MORIAH COLLEGE
MATHEMATICS DEPARTMENT
Year 11 Enrichment Mathematics

Question 1. (14 marks)

a) Factorise $x^2 + 6x + 13$.

b) i) Find all values of z such that $z^2 = 5 - 12i$
   ii) Solve $x^2 - 3x + (1+3i) = 0$

c) i) Use de Moivre's Theorem to find the three cube roots of unity in the form $\cos\theta + i\sin\theta$
   ii) Plot the three roots on an Argand diagram, demonstrating that they are the vertices of an equilateral triangle on the unit circle.

Question 2. (10 marks)

a) Express the following in the form $a + ib$:

i) $\dfrac{1}{1+2i} + \dfrac{1}{3-i}$ ii) $\dfrac{\left(\sqrt{3}-i\right)^7}{\left(1+i\sqrt{3}\right)^5}$

b) i) On the Argand diagram, draw a clear sketch to show the important features of the curve defined by

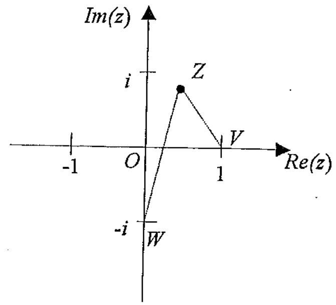
$$|z - A| = 3$$

where $A = 3 + 4i$.

ii) Also, for z on this curve, find the maximum value of $|z|$ and $\arg(z)$.

Question 3. (8 marks)

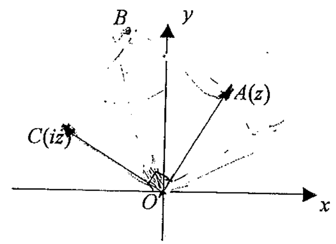
a) $Z(z)$ moves so that $Arg\left(\dfrac{z-1}{z+i}\right) = \dfrac{\pi}{4}$.



i) Explain why $\angle WZV = \dfrac{\pi}{4}$

ii) Describe precisely the locus of Z, giving reasons.

b) Let $OABC$ be a square on the Argand diagram where O is the origin.



The points A and C represent the complex numbers z and iz respectively.

i) Find the complex number representing B.

ii) The square is now rotated about O through 45° in a *clockwise* direction to $OA'B'C'$. Find the complex number representing C'.

Question 4. (4 marks)

a) Prove that $(\sin\theta + i\cos\theta)(1 + \sin\theta - i\cos\theta) = 1 + \sin\theta + i\cos\theta$.

b) Deduce that $\left(1+\sin\dfrac{\pi}{3}+i\cos\dfrac{\pi}{3}\right)^3 + i\left(1+\sin\dfrac{\pi}{3}-i\cos\dfrac{\pi}{3}\right)^3 = 0$

30/36

michael Hazan

Q1.

a) $x^2+6x+13$

$$x = \frac{-6 \pm \sqrt{36-52}}{2}$$

$$= \frac{-6 \pm \sqrt{-16}}{2}$$

$$= \frac{-6 \pm 4i}{2}$$

$$= -3 \pm 2i$$

factors?

b) i) let $z = x+iy$

$z^2 = x^2+2ixy-y^2$

$\therefore$ ~~x^2~~ $x^2+2ixy-y^2 = 5-12i$

① $x^2-y^2 = 5$

$2ixy = -12i$

$xy = -6$

$y = \frac{-6}{x}$

$\therefore$ $x^2 - \frac{36}{x^2} = 5$

$x^4 - 36 = 5x^2$

$x^4-5x^2-36=0$

let $x^2=m$

$m^2-5m-36=0$

$$m = \frac{5 \pm \sqrt{25+144}}{2}$$

$$= \frac{5 \pm \sqrt{169}}{2}$$

$$= \frac{5 \pm 13}{2}$$

$m = 9$ or $m = -4$

$\therefore$ $x = \pm 3$. $y = \mp 2$.

$\therefore$ $z = \pm 3 \mp 2i$

✓ 4

michael Hazan

ii) $x^2-3x+(1+3i)=0$

$$x = \frac{3 \pm \sqrt{9-4(1+3i)}}{2}$$

$$= \frac{3 \pm \sqrt{9-4-12i}}{2}$$

$$= \frac{3 \pm \sqrt{5-12i}}{2}$$

$$= \frac{3 \pm 3 \mp 2i}{2}$$

$x = \frac{6-2i}{2}$ or $x = \frac{2i}{2}$

$= 3-i$ $\quad = i$

✓ 3

c). let $z^3 = 1$

$z^3 = cis(0+2k\pi)$

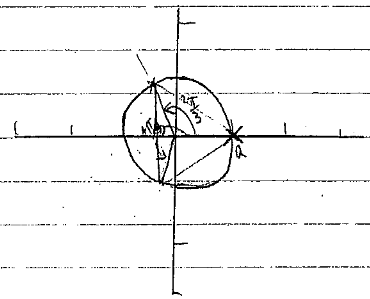
$z = 1^{\frac{1}{3}} cis\left(\frac{2k\pi}{3}\right)$

let $k = 0, 1, 2$

$z = \underset{a}{cis 0}, \underset{b}{cis \frac{2\pi}{3}}, \underset{c}{cis \frac{4\pi}{3}}$

✓ 3

ii



Since all internal ∠'s are equal the Δ is equilateral $= 120$ $\frac{2\pi}{3}$.

✓ 2

8

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Q2

i) $\frac{1}{1+2i} + \frac{1}{3-i} = \frac{3-i+1+2i}{(1+2i)(3-i)}$

$= \frac{4+i}{3-i+6i+2}$ careless!

$= \frac{4+i}{3+5i}$

$= \frac{(4+i)(3-5i)}{9+25}$

$= 12-20i+3i+5$

$= \frac{17-17i}{34}$

$= \frac{17}{34} - i\frac{17}{34}$ $\quad = \frac{1}{2}(1-i)$

$= \frac{17}{34}(1-i)$

ii) $\frac{z(\sqrt{3}-i)^7}{u(1+i\sqrt{3})^5}$

$\Rightarrow |z| = \sqrt{3+1}$

$= 2$

$\text{Arg}(z) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$\text{Arg}(z) = \frac{-\pi}{6}$

4

$z = 2\,\text{cis}\left(\frac{-\pi}{6}\right)$

$|u| = \sqrt{1+3}$

$= 2$

$\text{Arg}(u) = \tan^{-1}\frac{\sqrt{3}}{1}$

$= \frac{\pi}{3}$

$\therefore u = 2\,\text{cis}\frac{\pi}{3}$

$\frac{(z)^7}{(u)^5} = \frac{(2\,\text{cis}\,\frac{-\pi}{6})^7}{(2\,\text{cis}\,\frac{\pi}{3})^5}$

$= \frac{2^7\,\text{cis}\,\frac{-7\pi}{6}}{2^5 \ldots}$

$= \frac{128}{32}\,\text{cis}\,\frac{-7\pi}{6} - \frac{5\pi}{3}$

$= 4\,\text{cis}\,\frac{-17\pi}{6}$

$= 4\cos\frac{-17\pi}{6} + i\sin\frac{-17\pi}{6}$

$= 4\left(\frac{-\sqrt{3}}{2} - \frac{i}{2}\right)$

$= -2\sqrt{3} - 2i$

5

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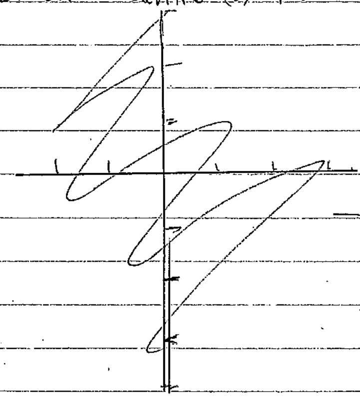
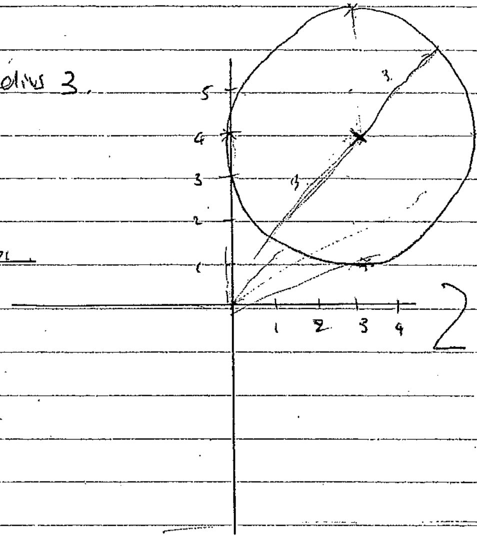
b i).

$|z - (3+4i)| = 3$

locus of a circle. centre (3,4) radius 3.





2

Max $|z| = 3+4i + |3|$.

$= \sqrt{9+16} + |3|$

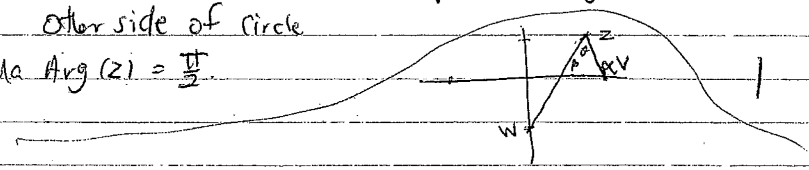
$= \sqrt{25} + 3$

$= 28$

~~max |z| = 3 constant~~

Max $|z|$ is the point that passes through centre to other side of circle 2

Ma $\text{Arg}(z) = \frac{\pi}{2}$.



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Q3.

i) because: $\underset{\alpha}{\text{Arg}(z-1)} - \underset{\beta}{\arg(z+i)} = \frac{\pi}{4}$

$\beta + \theta = \alpha$ (Ext $\angle$ Δ thm)

$\alpha - \beta = \theta$

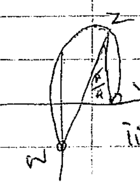
$\theta = \frac{\pi}{4}$

$\therefore \alpha - \beta = \frac{\pi}{4}$

$\therefore \text{Arg}(z-1) - \arg(z+i) = \frac{\pi}{4}$

$\therefore \angle WZV = \frac{\pi}{4}$

2



ii) The locus of z is that of the major arc of a circle because $0 < \arg z < \frac{\pi}{2}$, it does not pass through

$7\frac{1}{2}$

michael Hazan

b.

i) $B = z + iz$ ✓ 1

ii) A^* in new position is ie A' same modulus just translated $\frac{\pi}{4}^c$ clockwise.

$z = r\text{cis}\theta \times \text{cis}-\frac{\pi}{4}$

$= r\text{cis}(\theta - \frac{\pi}{4})$

$\therefore$ ~~i in new pt~~ $= i(r\text{cis}(\theta - \frac{\pi}{4}))$

$= r i\cos(\theta - \frac{\pi}{4}) + r i\sin(\theta - \frac{\pi}{4})$

$= ir\cos(\theta - \frac{\pi}{4}) - r\sin(\theta - \frac{\pi}{4})$

Use the fact that Cis $\frac{\pi}{2}$ $1\frac{1}{2}$

Q4.

a) RTP $(\sin\theta + i\cos\theta)(1 + \sin\theta - i\cos\theta) = 1 + \sin\theta + i\cos\theta$

~~RTP~~ LHS $= \sin\theta + \sin^2\theta - \sin\theta\, i\cos\theta + i\cos\theta + \sin i\cos\theta + \cos^2\theta$

$= \sin\theta + \sin^2\theta + i\cos\theta + \cos^2\theta$

$= \sin^2\theta + \cos^2\theta + \sin\theta + i\cos\theta$

$= 1 + \sin\theta + i\cos\theta$

$=$ RHS

Q.E.D ✓ 2

b).

~~$(\sin\theta + i\cos\theta)(1 + \sin\theta - i\cos\theta) = \sin^3\theta + i\cos^3\theta$~~

$\left(1 + \left(\sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)\right)^3 + i\left(1 + \sin\frac{\pi}{3} - i\cos\frac{\pi}{3}\right)^3 = 0$

~~Since opposite signs all terms would cancel out, but since using de Moivres $\frac{3\pi}{3}$ becomes π, $\sin\pi = 0$ $\cos\pi = -1$ since other has opp sign they will cancel.~~

PTO

$4\frac{1}{2}$

michael Hazan

Difference two cubes.

$\text{LHS} = \left(1 + \sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)^3 + i\left(1 + \sin\frac{\pi}{3} - i\cos\frac{\pi}{3}\right)^3$

$= \left(1 + \sin\frac{\pi}{3} + i\cos\frac{\pi}{3} + 1 + \sin\frac{\pi}{3} - i\cos\frac{\pi}{3}\right)\left[\left(1 + \sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)^2 + \left(1 + \sin\frac{\pi}{3} + i\cos\frac{\pi}{3} \times 1 + \sin\frac{\pi}{3} - i\cos\frac{\pi}{3}\right) + \left(1 + \sin\frac{\pi}{3} - i\cos\frac{\pi}{3}\right)^2\right]$

$= \left(1 + 2\sin\frac{\pi}{3}\right)\left[\left(\sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)^2 + \left(\sin\frac{\pi}{3} + i\cos\frac{\pi}{3} \times \sin\frac{\pi}{3} - i\cos\frac{\pi}{3}\right) + \left(\sin\frac{\pi}{3} - i\cos\frac{\pi}{3}\right)^2\right]$

$= (1 + \sqrt{3})\left[\left(\sin\frac{2\pi}{3} + i\cos\frac{2\pi}{3}\right) + \left(\sin^2\frac{\pi}{3} + \cos^2\frac{\pi}{3}\right) + \sin\frac{2\pi}{3} - i\cos\frac{2\pi}{3}\right]$

$= (1 + \sqrt{3})\left[\sin\frac{2\pi}{3} + i\cos\frac{2\pi}{3} + 1 + \sin\frac{2\pi}{3} - i\cos\frac{2\pi}{3}\right]$

$= 1 + \sqrt{3}\left(2\sin\frac{2\pi}{3} + 1.\right)$

Good effort