2005

DECEMBER ASSESSMENT

8th December

MATHEMATICS EXTENSION 1

Student Name:

Teacher:

Time allowed: 1 hour + 5 minutes

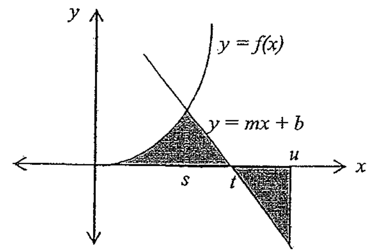
- Calculators may be used.
- Show all necessary working
- Start **each question** on a new page.
- Questions are not of equal value.

Question 1	/12
Question 2	/11
Question 3	/12
Question 4	/15
TOTAL	/50

Question 1. (12 marks)

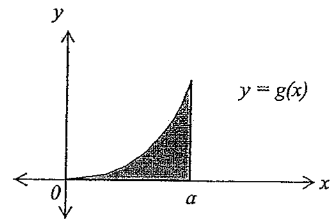
(a) If $y = x^3 e^{-2x+2}$, find $\frac{dy}{dx}$ in simplest form. 3

(b)



Write an expression to find the area of the shaded region. 2

(c)



The shaded area shown left is given by $B\,u^2$. 1

If $y = g(x)$ is an odd function, find $\int_{-a}^{a} g(x)\,dx$.

(d) $\frac{dy}{dx} = x^2 - 5x + a$. Find the equation of the curve given there is a stationary point at (1, 0). 3

(e) Find the following indefinite integral: 3

$\int \frac{5x}{\sqrt{5x^2 - 3}}\,dx$ by using the substitution $u = 5x^2 - 3$, or otherwise.

Question 2. (11 marks)

Start this question on a new page.

(a) Sketch the curves $y = 4 - x^2$ and $y = -2x - 4$. 5

Hence calculate the area enclosed by them.

(b) The portion of the curve $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$ is rotated around the x-axis. 6

(i) By using Simpson's rule with 5 function values, find its approximate volume.

(ii) Find its exact volume.

Question 3. (12 marks)

Start this question on a new page.

(a) (i) If $y = x\sqrt{4 - x}$, show that $\frac{dy}{dx} = \frac{8 - 3x}{2\sqrt{4 - x}}$ 5

(ii) Hence, evaluate $\int_0^2 \frac{8 - 3x}{\sqrt{4 - x}}\, dx$

(b) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points that lie on the parabola $x^2 = 4ay$. 7

(i) Show that the equation of the chord PQ is given by $y = \left(\frac{p + q}{2}\right)x - apq$

(ii) Find the condition for a chord PQ of the parabola $x^2 = 4ay$ to be a focal chord.

(iii) Find M the mid-point of PQ and hence show that the locus of M as P and Q moves on the parabola is given by $x^2 = 2a(y - a)$.

(iv) Prove that the tangents at P and Q intersect at right angles.

Question 4. (15 marks)

Start this question on a new page.

(a) Some bacteria are introduced into a supply of fresh milk. After t hours there are y grams of bacteria present, where 7

$$y = \frac{2e^{3t}}{1+2e^{3t}} \quad (t \geq 0)$$

(i) Show that $0 < y < 1$ for all values of t.

(ii) Prove that $\frac{dy}{dt} = 3y(1-y)$.

(b) For the curve $y = \frac{x}{e^x}$ 8

(i) Find all stationary points and determine their nature.

(ii) Find any points of inflexion.

(iii) Find $\lim_{x \to \infty} \frac{x}{e^x}$ and $\lim_{x \to -\infty} \frac{x}{e^x}$

(iv) Hence, sketch the curve $y = \frac{x}{e^x}$, showing all essential features.

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1 a) $y = x^2 e^{-2x+2}$

$\frac{dy}{dx} = -2x^2 e^{-2x+2} + 2x e^{-2x+2}$

$= 2x e^{-2x+2} [-x + 1]$

b) $A = \int_0^s f(x)\,dx + \int_s^t (mx+b)\,dx + \left| \int_t^u (mx+b)\,dx \right|$

c) 0

d) $\frac{dy}{dx} = 0$ at $x = 1$

$0 = 1 - 5 + a$

$a = 4$

$y = \frac{x^3}{3} - \frac{5x^2}{2} + 4x + C$

$0 = \frac{1}{3} - \frac{5}{2} + 4 + C$

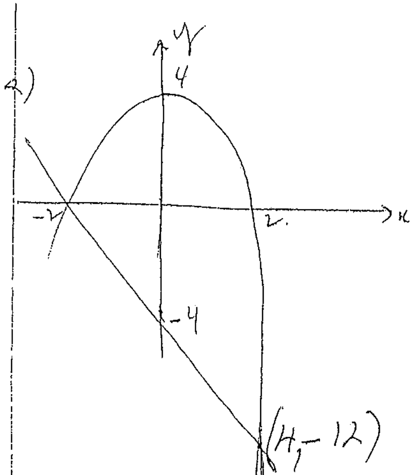
$\therefore C = -1\frac{5}{6}$

$\therefore y = \frac{x^3}{3} - \frac{5}{2}x^2 + 4x - 1\frac{5}{6}$

e) $\frac{1}{2}\int 2\times 5x(5x^2-3)^{-1/2}\,dx = \frac{2}{2}(5x^2-3)^{1/2} + C$

$= \sqrt{5x^2-3} + C$

2 a)



$4 - x^2 = -2x - 4$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4$ or $x = -2$

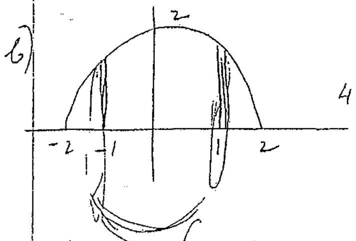
$A = \int_{-2}^{4} 4 - x^2 - (-2x-4)\,dx$

$= \int_{-2}^{4} 4 - x^2 + 2x + 4\,dx$

$= \left[8x + x^2 - \frac{x^3}{3} \right]_{-2}^{4}$

$= 26\frac{2}{3} - (\frac{28}{3}) = 36 u^2$

b)



x	-1	-½	0	½	1
$4-x^2=y^2$	3	$3\frac{3}{4}$	4	$3\frac{3}{4}$	3
$\cancel{x}$	1	4	2	4	1

$V = \pi \int y^2\,dx$

$= \pi \int_{-1}^{1} 4 - x^2\,dx$

$\doteq \frac{\frac{1}{2}\pi}{3}\left[3 + 3 + 4(3\frac{3}{4} + 3\frac{3}{4}) + 2\times 4\right]$

$\doteq 44\pi \times \frac{1}{6}$

$\doteq \frac{22}{3}\pi$

Exact Volume

$V = \pi \left[4x - \frac{x^3}{3} \right]_{-1}^{1}$

$= \pi \left[4 - \frac{1}{3} - (-4 + \frac{1}{3}) \right]$

$= \frac{22}{3}\pi$

Simpson's rule is exact as we have a quadratic function.

Q3 a) i) $y = x\sqrt{4-x}$

$\frac{dy}{dx} = \frac{-x}{2\sqrt{4-x}} + \sqrt{4-x}$

$= \frac{2(4-x) - x}{2\sqrt{4-x}}$

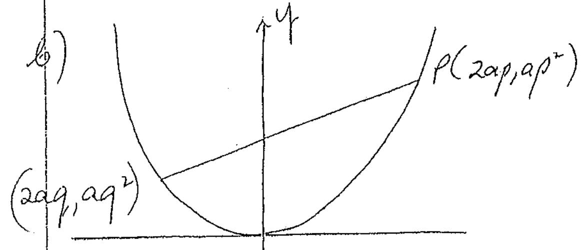
$= \frac{8 - 3x}{2\sqrt{4-x}}$

ii) $\int_0^2 \frac{8-3x}{\sqrt{4-x}}\,dx = 2\int_0^2 \frac{8-3x}{2\sqrt{4-x}}\,dx$

$= 2\left[x\sqrt{4-x} \right]_0^2$

$= 2[2\sqrt{2}] = 4\sqrt{2}$

b)



i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq} = \dfrac{a(p+q)}{2a}$

Eq of chord PQ $\quad y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$y = \left(\frac{p+q}{2}\right)x - ap(p+q) + ap^2$

$= \left(\frac{p+q}{2}\right)x - apq$.

ii) If focal chord (0, a) lies on line.

∴ $a = -apq$

∴ $pq = -1$ is cond.

iii) M $\quad x = a(p+q)$

$y = \frac{a}{2}(p^2+q^2)$

LHS: $\quad x^2 = a^2(p^2+q^2+2pq)$

$= a^2\left(\frac{2y}{a} - 2\right)$

$= 2ay - 2a^2$

$= 2a(y-a) = $ RHS.

iv. gradient of tangent at P is p.

since $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$= \frac{2ap}{2a}$ at P(2ap, ap²)

$= p$

gradient of tangent at Q is q.

Since in ii) we proved $pq = -1$ gradient are at right angles

Q4. a) $y = \dfrac{2e^{3t}}{1+2e^{3t}}$

i) as $2e^{3t} > 0$ ∘ $1 + 2e^{3t} > 0$

∴ $y > 0$

since $1 + 2e^{3t} > 2e^{3t}$

$1 > \frac{2e^{3t}}{1+2e^{3t}} \Rightarrow 1 > y$

∴ $y < 1$ for all t

ii) $\frac{dy}{dt} = 3y(1-y)$

LHS $= \frac{dy}{dt} = \dfrac{(1+2e^{3t})6e^{3t} - 2e^{3t} \cdot 6e^{3t}}{(1+2e^{3t})^2}$

$= \dfrac{6e^{3t}}{(1+2e^{3t})^2}$

RHS $= 3y(1-y)$

$= \dfrac{6e^{3t}}{1+2e^{3t}}\left(1 - \dfrac{2e^{3t}}{1+2e^{3t}}\right)$

$= \dfrac{6e^{3t}}{1+2e^{3t}}\left[\dfrac{1+2e^{3t} - 2e^{3t}}{1+2e^{3t}}\right]$

$= \dfrac{6e^{3t}}{(1+2e^{3t})^2} = $ LHS.

b) $y = \dfrac{x}{e^x}$

i) $\dfrac{dy}{dx} = \dfrac{e^x - xe^x}{(e^x)^2}$

$= \dfrac{1-x}{e^x} = 0$ for SP.

$x = 1$

$\dfrac{d^2y}{dx^2} = \dfrac{-e^x - (1-x)e^x}{(e^x)^2}$

$= \dfrac{-2+x}{e^x}$

< 0 when $x = 1$

∴ Max T.P. $\{y = \frac{1}{e} = 0.368$

ii) when $x = 2$ possible PI

x	1½	2	2½
y''	$\frac{-\frac{1}{2}}{e^{1\frac{1}{2}}}$	0	$\frac{\frac{1}{2}}{e^{2\frac{1}{2}}}$
	<0		>0

∴ concavity change

∴ $\left(2, \frac{2}{e^2}\right)$ is PI. (2, 0.27)

iii) $\lim_{x\to\infty} \frac{x}{e^x} = 0 \qquad \lim_{x\to-\infty} \frac{x}{e^x} \to -\infty$