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MORIAH COLLEGE

Year 11

MATHEMATICS ENRICHMENT

Class Test: Polynomials, Partial Fractions and Inequalities

Date: Tuesday 7th March, 2006

Time Allowed: 50 min + 5 minutes reading time

Examiners: J. Taylor

General Instructions

- Calculators may be used
- Show all necessary working.
- Start **Each Question** on a new page.
- Questions are not of equal value

Question 1 (8 marks)

a) Use long division to perform each of the following divisions. Express each result in the form $P(x) = D(x)Q(x) + R(x)$

i) $(x^3 + 4x^2 - 5x + 3) \div (x+1)$ 3

ii) $(2x^3 + 5) \div (x^2 + x + 1)$ 2

b) If $x^4 - 4x^3 - 14x^2 + ax + b$ is exactly divisible by $x^2 - 6x + 3$, find a and b 3

Question 2 (11 marks)

a) Express as a sum of partial fractions $\frac{6}{2x^2 - 5x + 2}$. 3

b) Express as a sum of partial fractions $\frac{x^2 + x + 2}{x(x+1)}$. 4

c) Express as a sum of a partial fractions $\frac{3x^2 - 3x + 2}{(2x-1)(x^2+1)}$. 4

Question 3 (12 marks)

a) If n and r are positive integers such that $n > r > 1$, show that 2

$$\frac{n+1-r}{n-r} > \frac{n+1}{n}$$

b) i) Prove that, for any real numbers a, b, $a^2 + b^2 \geq 2ab$. 2

Hence or otherwise prove that:

ii) $bc(b + c) + ca(c + a) + ab(a + b) \geq 6abc$ 3

iii) $a^3 + b^3 \geq ab(a + b)$ 2

iv) $2(a^3 + b^3 + c^3) \geq ab(a + b) + bc(b + c) + ca(c + a)$ 1

v) $\frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} \geq \frac{3}{xyz}$ for any real x, y, z. 2

Question 4 (11 marks)

a) i) Given m, n are positive integers, prove that $\dfrac{m+2n}{m+n}+\dfrac{m}{n}=\dfrac{m+n}{n}+\dfrac{n}{m+n}$. **2**

ii) Deduce that $\dfrac{m+2n}{m+n}+\dfrac{m}{n}>2$. **2**

b) Prove that if $\dfrac{m}{n}<\sqrt{2}$ then

$\dfrac{m+2n}{m+n} > 2 - \dfrac{m}{2}$

i) $\dfrac{m+n}{n}<\sqrt{2}+1$ **1**

$1+\dfrac{n}{m+n} < 2 - \sqrt{}$

ii) $\dfrac{n}{m+n}>\sqrt{2}-1$ **2**

$-1+\sqrt{2} < n$

iii) $\dfrac{m+2n}{m+n}>\sqrt{2}$ **2**

$\sqrt{2} + \sqrt{2} - 1$ $\sqrt{2}$.

c) Explain how a) above and b)iii) lead to the conclusion that **2**

$$2<\frac{m+2n}{m+n}+\frac{m}{n}<2\sqrt{2}$$

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Q1 a) i)

$$\begin{array}{r l} & x^2+3x-8 \\ x+1 \,\big) & x^3+4x^2-5x+3 \\ & \underline{x^3+x^2} \\ & 3x^2-5x+3 \\ & \underline{3x^2+3x} \\ & -8x+3 \\ & \underline{-8x-8} \\ & 11 \end{array}$$

$\therefore x^3+4x^2-5x+3 = (x+1)(x^2+3x-8) +$

ii)

$$\begin{array}{r l} & 2x-2 \\ x^2+x+1 \,\big) & 2x^3+0x^2+0x+5 \\ & \underline{2x^3+2x^2+2x} \\ & -2x^2-2x+5 \\ & \underline{-2x^2-2x-2} \\ & 7 \end{array}$$

$\therefore 2x^3+5 = (x^2+x+1)(2x-2) + 7$

b)

$$\begin{array}{r l} & x^2+2x-5 \\ x^2-6x+3 \,\big) & x^4-4x^3-14x^2+ax+b \\ & \underline{x^4-6x^3+3x^2} \\ & 2x^3-17x^2+ax+b \\ & \underline{2x^3-12x^2+6x} \\ & -5x^2+(a-6)x+b \\ & \underline{-5x^2+30x-15} \\ & (a-36)x+b+15 \end{array}$$

Remainder $= (a-36)x + b+15$

Since the remainder must be 0, it follows that

$a = 36$ and $b = -15$

Q2 a)

$$\frac{6}{2x^2-5x+2} = \frac{6}{(2x-1)(x-2)} \equiv \frac{a}{2x-1} + \frac{b}{x-2}$$

$\therefore \quad 6 = a(x-2) + b(2x-1)$

Sub $x=\frac{1}{2}$: $\quad 6 = a \times -1\frac{1}{2}$

$\therefore a = -4$

Sub $x=2$: $\quad 6 = 3b$

$\therefore b = 2$

$$\therefore \frac{6}{2x^2-5x+2} = \frac{-4}{2x-1} + \frac{2}{x-2}$$

b)

$$\frac{x^2+x+2}{x(x+1)} = \frac{x^2+x+2}{x^2+x} = 1 + \frac{2}{x^2+x}$$

Consider $\dfrac{2}{x(x+1)} \equiv \dfrac{a}{x} + \dfrac{b}{x+1}$

$2 = a(x+1) + bx$

Sub $x=0$: $\quad a = 2$

Sub $x=-1$: $\quad b = -2$

$$\therefore \frac{x^2+x+2}{x(x+1)} = 1 + \frac{2}{x} - \frac{2}{x+1}$$

c) Let $\dfrac{3x^2-3x+2}{(2x-1)(x^2+1)} \equiv \dfrac{a}{2x-1} + \dfrac{bx+c}{x^2+1}$

$\therefore 3x^2-3x+2 = a(x^2+1) + (2x-1)(bx+c)$

Sub $x=\frac{1}{2}$: $\quad \frac{5}{4} = \frac{5}{4}a$

$\therefore a = 1$

Coeff x^2: $\quad 3 = a+2b$

$\therefore b = 1$

Sub $x=0$: $\quad 2 = a - 1(0+c)$

$2 = a-c$

$\therefore c = -1$

$$\therefore \frac{3x^2-3x+2}{(2x-1)(x^2+1)} = \frac{1}{2x-1} + \frac{x-1}{x^2+1}$$

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a) Consider $\frac{n+1-r}{n-r} - \frac{n+1}{n} = \frac{n^2+n-nr-n^2+nr-n+r}{n(n-r)}$ ✓

$= \frac{r}{n(n-r)}$

> 0 since $r>0,\ n>r$ ✓

$\therefore \frac{n+1-r}{n-r} > \frac{n+1}{n}$

b) i) Consider $a^2+b^2-2ab = (a-b)^2$ ✓

≥ 0 ✓

$\therefore a^2+b^2 \ge 2ab$ ✓

ii) LHS $= b^2c+bc^2+c^2a+ca^2+a^2b+ab^2$ ✓

$= (b^2+a^2)c + (a^2+c^2)b + (b^2+c^2)a$ ✓

$\ge 2abc + 2acb + 2bca$ ✓

$= 6abc$

$\therefore bc(b+c) + ca(c+a) + ab(a+b) \ge 6abc$

iii) $a^3+b^3 = (a+b)(a^2+b^2-ab)$ ✓

$\ge (a+b)(2ab-ab)$

$= (a+b)ab$ ✓

$\therefore a^3+b^3 \ge ab(a+b)$

iv) Adding similar results:

$a^3+b^3+b^3+c^3+c^3+a^3 \ge ab(a+b)+bc(b+c)+ca(a+c)$ ✓

$\therefore 2(a^3+b^3+c^3) \ge ab(a+b)+bc(b+c)+ca(a+c)$

v) Combining (iv) and (ii) gives $a^3+b^3+c^3 \ge 3abc$ ✓

Replace $a \to \frac{1}{x}$ $b \to \frac{1}{y}$ $c \to \frac{1}{z}$: ✓

$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3} \ge \frac{3}{xyz}$$

Question 4

a) i) LHS − RHS $= \frac{m+2n-n}{m+n} + \frac{m-(m+n)}{n}$ 1

$= \frac{m+n}{m+n} + -\frac{n}{n}$

$= 1-1$

$= 0$ 1

$\therefore \frac{m+2n}{m+n}+\frac{m}{n} = \frac{m+n}{n}+\frac{n}{m+n}$

ii) For a a positive integer, $a+\frac{1}{a} \ge 2$ 1

$\therefore \frac{m+n}{n}+\frac{n}{m+n} > 2$ if $m \ne n$ 1

$\therefore \frac{m+2n}{m+n}+\frac{m}{n} > 2$

b) i) $\frac{m+n}{n} = \frac{m}{n}+\frac{n}{n}$ ✓

$< \sqrt{2}+1$

ii) $\frac{n}{m+n} > \frac{1}{\sqrt{2}+1}$ ✓

$= \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}$

$= \frac{\sqrt{2}-1}{1}$ ✓

$\therefore \frac{n}{m+n} > \sqrt{2}-1$

iii) $\frac{m+2n}{m+n} = \frac{m+n+n}{m+n}$

$= \frac{m+n}{m+n}+\frac{n}{m+n}$ ✓

$> 1+\sqrt{2}-1$

$= \sqrt{2}$

$\therefore \frac{m+2n}{m+n} > \sqrt{2}$ ✓

c) Combining $\frac{m}{n} > \sqrt{2}$ (given) with iii) we have

$\frac{m+2n}{m+n}+\frac{m}{n} > \sqrt{2}+\sqrt{2}$ 1

$= 2\sqrt{2}$

Using a) ii) as well:

$2 < \frac{m+2n}{m+n}+\frac{m}{n} < 2\sqrt{2}$ 1