Student Number: ....

2005

2 UNIT ASSESSMENT TASK

Examination Paper

MATHEMATICS

Directions to Candidates

Reading time – 5 minutes
Working time – Two (2) hours

- Attempt ALL questions 1– 5.
- The marks for each question are clearly indicated on the paper.
- All necessary working must be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- **Each question is to be started on a new page.**
- This examination paper must NOT be removed from the examination room.

Question 1. (18 marks)

(i) Let $f(x)=x^3+3x^2+2$ Find

(a) $f'(x)$ [1]

(b) $f'(2)$ [1]

(c) $f'(-3)$ [1]

(d) the value(s) of k for which $f'(k)=24$ [2]

(ii) Differentiate:

(a) $y=5x^4-5x+7$ [1]

(b) $y=\frac{1}{x}+\frac{5}{x^3}$ [2]

(c) $y=x\sqrt{x}$ [2]

(d) $y=(2x^2+7)^4$ [2]

(e) $y=\frac{3x+5}{4x+1}$ [2]

(iii) Find the equation of the normal to the curve $y=\sqrt{x^3+x+6}$ at the point where $x=3$. Give your answer in the form: $ax+by+c=0$ [4]

Question 2. (14 marks)

Solve the equations, writing, where relevant, your answers correct to 2dp:

(i) $2x^4-3x^2-20=0$ [3]

(ii) $(x+\frac{4}{x})^2-(x+\frac{4}{x})=20$ [3]

(iii) (a) Express $4x^2-24x+38$ in the form: $m(x-n)^2+p$ [3]

(b) Hence, find the minimum value of the function: $y=4x^2-24x+38$ [1]

(iv) The line $y=x+c$ intersects the circle $x^2+y^2=8$
(a) Show that the x coordinate of the point of intersection satisfies the equation:
$2x^2+2cx+c^2-8=0$ [2]

(b) For what value(s) of c will the line be a tangent to the circle? [2]

Question 3. (10 marks)

(i) Let α and β be the roots of the quadratic equation: $x^2+4x-6=0$.
Find the values of :

(a) $\alpha+\beta$ [1]

(b) $\alpha\beta$ [1]

(c) $(\alpha-1)(\beta-1)$ [1]

(d) $\alpha^2\beta+\alpha\beta^2$ [1]

(ii) Find the equation, in the form: $x^2+bx+c=0$, of the quadratic with roots

(a) 2, -8 [1]

(b) $\dfrac{1}{2+\sqrt{3}}, \dfrac{1}{2-\sqrt{3}}$ [2]

(iii) (a) Find the discriminant of the quadratic equation $x^2+(m-2)x+4=0$ [1]

(b) Hence, or otherwise, determine the values of m for which the graph of $y=x^2+(m-2)x+4$ does not cross the x-axis [2]

Question 4. (13 marks)

(i) Given the fixed points $A(1,2)$ and $B(4,-1)$. Show that the locus of the point P is:

(a) a line if $PA=PB$. Find the equation of the line. [3]

(b) a circle if $PA=2PB$. Find the centre and radius of the circle. [3]

(ii) For the parabola $x^2=8y$ find:

(a) the coordinates of the focus [1]

(b) the equation of the directrix [1]

(iii) For of the parabola with equation: $y^2+8y-8x+24=0$ find

(a) the coordinates of the vertex [2]

(b) the coordinates of the focus [1]

(c) the equation of the directrix [1]

(d) the equation of the axis [1]

Question 5. (16 marks)

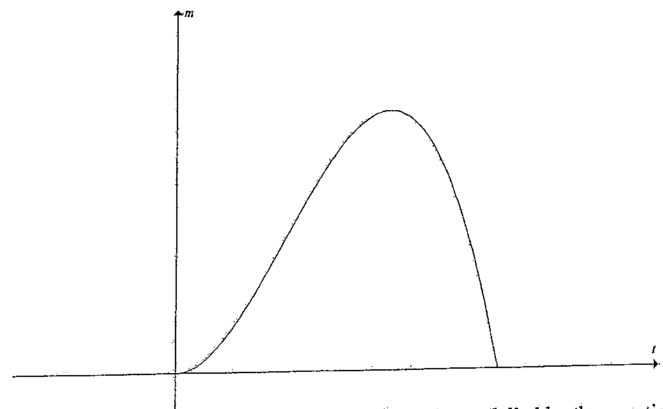
(i) Show that the graph of $y=2x^3+3x+4$ has no stationary points. [2]

(ii) For the curve $y=-x^4+4x^3+10$

(a) Find the stationary points and determine their nature. [4]

(b) Sketch the curve, indicating stationary points. [3]

(iii) Mr White had to take some medicine to calm his nerves. His doctor measured the amount of medicine (in grams) present in his bloodstream, t hours after he took the medication. The results were then drawn on a graph as follows:



The doctor discovered that the graph can be modelled by the equation: $m=At^2-Bt^3$ where A and B are constants.

It is known that Mr White had no traces of medicine in his blood at $t=0$ and that the medication level was back to zero exactly four hours later. It is also known that after two hours the level of medication in his blood was 8 grams.

(a) Find the values of A and B. [3]

(b) How long after taking the medication did the amount of medicine in Mr. White's bloodstream reach its highest level and what was the amount? [4]

END

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2 UNIT ASSESSMENT TASK- MATHEMATICS

SOLUTIONS

Question 1. (18 marks)

(i) Let $f(x)=x^3+3x^2+2$ Find

(a) $f'(x)=3x^2+6x$ [1]

(b) $f'(2)=12+12=24$ [1]

(c) $f'(-3)=-27-18=-45$ [1]

(d) the value(s) of k for which $f'(k)=24$

$24=3k^2+6k$

$k^2+2k-8=0$

$(k+4)(k-2)=0 \quad \Rightarrow \quad k=2$ or $k=-4$ [2]

(ii) Differentiate:

(a) $y'=20x^3-5$ [1]

(b) $y=x^{-1}+5x^{-3}$

$y'=-1x^{-2}-15x^{-4}$ or: $y'=\frac{-1}{x^2}-\frac{15}{x^4}$ [2]

(c) $y=x^{1.5}$

$y'=1.5x^{0.5}$ or $y'=\frac{3}{2}\sqrt{x}$ [2]

(d) $y'=4(2x^2+7)^3\times 4x$

$y'=16x(2x^2+7)^3$ [2]

(e) $y'=\frac{3(4x+1)-(3x+5)4}{(4x+1)^2}$

$y'=\frac{-17}{(4x+1)^2}$ [2]

(iii) $y=\left(x^3+x+6\right)^{\frac{1}{2}}$. When $x=3$, $y=\sqrt{3^3+3+6}=6 \Rightarrow (3,6)$

$y'=\frac{1}{2}\left(x^3+x+6\right)^{-\frac{1}{2}}\times\left(3x^2+1\right)$ $y'=\frac{3x^2+1}{2\sqrt{x^3+x+6}}$

At $x=3$ $y'=\frac{3\times 9+1}{2\sqrt{36}}=\frac{14}{6}=\frac{7}{3}$ Therefore, gradient of normal: $-\frac{3}{7}$

$y-6=-\frac{3}{7}(x-3)$

$3x+7y-51=0$ [4]

Question 2. (14 marks)

Solve the equations, writing, where relevant, your answers correct to 2dp:

(i) Substituting $m = x^2$:

$2m^2 - 3m - 20 = 0$

$(2m+5)(m-4) = 0 \quad \Rightarrow m = 4 \quad m = -\frac{5}{2}$

So, either: $x^2 = -\frac{5}{2} \quad \Rightarrow$ no solution

or: $x^2 = 4 \Rightarrow x = \pm 2$ [3]

(ii) Substituting $k = (x + \frac{4}{x})$ yields:

$k^2 - k - 20 = 0$

$(k-5)(k+4) = 0 \quad \Rightarrow k = 5 \quad k = -4$

So: $x + \frac{4}{x} = 5 \quad \Rightarrow \quad x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0 \quad \Rightarrow \quad x = 4$ or $x = 1$ [3

$x + \frac{4}{x} = -4 \quad \Rightarrow \quad x^2 + 4x + 4 = 0 \quad \Rightarrow \quad x = 2$

$(x-4)(x-1) = 0 \quad \Rightarrow \quad x = 4$ or $x = 1$

(iii) (a) Express $4x^2 - 24x + 38 \equiv m(x^2 - 2nx + n^2) + p$

$4x^2 - 24x + 38 \equiv mx^2 - 2mnx + mn^2 + p$

x^2: $4 = m$

x: $-24 = -2mn \quad 3 = n$

$const$: $38 = mn^2 + p \quad 2 = p$ [3]

(b) $y = 4(x-3)^2 + 2$ Therefore the minimal value is 2 [1]

(iv) (a) Solve: $y = x + c$ and $x^2 + y^2 = 8$ simultaneously:

$x^2 + (x+c)^2 = 8$

$x^2 + x^2 + 2c + c^2 = 8$

$2x^2 + 2cx + c^2 - 8 = 0$ [2]

(b) For one solution (tangent): $\Delta = 0$

$(2c)^2 - 4 \times 2(c^2 - 8) = 0 \quad \Rightarrow -8c^2 + 32 = 0$

$c = \pm 2$ [2]

Question 3. (10 marks)

(a) $\alpha + \beta = -4$ [1]

(b) $\alpha\beta = -6$ [1]

(c) $(\alpha - 1)(\beta - 1) \equiv \alpha\beta - (\alpha + \beta) + 1$

$= -1$ [1]

(d) $\alpha^2\beta + \alpha\beta^2 \equiv \alpha\beta(\alpha + \beta)$

$= 24$ [1]

(ii) Find the equation, in the form: $x^2 + bx + c = 0$, of the quadratic with roots

(a) $x^2 - (2-8)x - 16 = 0$

$x^2 + 6x - 16 = 0$ [1]

(b) $\alpha + \beta = \frac{1}{2+\sqrt{3}} + \frac{1}{2-\sqrt{3}} = 4$

$\alpha\beta = \frac{1}{2+\sqrt{3}} \times \frac{1}{2-\sqrt{3}} = 1$

$x^2 - 4x + 1 = 0$ [2]

(iii) (a) $\Delta = (m-2)^2 - 16 = m^2 - 4m - 12$ [1]

(b) $\Delta < 0 \quad \Rightarrow m^2 - 4m - 12 < 0$

$(m-6)(m+2) < 0$

$-2 < m < 6$ [2]

Question 4. (13 marks)

(i) (a) $\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x-4)^2 + (y+1)^2} \quad /(\quad)^2$

$x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 - 8x + 16 + y^2 + 2y + 1$

$6x - 6y - 12 = 0$

$y = x - 2$ Equation of a line [3]

(b) $\sqrt{(x-1)^2 + (y-2)^2} = 2\sqrt{(x-4)^2 + (y+1)^2} \quad /(\quad)^2$

$(x-1)^2 + (y-2)^2 = 4[(x-4)^2 + (y+1)^2]$

$x^2 - 2x + 1 + y^2 - 4y + 4 = 4[x^2 - 8x + 16 + y^2 + 2y + 1]$

$x^2 - 2x + y^2 - 4y + 5 = 4[x^2 - 8x + y^2 + 2y + 17]$

$x^2 - 2x + y^2 - 4y + 5 = 4x^2 - 32x + 4y^2 + 8y + 68$

$0 = 3x^2 - 30x + 3y^2 + 12y + 63 \quad \Rightarrow \quad x^2 - 10x + y^2 + 4y + 21 = 0$

$(x-5)^2 - 25 + (y+2)^2 - 4 + 21 = 0 \quad \Rightarrow \quad (x-5)^2 + (y+2)^2 = 8$

Equation of circle with centre (5,-2) and radius $\sqrt{8}$

[3]

(ii) (a) $a=2$ and, hence, the coordinates of $F:(0,2)$ [1]

(b) $y=-2$ [1]

(iii) $(y+4)^2-16=8x-24$

$(y+4)^2=8x-8$

$(y+4)^2=8(x-1)$

(a) the coordinates of the vertex : $(1,-4)$ [2]

(b) the coordinates of the focus: $(3,-4)$ [1]

(c) the equation of the directrix: $x=-1$ [1]

(d) the equation of the axis: $y=-4$ [1]

Question 5. (16 marks)

(i) $y'=6x^2+3$

for stationary points, solve: $6x^2+3=0$

No real solutions, hence no stationary points [2]

(ii) $y'=-4x^3+12x^2$

(a) $0=-4x^3+12x^2$

$0=4x^2(-x+3)$

$x=0 \quad x=3$ [4]

Substitute the x-values in the equation to find the y-values:

$x=0 \quad \Rightarrow y=0 \qquad A:(0,0)$

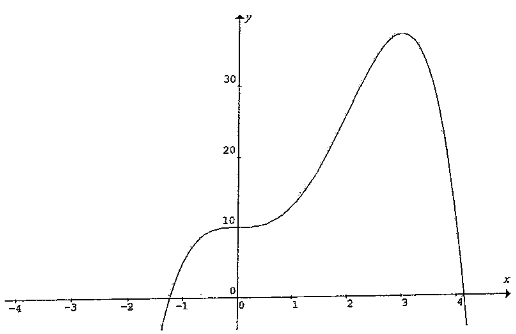
$x=3 \quad \Rightarrow y=37 \qquad B:(3,37)$

Nature of the points: $y''=-12x^2+24x$

At point B $\quad x=3 \quad \Rightarrow y''=-36$ hence maximum

At point A $\quad x=0 \quad \Rightarrow y''=0$

$y'''=-24x+24$ and at $x=0$ $y'''\neq 0$ hence point of inflexion [3]



(iii) (a)

Substituting $t=2 \quad m=8$ in $m=At^2-Bt^3$ yields:

$8=4A-8B$

$2=A-2B$ [I]

Substituting $t=4 \quad m=0$ in $m=At^2-Bt^3$ yields:

$0=16A-64B$

$0=A-4B$ [II]

Solving [I] and [II] simultaneously:

$$\begin{cases} A-2B=2 \\ A=4B \end{cases} \Rightarrow 2B=2 \qquad B=1 \text{ and } A=4$$

Hence the equation is: $m=4t^2-t^3$ [3]

(b) $m'=8t-3t^2$

To find stationary point, solve: $0=8t-3t^2$

$0=t(8-3t)$

$t=0$ [inappropriate]

$t=\frac{8}{3}$ (or 2 hours and 40 minutes)

Check: $m''=8-6t$, at $t=\frac{8}{3}\Rightarrow m''=-8$ Hence maximum (can also look at given graph)

Value of function at $t=\frac{8}{3}$: $m=4(\frac{8}{3})^2-(\frac{8}{3})^3=\frac{256}{27}$

Therefore, after 2 hours and 40 minutes, the medication in Mr White's blood reached its highest level, which was $\frac{256}{27}gr$

[4]

END