Student Number .... Teacher ....

MORIAH COLLEGE

Year 12

EXTENSION 1 MATHEMATICS
Assessment 1
December 2006

Time Allowed: 1.5 hours plus 5 minutes reading time.

Examiners: E. Apfelbaum, J. Cohen, H. Dalakiaris, G. Wagner, C. White

Instructions:

- Answer every question.
- Start each question on a new page.
- Show all necessary working. Draw clear well labeled diagrams.

Question 1	
Question 2	
Question 3	
Question 4	
Question 5	
Total	

QUESTION 1: (Start a new page)

(a) Differentiate the following:

(i) $y=\sqrt{3x^2-2x}$ 2

(ii) $y=3e^{x^2}$ 1

(b) Find the following indefinite integrals:

(i) $\int x^4+\frac{2}{x^2}\ dx$ 2

(ii) $\int\frac{5}{(2x+3)^3}\ dx$ 2

(c) The gradient function of a particular curve is given by $\frac{dy}{dx}=\frac{1}{\sqrt{4-x}}$ 3

The curve passes through the point $(0,0)$.

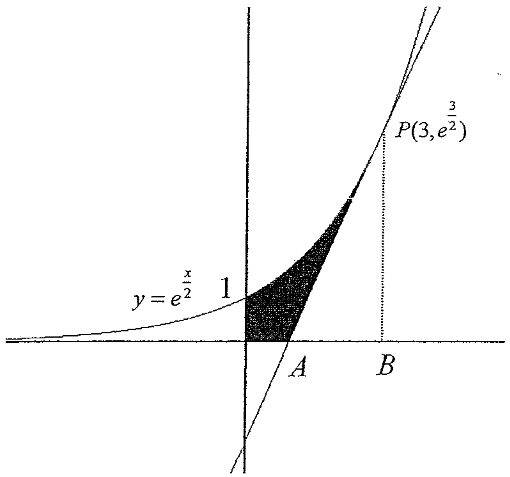
Find the equation of the curve.

(d) Evaluate the following definite integral, giving your answer in terms of e. 3

$$\int_{-1}^{1}\left(e^x-e^{-x}\right)^2\ dx$$

QUESTION 2: (Start a new page)

(a) The diagram below shows the graph of $y = e^{\frac{x}{2}}$. The tangent to the curve at point $P(3, e^{\frac{3}{2}})$ is shown cutting the x-axis at A. The point B has coordinates $(3,0)$.



(i) Show the equation of the tangent at P is $e^{\frac{3}{2}}x - 2y - e^{\frac{3}{2}} = 0$. 2

(ii) Find the coordinates of A. 1

(iii) Hence, find the shaded area in terms of e. 3

(b) (i) Sketch the region in the first quadrant bounded by the curve $y = \sqrt{16 - x^2}$, the x-axis and the y-axis. 1

(ii) By inspecting the graph, find the exact area of this region. 1

(iii) Write down a definite integral that would find the area of this region. There is no need to try and evaluate this definite integral. 1

(iv) Use the Trapezoidal Rule with 5 function values to find an approximation for the area of this region. 2

(v) Explain briefly why the Trapezoidal rule gives an under-estimate for the area of this solid. 1

QUESTION 3: (Start a new page)

(a) $P(2ap, ap^2)$ is a variable point on the parabola $x^2 = 4ay$, where a is the focus.

(i) Show that the gradient of the tangent at $P(2ap, ap^2)$ is p. 2

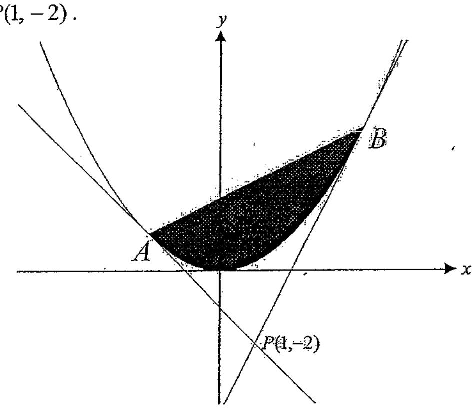
(ii) Show that the equation of the normal to the parabola at the point $P(2ap, ap^2)$ is given by: $x + py = 2ap + ap^3$. 2

(iii) If the normal at P cuts the y-axis at Q, show that the coordinates of Q are given by: $(0,\ 2a + ap^2)$. 1

(iv) The point R is the midpoint of PQ. As P moves on the parabola, the locus of R is another parabola. Find the Cartesian equation of this locus. 2

(v) Find the focal length of the parabola found in part (iv). 2

(b) The diagram shows the parabola $x^2 = 4y$ and the chord of contact AB from the external point $P(1, -2)$.



The Cartesian equation of the chord of contact to the curve $x^2 = 4ay$ from an external point (x_0, y_0) is $xx_0 = 2a(y + y_0)$.

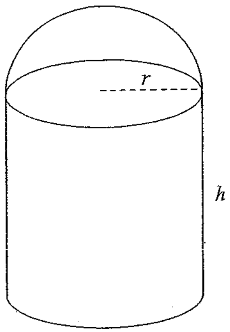
(i) Show that the equation of the chord of contact is $x - 2y + 4 = 0$. 1

(ii) Show that the endpoints of the chord of contact are $A(-2, 1)$ and $B(4, 4)$. 2

(iii) Find the area of the shaded region. 3

QUESTION 4: (Start a new page)

A specialised postage container in the shape of a cylinder with a hemispherical end (as shown in the diagram) has a volume of 1000 cm^3.



NOT TO SCALE

(i) Given that the volume of a sphere is $\frac{4}{3}\pi r^3$ show that the volume of the container is $V = \pi r^2 h + \frac{2}{3}\pi r^3$. 1

(ii) Hence show that the height h of the container can be given by 2

$$h = \frac{1000}{\pi r^2} - \frac{2r}{3}$$

(iii) Given that the surface area of a sphere is $4\pi r^2$, show that the total surface area of the container, including the base, is given by 2

$$A = \frac{5\pi r^2}{3} + \frac{2000}{r}$$

(iv) Find the radius of this container if the surface area is to be a minimum. Give your answer correct to the nearest millimetre. 3

QUESTION 5: (Start a new page)

Consider the function $y = \frac{x}{e^{2x}}$.

(i) Use the quotient rule to show that $\frac{dy}{dx} = \frac{1-2x}{e^{2x}}$ and find the coordinates of the **one** stationary point on the graph of $y = \frac{x}{e^{2x}}$. 3

(ii) Determine the nature of this stationary point. Show appropriate working out to support your answer. 2

(iii) As $x \to -\infty$, $\frac{x}{e^{2x}} \to -\infty$ but as $x \to \infty$, $\frac{x}{e^{2x}}$ has a limit. 1
Find this limit.

(iv) Hence, sketch the curve of the function $y = \frac{x}{e^{2x}}$ 2

(v) Using part (i) or otherwise, show that $\frac{d}{dx}\left(\frac{1}{2}e^{-2x} + \frac{x}{e^{2x}}\right) = \frac{-2x}{e^{2x}}$. 2

(vi) Hence, find the area bounded by the curve $y = \frac{x}{e^{2x}}$, the x- axis and the line $x = 1$. 3

2006 (ask 1

lost ½ mark once for forgetting '+c'

Question 1:

(a) (i) $y = (3x^2-2x)^{\frac{1}{2}}$

$y' = \frac{1}{2}(3x^2-2x)^{-\frac{1}{2}}.(6x-2)$ ✓

$= \dfrac{2(3x-1)}{2\sqrt{3x^2-2x}}$

$= \dfrac{3x-1}{\sqrt{3x^2-2x}}$ ✓

(ii) $y = 3e^{x^2}$

$y' = 3(2x)e^{x^2}$

$= 6x e^{x^2}$ ✓

(b)(i) $\int x^4 + 2x^{-2} \, dx$

$= \dfrac{x^5}{5} + \dfrac{2x^{-1}}{-1} + c$ ✓

$= \dfrac{x^5}{5} - \dfrac{2}{x} + c.$ ✓

(ii) $5\int (2x+3)^{-3} \, dx.$ ✓

$= 5 \dfrac{(2x+3)^{-2}}{-2 \cdot 2} + c$

$= \dfrac{-5}{4(2x+3)^2} + c$ ✓

(c) $\dfrac{dy}{dx} = (4-x)^{-\frac{1}{2}}$

$y = -2(4-x)^{\frac{1}{2}} + c.$ ✓

$\therefore y = -2\sqrt{4-x} + c.$

At $(0,0)$: $0 = -2\sqrt{4} + c$

$\therefore c = 4$ ✓

$\therefore y = -2\sqrt{4-x} + 4$ ✓

(d) $\int_{-1}^{1} (e^x - e^{-x})^2 \, dx$

$= \int_{-1}^{1} (e^{2x} - 2e^x.e^{-x} + e^{-2x}) \, dx$ ✓

$= \int_{-1}^{1} (e^{2x} - 2 + e^{-2x}) \, dx$

$= \left[\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right]_{-1}^{1}$ ✓

$= \left(\frac{e^2}{2} - 2 - \frac{e^{-2}}{2}\right) - \left(\frac{e^{-2}}{2} + 2 - \frac{e^2}{2}\right)$

$= \dfrac{2e^2}{2} - 4 - \dfrac{2e^{-2}}{2}$

$= e^2 - 4 - e^{-2}.$ ✓

Question 2:

(a) (i) $y = e^{x/2}$

$$\frac{dy}{dx} = \frac{1}{2} e^{x/2}$$

At $P(3, e^{3/2})$ m of tangent $= \dfrac{e^{3/2}}{2}$

Eqn: $y - e^{3/2} = \dfrac{e^{3/2}}{2}(x-3)$

$$2y - 2e^{3/2} = e^{3/2}x - 3e^{3/2}$$

$$\therefore e^{3/2}x - 2y - e^{3/2} = 0.$$

(ii) At A, $y = 0$.

$$\therefore e^{3/2}x = e^{3/2}$$

$$\therefore x = 1$$

$$\therefore A(1,0).$$

(iii) $$\text{Area} = \int_0^3 e^{x/2}\,dx - \int_1^3 \frac{e^{3/2}x - e^{3/2}}{2}\,dx$$

$$= \left[2e^{x/2}\right]_0^3 - \frac{e^{3/2}}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$= 2e^{3/2} - 2 - \frac{e^{3/2}}{2}\left[\left(\frac{9}{2}-3\right) - \left(\frac{1}{2}-1\right)\right]$$

$$= 2e^{3/2} - 2 - \frac{e^{3/2}}{2}\left(\frac{3}{2} + \frac{1}{2}\right)$$

$$= 2e^{3/2} - 2 - \frac{e^{3/2}}{2} \times 2$$

$$= e^{3/2} - 2$$

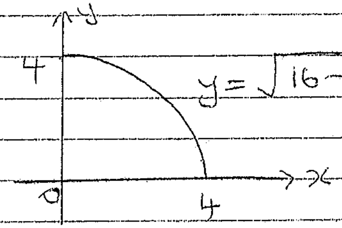
OR

$$\text{Area} = \int_0^3 e^{x/2}\,dx - \text{Area } \triangle ABP$$

$$= 2e^{3/2} - 2 - \frac{1}{2} \times 2 \times e^{3/2}$$

$$= e^{3/2} - 2.$$

(b) (i)



(ii) $$\text{Area} = \frac{1}{4} \times \pi \times 4^2$$

$$= 4\pi \text{ units}^2$$

(iii) $$\text{Area} = \int_0^4 (16 - x^2)^{1/2}\,dx$$

(iv)

x	0	1	2	3	4
y	4	$\sqrt{15}$	$\sqrt{12}$	$\sqrt{7}$	0

$$\text{Area} \doteqdot \frac{1}{2} \times 1 \times \left[4 + 2\sqrt{15} + 2\sqrt{12} + 2\sqrt{7} + 0\right]$$

$$= 11.98\,u^2 \text{ (to 2 dp)}$$

(v) Function decreasing so ~~rectangles~~ Trapezoids are

Question 3:

(a) (i) $x^2 = 4ay$ or $P(2ap, ap^2)$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$ ✓

$\frac{dy}{dx} = \frac{dy/dp}{dx/dp}$

$= \frac{2ap}{2a}$ ✓

$= p$

when $x = 2ap$.

$\frac{dy}{dx} = \frac{2ap}{2a}$

$= p$.

$\therefore$ gradient at P is p.

$\therefore$ gradient at P is p.

(ii) gradient of normal $= -\frac{1}{p}$.

Eqn of normal:

$y - ap^2 = -\frac{1}{p}(x - 2ap)$ ✓

$yp - ap^3 = -x + 2ap$.

$\therefore x + py = 2ap + ap^3$ ✓

(iii) cuts y-axis when $x = 0$

$\therefore py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore Q(0, 2a + ap^2)$ ✓

(iv) $P(2ap, ap^2)$ $Q(0, 2a+ap^2)$

$R = \left(\frac{2ap + 0}{2}, \frac{ap^2 + 2a + ap^2}{2}\right)$

$\therefore R = (ap, a + ap^2)$ ✓

$\therefore x = ap$

$p = \frac{x}{a}$

$y = a + ap^2$

$y = a + a\left(\frac{x}{a}\right)^2$

$y = a + \frac{x^2}{a}$ ✓

$\therefore y - a = \frac{x^2}{a}$

$x^2 = a(y - a)$ ✓

(v) focal length is $\frac{a}{4}$. ✓

(b) (i) $(x_0, y_0) \Rightarrow (1, -2)$

$a \Rightarrow 1$

$x x_0 = 2a(y + y_0)$

$x \cdot 1 = 2(y - 2)$ ✓

$x = 2y - 4$

$x - 2y + 4 = 0$.

(ii) $x = 2y - 4$. — ①

$x^2 = 4y$ — ②

sub ① in ②:

$(2y-4)^2 = 4y$

$4y^2 - 16y + 16 = 4y$.

$y^2 - 5y + 4 = 0$

$(y-4)(y-1) = 0$.

$\therefore y = 4, 1$.

$x = 4, -2$

$\therefore$ points are $A(-2,1)$ $B(4,4)$

(iii) Area $= \int_{-2}^{4} \left(\frac{x}{2} + 2\right) dx - \int_{-2}^{4} \frac{x^2}{4} dx$

$= \left[\frac{x^2}{4} + 2x - \frac{x^3}{12}\right]_{-2}^{4}$

$= \left(4 + 8 - \frac{16}{3}\right) - \left(1 - 4 + \frac{2}{3}\right)$

$= 9$

Question 4:

(i) $V = \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \pi r^2 h + \frac{2}{3}\pi r^3$

(ii) $1000 = \pi r^2 h + \frac{2}{3}\pi r^3$

$\pi r^2 h = 1000 - \frac{2}{3}\pi r^3$

$h = \frac{1000}{\pi r^2} - \frac{2\pi r^3}{3\pi r^2}$

$\therefore h = \frac{1000}{\pi r^2} - \frac{2r}{3}$ — ①

(iii) Surface Area $A = \frac{1}{2} \times 4\pi r^2 + \pi r^2 + 2\pi r h$

$A = 2\pi r^2 + \pi r^2 + 2\pi r h$

$A = 3\pi r^2 + 2\pi r h$

sub in ①:

$A = 3\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2} - \frac{2r}{3}\right)$

$= 3\pi r^2 + \frac{2000}{r} - \frac{4\pi r^2}{3}$

$= \frac{5\pi r^2}{3} + \frac{2000}{r}$.

(iv) $\frac{dA}{dr} = \frac{10\pi r}{3} - \frac{2000}{r^2}$ ✓

For minimum $\frac{dA}{dr} = 0$. ✓

$$\Rightarrow \frac{10\pi r}{3} - \frac{2000}{r^2} = 0$$

$$\frac{10\pi r}{3} = \frac{2000}{r^2}$$

$$10\pi r^3 = 6000.$$

$$r^3 = \frac{600}{\pi}.$$

$$\therefore r = \sqrt[3]{\frac{600}{\pi}} \text{ cm.}$$ ✓

$\therefore r \doteqdot 5.8$ cm

Question 5: ✓ ✓

(i) $\frac{dy}{dx} = \frac{e^{2x}.1 - x.2e^{2x}}{(e^{2x})^2} = \frac{e^{2x}(1-2x)}{(e^{2x})^2}$

$$= \frac{1-2x}{e^{2x}}$$

Stationary point $\frac{dy}{dx} = 0 \Rightarrow \frac{1-2x}{e^{2x}} = 0.$

$2x = 1$

$x = \frac{1}{2}$ ✓

$\therefore y = \frac{1}{2} \div e^{2(1/2)} = \frac{1}{2e}.$ ✓

$\therefore$ Stat pt $= \left(\frac{1}{2}, \frac{1}{2e}\right)$

(ii) At $\left(\frac{1}{2}, \frac{1}{2e}\right)$

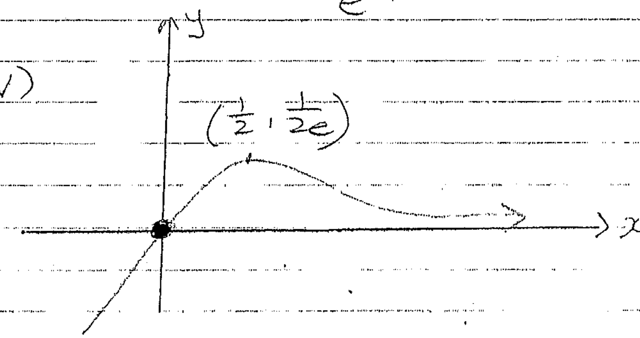
x	0	$\frac{1}{2}$	$3/4$
y'	>0	0	<0

✓ correct values needed to be shown in y'.

$\therefore$ local maximum. ✓
turning point

(iii) $x \to \infty$, $\frac{x}{e^{2x}} \to 0$ ✓

(iv) ✓✓



(v) $\frac{d}{dx}\left(\frac{1}{2}e^{-2x} + \frac{x}{e^{2x}}\right) = -e^{-2x} + \frac{1-2x}{e^{2x}}$ ✓

$= \frac{-1+1-2x}{e^{2x}}$ ✓

$= \frac{-2x}{e^{2x}}$

(vi) Area $= \int_0^1 \frac{x}{e^{2x}}\,dx$

$= -\frac{1}{2}\left[\frac{1}{2}e^{-2x} + \frac{x}{e^{2x}}\right]_0^1$ from part (v) ✓

$= -\frac{1}{2}\left[\frac{1}{2e^2} + \frac{1}{e^2} - \frac{1}{2}\right]$ ✓

$= \frac{1}{4e^2}\left(e^2 - 3\right)$ ✓