqrt{3}\sin x\equiv R\left[\cos x\cos\alpha+\sin\alpha\sin x\right]$

Comparing coefficients: $1=R\cos\alpha \quad [1]$

$\sqrt{3}=R\sin\alpha \quad [2]$

Dividing [2] by [1] yields: $\tan\alpha=\sqrt{3} \qquad \alpha=\frac{\pi}{3}$

Adding the squares of [2] and [1] yields: R=2

$\boxed{\therefore \cos x+\sqrt{3}\sin x\equiv 2\cos(x-\frac{\pi}{3})}$

ii. Rewrite: $\cos x+\sqrt{3}\sin x=\sqrt{2}$ as: $2\cos(x-\frac{\pi}{3})=\sqrt{2}$

$\cos(x-\frac{\pi}{3})=\frac{\sqrt{2}}{2} \qquad x-\frac{\pi}{3}=\frac{\pi}{4}+2\pi k \quad\Rightarrow\quad x=\frac{7\pi}{12}+2\pi k$

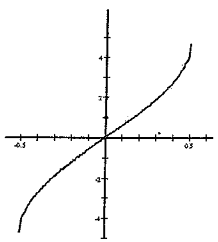
or: $\qquad x-\frac{\pi}{3}=-\frac{\pi}{4}+2\pi k \quad\Rightarrow\quad x=\frac{\pi}{12}+2\pi k$

iii. In the interval $-\pi\le x\le\pi$ the solutions are: $x=\frac{7\pi}{12},\frac{\pi}{12}$

Question 4

a.

i.



ii. $y=3\sin^{-1}2x$

At $x=\frac{1}{4}$ $\ y=3\sin^{-1}\frac{1}{2}$

Point is: $\left(\frac{1}{4},\frac{\pi}{2}\right)$

$y'=\frac{6}{\sqrt{1-4x^2}}$

At $x=\frac{1}{4} \qquad y'=\frac{6}{\sqrt{1-\frac{1}{4}}}=\frac{12}{\sqrt{3}}=4\sqrt{3}$

$y-\frac{\pi}{2}=4\sqrt{3}\left(x-\frac{1}{4}\right) \quad\Rightarrow\quad y=4\sqrt{3}x+\frac{\pi}{2}-\sqrt{3}$

iii. Check at $x=-\frac{1}{2}$ for both tangent and curve: The y-value for the curve is

$y_{curve}=-\frac{3\pi}{2}\approx -4.71,\ y_{line}=-3\sqrt{3}+\frac{\pi}{2}\approx -3.63$

So, at $x=-\frac{1}{2}$ the line is *above* the curve, but at $x=\frac{1}{4}$ the curve is concave up and the line is under the curve. Therefore the line must cross the curve at an additional point.

b. <u>Step 1</u>: Check for n=1

LHS: $1^2=1$

RHS: $\frac{1}{3}\times 1\times(2-1)\times(2+1)=1$

LHS=RHS

<u>Step 2</u>: Assume true for n=k, ie:

$1^2+3^2+...+(2k-1)^2=\frac{1}{3}k(2k-1)(2k+1)$

<u>Step 3</u>: Show true for $n=k$+1

$\underbrace{1^2+3^2+...+(2k-1)^2}_{assumption}+(2k+1)^2=\frac{1}{3}(k+1)(2k+1)(2k+3)$

$\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2=\frac{1}{3}(k+1)(2k+1)(2k+3)$

LHS: $\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$

$(2k+1)\left[\frac{1}{3}k(2k-1)+(2k+1)\right]$

$\frac{1}{3}(2k+1)\left[k(2k-1)+3(2k+1)\right]$

$\frac{1}{3}(2k+1)\left[2k^2+5k+3\right]$

$\frac{1}{3}(2k+1)\left[(2k+3)(k+1)\right]$ LHS = RHS

Step 4

Shown by mathematical induction that the equality is true.

i. The required sum can be rewritten as:

$[1^2+3^2+...+71^2]-\left[1^2+3^2+...+15^2\right]$

$1^2+2^2+...+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$

$\frac{1}{3}\times 36(72-1)(72+1)-\frac{1}{3}\times 8(16-1)(16+1)$

$\frac{1}{3}\times 36\times 71\times 73-\frac{1}{3}\times 8\times 15\times 17=61516$

Question 5

a.

i. $AC=\frac{100}{\tan 20}=274.75m$

ii. $AB^2=AC^2+BC^2-2\times AC\times BC\times\cos 60^\circ$

$AB=520m$

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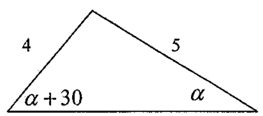
b. i. Diagram (not to scale):



ii. $\frac{5}{\sin(\alpha+30)}=\frac{4}{\sin\alpha}$

$5\sin\alpha=4\sin(\alpha+30)$

$5\sin\alpha=4\sin\alpha\cos 30+4\cos\alpha\sin 30$

$5\sin\alpha=4\sin\alpha\times\frac{\sqrt{3}}{2}+4\cos\alpha\times\frac{1}{2}$

$5\sin\alpha=2\sqrt{3}\sin\alpha+2\cos\alpha$

$5\sin\alpha-2\sqrt{3}\sin\alpha=2\cos\alpha$

$(5-2\sqrt{3})\sin\alpha=2\cos\alpha$

$(5-2\sqrt{3})\sin\alpha=2\cos\alpha$

$\tan\alpha=\frac{2}{5-2\sqrt{3}}$

$\tan\alpha=\frac{2\times\left(5+2\sqrt{3})\right)}{\left(5-2\sqrt{3}\right)\times\left(5+2\sqrt{3})\right)}=\frac{10+4\sqrt{3})}{13}$

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c. The volume V of a sphere of radius r mm is increasing at a constant rate of 200mm^3 per second.

Given: $\frac{dV}{dt}=200$ $\quad V=\frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr}=4\pi r^2$

$S=4\pi r^2 \Rightarrow \frac{dS}{dr}=8\pi r$

i. $\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$

$\frac{dr}{dt}=\frac{1}{4\pi r^2}\times 200=\frac{50}{\pi r^2}$ mm/second

ii. $\frac{dS}{dt}=\frac{dS}{dr}\times\frac{dr}{dt}=8\pi r\times\frac{50}{\pi r^2}=\frac{400}{r}$

At $r=50$. $\frac{dS}{dt}=\frac{400}{50}=8mm^2/\text{second}$

Question 6

a. Rewrite: $\sin^2\theta+2\cos 2\theta=\frac{1}{2}$: $\quad \sin^2\theta+2(1-2\sin^2\theta)=\frac{1}{2}$

$2-3\sin^2\theta=\frac{1}{2}$

$4-6\sin^2\theta=1$

$\sin^2\theta=\frac{1}{2}$

$\sin\theta=\pm\frac{\sqrt{2}}{2}$

$\sin\theta=\frac{\sqrt{2}}{2} \Rightarrow \theta=45^\circ+360k$

$\theta=135^\circ+360k$

$\sin\theta=-\frac{\sqrt{2}}{2} \Rightarrow \theta=-45^\circ+360k$

$\theta=225^\circ+360k$

b.

i.

The tangent at P is $y = px - ap^2$

The tangent at Q is $y = qx - aq^2$

Solve simultaneously: $px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p+q)(p-q)$

$x = a(p+q)$ substituting in one of the functions, yields:

$y = pa(p+q) - ap^2$ $\boxed{T:[a(p+q), apq]}$

$y = apq$

ii. $S:(0,a)$ $P:(2ap, ap^2)$

$SP = \sqrt{4a^2p^2 + (ap^2 - a)^2}$

$SP = \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$

$SP = \sqrt{a^2p^4 + 2a^2p^2 + a^2}$

$SP = \sqrt{a^2(p^4 + 2p^2 + 1)}$

$SP = \sqrt{a^2(p^2+1)^2}$

$SP = a(p^2+1)$

iii. We can conclude that $SQ = a(q^2+1)$ using the same method

Therefore $SP + SQ = a(p^2+1) + a(q^2+1)$

But we also know that $SP + SQ = 4a$, so: $4a = a(p^2+1) + a(q^2+1)$ Hence:

$4 = (p^2+1) + (q^2+1)$

$2 = p^2 + q^2$

$2 = (p+q)^2 - 2pq$ [1]

At T: $x_t = a(p+q)$ $y_t = apq$

Rewriting [1]:

$2 = \left(\frac{x_t}{a}\right)^2 - 2\left(\frac{y_t}{a}\right)$

$2 = \frac{x_t^2}{a^2} - \frac{2y_t}{a}$

$2a^2 = x_t^2 - 2ay_t$

$2ay_t = x_t^2 - 2a^2$

$\boxed{y_t = \frac{x_t^2}{2a} - a}$ This is the equation of a parabola

c. Evaluate: Let $u = 1 + 2x$ $\Rightarrow \frac{du}{dx} = 2$

$\Rightarrow \frac{du}{2} = dx$

$$\int_0^{\frac{1}{2}} \frac{4x}{(1+2x)^4}dx = \int \frac{2u-2}{u^4} \times \frac{du}{2} = \int \left(\frac{u-1}{u^4}\right)du = \int (u^{-3} - u^{-4})\,du = \left[\frac{u^{-2}}{-2} - \frac{u^{-3}}{-3}\right] = \left[\frac{1}{-2u^2} + \frac{1}{3u^3}\right]$$

$$\int_0^{\frac{1}{2}} \frac{4x}{(1+2x)^4}dx = \left[\frac{1}{-2(1+2x)^2} + \frac{1}{3(1+2x)^3}\right]_0^{\frac{1}{2}} = \left[\frac{1}{-8} + \frac{1}{24}\right] - \left[\frac{1}{-2} + \frac{1}{3}\right] = \frac{1}{12}$$

Question 7

$f(x) = \frac{e^x}{3 + e^x}$

i. $f'(x) = \frac{3e^x}{(3+e^x)^2}$. Since $e^x > 0$ for all real x, both the numerator and the denominator are positive for all real x. Hence $f'(x) = 0$ has no solution, and therefore there are no stationary points.

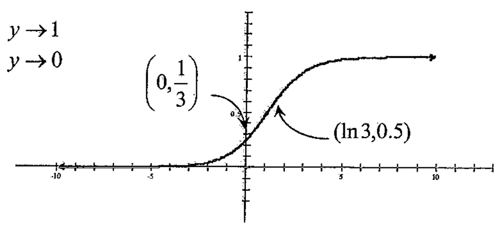
ii. $f''(x) = \frac{3e^x(3+e^x)(3-e^x)}{(3+e^x)^4}$ For point of inflexion: $f''(x) = 0$

$\Rightarrow$ $3 - e^x = 0 \Rightarrow$ $x = \ln 3$ POI: $(\ln 3, 0.5)$

iii. As $x \to \infty$ $y \to 1$

As $x \to -\infty$ $y \to 0$

iv.



v. The graph is continuous with no turning points. Passes the horizontal line test. Hence it has an inverse.

vi. $f^{-1}(x) = \ln\left(\frac{3x}{1-x}\right)$ Domain: $0 < x < 1$