MORIAH COLLEGE

Year 12 2009 Pre-Trial

Extension 1 MATHEMATICS

Time Allowed: 2 hours plus 5 minutes reading time

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Instructions:

- Answer every question. All questions are of equal value.
- Show all necessary working. Draw clear, well labelled diagrams

Student Number: ____________________ **Teacher:** _________

Question 1. (12 Marks) Use a SEPARATE Booklet **Marks**

(a) Find the exact value of $\sin\left(\frac{5\pi}{4}\right)$. **1**

(b) Find:

(i) $\int \frac{x}{x^2 - 2}\,dx$ **1**

(ii) $\frac{d}{dx}(\cos^{-1}\frac{x}{2})$. **1**

(c) Find the exact value of $\int_{-1}^{3} \frac{1}{3 + x^2}\,dx$. **2**

(d) The point P divides the line AB externally in the ratio $3:2$. Find P if A is (2, -5) and B is (6, 1). **2**

(e) Find the values of x which satisfy $\frac{1}{x+1} \geq 3$. **3**

(f) If $t = \tan\frac{x}{2}$, express $\frac{1 - \cos x}{1 + \cos x}$ in terms of t, in simplest form. **2**

Question 2. (12 Marks) Use a SEPARATE Booklet — **Marks**

(a) Find $\lim_{x \to 1} \frac{(x^2 - 1) + \sin(x - 1)}{x - 1}$ showing all working. **2**

(b) P is the point, other than the origin, where $y = ax^2$ meets the line $y = x$.

(i) Find the coordinates of P. **1**

(ii) Find, to the nearest minute, the size of the acute angle formed by the line $y = x$ and the tangent to $y = ax^2$ at P. **2**

(c) Suppose $x^3 - 2x^2 + a \equiv (x + 2)Q(x) + 3$ where $Q(x)$ is a polynomial. Find the value of a. **2**

(d) Evaluate $\int_0^{\frac{\pi}{3}} 3\sin x \cos^2 x \, dx$ **3**

(e) Sketch the graph of $y = 3\sin^{-1} 2x$ **2**

Question 3. (12 Marks) Use a SEPARATE Booklet — **Marks**

(a) (i) Sketch the graph of $P(x) = x^2(x-1)(5-x)$. **1**

(ii) Hence solve $x^2(x-1)(5-x) < 0$. **1**

(b) (i) Express $\sqrt{3}\cos 2t - \sin 2t$ in the form $A\cos(2t + \alpha)$, with $A > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Find, in exact form, the general solutions to $\sqrt{3}\cos 2t - \sin 2t = 1$. **2**

(c) $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$, whose focus is S. $Q(x, y)$ divides the interval from P to S in the ratio $t^2 : 1$ [i.e, $PQ{:}QS = t^2 : 1$]

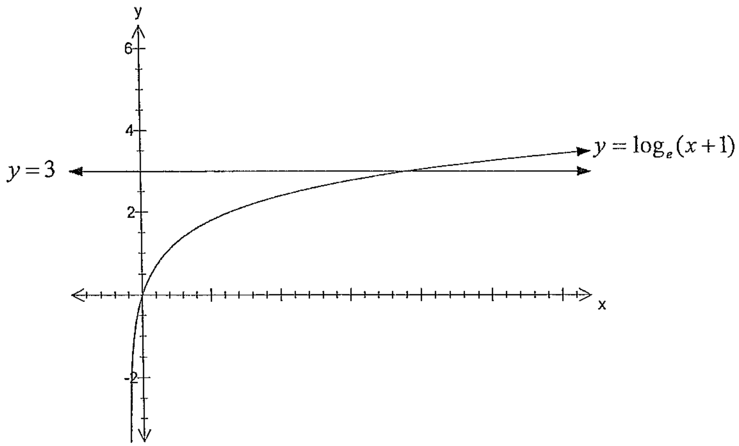
(i) Find x and y in terms of a and t. **2**

(ii) Verify that $\frac{y}{x} = t$. **1**

(iii) Prove that, as P moves on the parabola, Q moves on a circle, and state its centre and radius. **3**

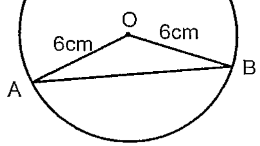
Question 4. (12 Marks) Use a SEPARATE Booklet **Marks**

(a) (i) Find the area of the region enclosed by the curve $y = \log_e(x+1)$, the y-axis, and the line $y = 3$. **3**



(ii) If this region is rotated about the y-axis, find the volume of the solid formed. Leave the answer in simplified exact form. **3**

(b) O is the centre of a circle with radius 6cm and $< AOB = \theta$ radians.
θ is increasing at a rate of 0.2 radians/second.



(i) Find the rate of change of the area of ΔAOB **3**

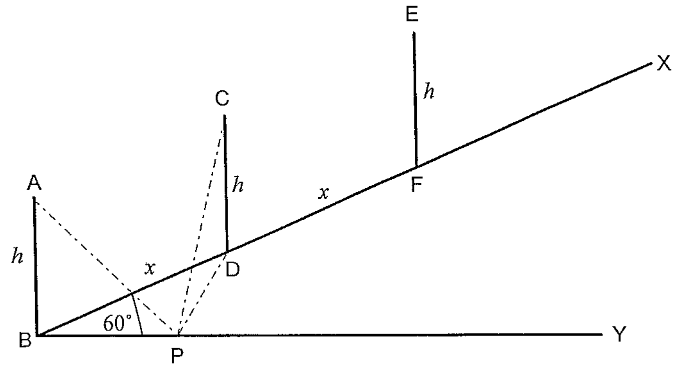
(ii) Find the rate of change of the area of the minor segment formed by AB when $< AOB = \frac{2\pi}{3}$. **3**

Question 5. (12 Marks) Use a SEPARATE Booklet **Marks**

(a) (i) If $\theta = \tan^{-1} A + \tan^{-1} B$ show that $\tan\theta = \frac{A+B}{1-AB}$. **1**

(ii) Hence solve the equation $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$. **3**

(b)



In the above diagram, BX and BY represent two roads intersecting at an angle of 60°.

On the road BX are situated three telegraph poles AB, CD and EF, all of equal height (h metres) and the same distance, x metres apart. (i.e. BD = DF= x).

P is a point on the road BY and the angles of elevation of A and C from P are 45° and 30° respectively.

(i) Show that BP = h and DP = $h\sqrt{3}$. **2**

(ii) By using the Sine Rule in ΔBDP, show that <BDP = 30° **2**
Hence show that ΔBDP is right angles at P.

(iii) Prove that $x = 2h$ **2**

(iv) By using the Cosine Rule in ΔPDF, show that PF = $h\sqrt{13}$. **2**
Hence show that the angle of elevation of E from P is approximately 15.5°.

Question 6. (12 Marks) Use a SEPARATE Booklet **Marks**

(a) (i) Prove that $\int_0^{\frac{\pi}{4}} \sin^2 x \, dx = \frac{\pi}{8} - \frac{1}{4}$. **2**

(ii) Prove that $\frac{d}{dx}\left(x\sin^2 x\right) - \sin^2 x = x\sin 2x$. **2**

(iii) Hence, or otherwise, prove $\int_0^{\frac{\pi}{4}} x\sin 2x \, dx = \frac{1}{4}$ **2**

(b) (i) Use Mathematical Induction to show that if x is a positive integer then $(1 + x)^n - 1$ is divisible by x for all integers n such that $n \geq 1$. **3**

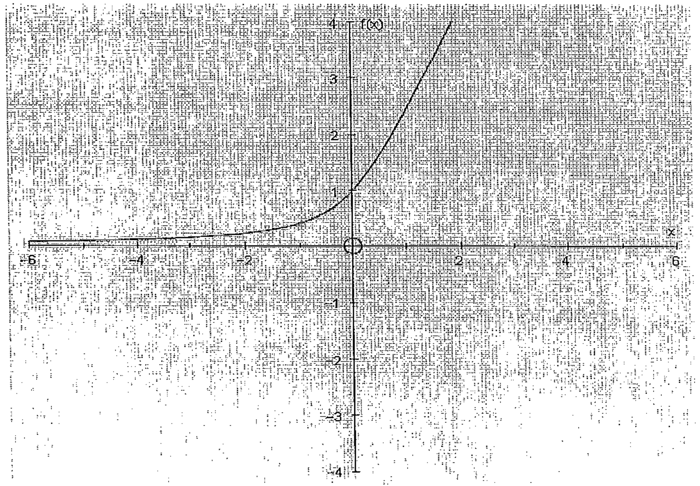
(ii) Fully factorise $12^n - 4^n - 3^n + 1$. **1**

(iii) Without using Mathematical Induction again, use the result from part (i) above, to deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers n. **2**

Question 7. (12 Marks) Use a SEPARATE Booklet **Marks**

(a) Use the substitution $u^2 = x$, $u > 0$, to evaluate $\int_1^{49} \frac{1}{x + \sqrt{x}} \, dx$, giving the answer in the form $\ln a$ for some positive integer a. **3**

(b) Consider the function $f(x) = x + \sqrt{x^2 + 1}$.



(i) State the range and domain of $f(x)$. **1**

(ii) Show that $f'(x) = \frac{f(x)}{\sqrt{x^2+1}}$ and, hence, show that $f'(x) > 0$ for all real x **2**

(iii) State the domain and range of $f^{-1}(x)$, the inverse function of $f(x)$ **1**

(iv) Show that $f^{-1}(x) = \frac{1}{2}(x - \frac{1}{x})$ **2**

(v) By comparing the graphs of $f(x)$ and $f^{-1}(x)$ or otherwise, show that:

$$\int_0^1 \left(x + \sqrt{x^2+1}\right) dx = \frac{1}{2}\left(1 + \sqrt{2} + \ln(1 + \sqrt{2})\right)$$ **3**

SOLUTIONS EXTENSION 1 PRE TRIAL 2009

Question 1

a) $\frac{-1}{\sqrt{2}}$

b) i) $\frac{1}{2}\ln(x^2-2)+C$

ii) $\frac{-1}{\sqrt{4-x^2}}$

c)

$\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_{-1}^{3}=\frac{1}{\sqrt{3}}(\frac{\pi}{3}-\frac{-\pi}{6})=\frac{\pi}{2\sqrt{3}}$

d) $A(2,-5)$ $B(6,1)$

3:-2

$\left(\frac{3\times6+-2\times2}{1},\frac{3\times1+-2\times-5}{1}\right)=(14,13)$

e) $x\neq-1$

Consider: $\frac{1}{x+1}=3\Rightarrow x=\frac{-2}{3}$

Solution is:

$-1<x\leq\frac{-2}{3}$

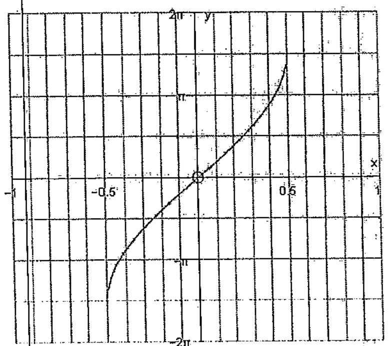
f)

If $t=\tan\frac{x}{2},\cos x=\frac{1-t^2}{1+t^2}$

$\frac{1-\cos x}{1+\cos x}=(1-\frac{1-t^2}{1+t^2})\div(1+\frac{1-t^2}{1+t^2})$

$=(\frac{1+t^2-1+t^2}{1+t^2})\div(\frac{1+t^2+1-t^2}{1+t^2})$

$=(\frac{2t^2}{1+t^2})\times(\frac{1+t^2}{2})=t^2$



Question 2

(a) $\lim_{x\to1}\frac{(x^2-1)+\sin(x-1)}{x-1}=$

$\lim_{x\to1}\frac{(x^2-1)}{x-1}+\lim_{x\to1}\frac{\sin(x-1)}{x-1}$

$\lim_{x\to1}\frac{(x-1)(x+1)}{x-1}+\lim_{x\to1}\frac{\sin(x-1)}{x-1}$

$\lim_{x\to1}(x+1)+1=2+1=3$

(b) $x=ax^2$

$x-ax^2=0$

$x(1-ax)=0$

x=0 (rule out, P cannot be at the origin, given)

$x=\frac{1}{a}\Rightarrow$ $y=\frac{1}{a}$ $\Rightarrow$ $P:(\frac{1}{a},\frac{1}{a})$

$y'=2ax$ At $x=\frac{1}{a}$ the gradient is: $y'=2$

Gradient of the line is: 1

$m_1=2$

$m_2=1$ Use: $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\frac{1}{3}$

$\theta=10°26'$

(c) Substitute x=-2 in: $x^3-2x^2+a\equiv(x+2)Q(x)+3$

$-8-8+a\equiv0+3$ $\Rightarrow$ $a=19$

(a) Evaluate $\int_0^{\frac{\pi}{3}}3\sin x\cos^2x\,dx$

$-3\int_0^{\frac{\pi}{3}}(-\sin x)\cos^2x\,dx=-3\left[\frac{\cos^3x}{3}\right]_0^{\frac{\pi}{3}}=-[\cos^3x]_0^{\frac{\pi}{3}}=\frac{7}{8}$

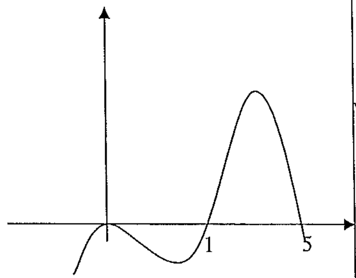
(e) Sketch the graph of $y=3\sin^{-1}2x$

D: $-\frac{1}{2}\leq x\leq\frac{1}{2}$

R: $-\frac{3\pi}{2}\leq y\leq\frac{3\pi}{2}$

Question three

a) i)



ii) $x>5,x<1,x\neq0$

b) i) $A\cos(2t+\alpha)=A\cos2t\cos\alpha-A\sin2t\sin\alpha$

$\Rightarrow\sqrt{3}=A\cos\alpha\ \&\ 1=A\sin\alpha$

So, $A=2$

$\cos\alpha=\frac{\sqrt{3}}{2},\sin\alpha=\frac{1}{2}\Rightarrow\alpha=\frac{\pi}{6}$

$\sqrt{3}\cos2t-\sin2t=2\cos(2t+\alpha)$

ii) $\cos(2t+\frac{\pi}{6})=\frac{1}{2}\Rightarrow2t+\frac{\pi}{6}=2n\pi\pm\frac{\pi}{3}$

$t=n\pi-\frac{\pi}{4},n\pi+\frac{\pi}{12}$

c) i)

$P(2at,at^2)$ $S(0,a)$

$t^2:1$

$x=\frac{2at}{t^2+1},y=\frac{2at^2}{t^2+1}$

ii) $\frac{y}{x}=\frac{2at^2}{t^2+1}\times\frac{t^2+1}{2at}=t$

iii) Eliminate t:

$x=\frac{2a\frac{y}{x}}{\frac{y^2}{x^2}+1}\times\frac{x^2}{x^2}\Rightarrow x=\frac{2ayx}{y^2+x^2}$

$1=\frac{2ay}{y^2+x^2}\Rightarrow y^2+x^2-2ay=0$

Complete the square;

$x^2+(y-a)^2=a^2$

Circle centre $(0,a)$,radius a.

Question four

a)i) Area $=\int_0^3x\,dy=\int_0^3e^y-1dy$

$=[e^y-y]_0^3=(e^3-3)-1=e^3-4u^2$

Volume $=\pi\int_0^3x^2dy=\pi\int_0^3(e^y-1)^2dy$

$=\pi\int_0^3e^{2y}-2e^y+1dy=\pi\left[\frac{1}{2}e^{2y}-2e^y+y\right]_0^3$

$=$

$\pi\left[(\frac{1}{2}e^6-2e^3+3)-(\frac{1}{2}-2)\right]=\frac{\pi}{2}(e^6-4e^3+9)$

b)

$\frac{dA}{dt}=\frac{dA}{d\theta}\times\frac{d\theta}{dt}$

$A=18\times\sin\theta,$

$\frac{d\theta}{dt}=0\cdot2\ \&\ \frac{dA}{d\theta}=18\times\cos\theta$

$\theta=\frac{2\pi}{3}\Rightarrow\frac{dA}{dt}=0\cdot2\times18\times\frac{-1}{2}=-1\cdot8\,cm^2/s$

ii) Area of minor segment (A)=

$\frac{1}{2}r^2(\theta-\sin\theta)$

$\frac{dA}{dt}=\frac{dA}{d\theta}\times\frac{d\theta}{dt}$

$=18(1-\cos\theta)\times0\cdot2$

$=3\cdot6\times(1-\frac{-1}{2})=5\cdot4\ cm^2/s$.

Question five

a)

Let $\tan^{-1}A=\alpha\ \&\ \tan^{-1}B=\beta\Rightarrow$

$\tan\alpha$=A and $\tan\beta$=B

So, $\tan\theta=\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$=\frac{A+B}{1-AB}$

b) i) In ΔABP: $\tan45°=\frac{h}{BP}\Rightarrow BP=h$

In Δ CDP:

$\tan30°=\frac{h}{PD}\Rightarrow PD=\frac{h}{\tan30°}=h\sqrt{3}$

ii)In ΔBDP;

$$\frac{\sin 60^\circ}{PD}=\frac{\sin\angle BDP}{BP}\Rightarrow\frac{\sqrt{3}}{2h\sqrt{3}}=\frac{\sin\angle BDP}{h}$$

$$\Rightarrow \sin\angle BDP=\frac{1}{2}\Rightarrow \sin\angle BDP=\frac{1}{2}$$

$$\angle BDP=30^\circ$$

In ΔBDP: $\angle BPD=180^\circ-\angle BDP-\angle DBP$

$$=180^\circ-30^\circ-60^\circ=90^\circ$$

iii)in ΔBDP: $\sin 30^\circ=\dfrac{BP}{BD}=\dfrac{h}{x}=\dfrac{1}{2}\Rightarrow x=2h$

iv) $\angle PDF=150^\circ$ (exterior angle of ΔBDP)

$$PF^2=PD^2+DF^2-2\times PD\times DF\times\cos\angle PDF$$

$$=3h^2+4h^2-2\times h\sqrt{3}\times 2h\times\frac{-\sqrt{3}}{2}=3h^2+4h^2+($$

$$\Rightarrow PF=h\sqrt{13}$$

$\angle$ of elevation is $\angle EPF=\theta$

$$\tan\theta=\frac{EF}{PF}=\frac{h}{h\sqrt{13}}=\frac{1}{\sqrt{13}}$$

$$\theta=\tan^{-1}\frac{1}{\sqrt{13}}\approx 15\cdot 5^\circ$$

Question six

a)i)

$$\sin^2 x=\frac{1}{2}(1-\cos 2x)$$

$$\therefore\int_0^{\frac{\pi}{4}}\sin^2 x\,dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\cos 2x)dx$$

$$=\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}=\frac{1}{2}\times(\frac{\pi}{4}-\frac{1}{2}\times 1)=\frac{\pi}{8}-\frac{1}{4}$$

As required!

ii)

Using the product rule:

$$\frac{d}{dx}(x\sin^2 x)=1\times\sin^2 x+x\times 2\sin x\times\cos x=\sin^2 x+x\sin 2x$$

$$\therefore x\sin 2x=\frac{d}{dx}(x\sin^2 x)-\sin^2 x$$

iii) From above:

$$\int_0^{\frac{\pi}{4}}x\sin 2x\,dx=\int_0^{\frac{\pi}{4}}\frac{d}{dx}(x\sin^2 x)dx-\int_0^{\frac{\pi}{4}}\sin^2 x\,dx$$

$$=\left[x\sin^2 x\right]_0^{\frac{\pi}{4}}-\left(\frac{\pi}{8}-\frac{1}{4}\right)\text{ from above!}$$

$$=\left(\frac{\pi}{8}\right)-\left(\frac{\pi}{8}-\frac{1}{4}\right)=\frac{1}{4}$$

b) i) Prove true for $n=1$.

$LHS=(1+x)-1=x=x\times 1\therefore$ divisible by x.

Assume true for all values up to $n=k$. i.e. Assume:

$(1+x)^k-1=x\times P(x)$

Prove true for $n=k+1$,

i.e. We want to prove:

$(1+x)^{k+1}-1=x\times Q(x)$

$LHS=$

$(1+x)(1+x)^k-1=(1+x)(x\times P(x)+1)-1$, by the above assumption.

$x\times P(x)+x^2\times P(x)+x+1-1=x(P(x)+xP(x)+1)$

$=x\times Q(x)$

i.e. If true for $n=k$, true for $n=k+1$

which proves the result by the induction.

Question six (cont'd)

$4^n(3^n-1)-1(3^n-1)$

$=(4^n-1)(3^n-1)$

$=(2^n-1)(2^n+1)(3^n-1)$

iii) From above:

$4^n-1=(1+3)^n-1$ is divisible by 3

And $3^n-1=(1+2)^n-1$ is divisible by 2.

Therefore their product is divisible by 6

For all positive integers n.

Question seven

(a) Use the substitution $u^2=x,\quad u>0$, to evaluate $\displaystyle\int_1^{49}\frac{1}{x+\sqrt{x}}\,dx$, giving the answer in the form $\ln a$ for some positi integer a.

$$x=u^2\qquad\frac{dx}{du}=2u\qquad\Rightarrow\qquad dx=2u\,du$$

$$\int\frac{1}{u^2+u}2u\,du=2\int\frac{u}{u^2+u}du=2\int\left(\frac{1}{u+1}\right)$$

$$=2\ln(u+1)$$

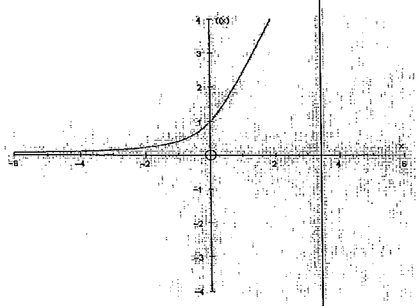
$$=2\left[\ln(1+\sqrt{x})\right]_1^{49}$$

$$=2(\ln 8-\ln 2)$$

$$\boxed{\int_1^{49}\frac{1}{x+\sqrt{x}}\,dx=2\ln 4=\ln 16}$$

Question seven (cont'd)

(b) Consider the function $f(x) = x + \sqrt{x^2+1}$.



(i) Range: $f(x) > 0$, Domain: all real values of x

(ii)
$$f'(x) = 1 + \frac{2x}{2\sqrt{x^2+1}} = 1 + \frac{x}{\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}} = \frac{f(x)}{\sqrt{x^2+1}}$$

Since $f(x)>0$ (from (i)), and $\sqrt{x^2+1}>0$ $f'(x) > 0$

(iii) Domain: $x>0$ Range: all real y

(iv) $f(x) = x + \sqrt{x^2+1}$

$y = x + \sqrt{x^2+1}$

$\Rightarrow \quad x = y + \sqrt{y^2+1}$

$\Rightarrow \quad x - y = \sqrt{y^2+1}$

$\Rightarrow \quad (x-y)^2 = y^2+1$

$\Rightarrow \quad x^2 - 2xy + y^2 = y^2 + 1$

$\Rightarrow \quad x^2 - 2xy = 1$

$\Rightarrow \quad x^2 - 1 = 2xy$

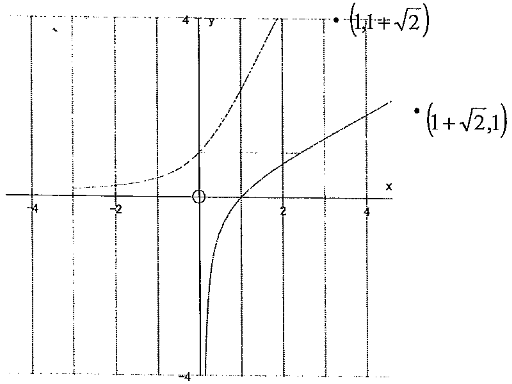
$\Rightarrow \quad \dfrac{x^2-1}{2x} = y$

$\Rightarrow \quad \dfrac{1}{2}\left(\dfrac{x^2-1}{x}\right) = y$

$$\boxed{f^{-1}(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)}$$

v) see next page!

(v) By comparing the graphs of $f(x)$ and $f^{-1}(x)$ or otherwise, show that:



By inspection of the two graphs:

$$\int_0^1 \left(x + \sqrt{x^2+1}\right)dx = \text{Area of rectangle} - \frac{1}{2}\int_1^{1+\sqrt{2}} \left(x - \frac{1}{x}\right)dx$$

$$A = 1 \times (1+\sqrt{2}) - \frac{1}{2}\left[x - \frac{1}{x}\right]_1^{1+\sqrt{2}}$$

$$A = 1 + \sqrt{2} - \frac{1}{2}\left[\frac{x^2}{2} - \ln x\right]_1^{1+\sqrt{2}}$$

$$A = 1 + \sqrt{2} - \frac{1}{2}\left[\frac{(1+\sqrt{2})^2}{2} - \ln(1+\sqrt{2}) - (\frac{1}{2})\right]$$

$$A = 1 + \sqrt{2} - \left[\frac{(1+\sqrt{2})^2}{4} - \frac{\ln(1+\sqrt{2})}{2} - (\frac{1}{4})\right]$$

$$A = 1 + \sqrt{2} - \left[\frac{3+2\sqrt{2}-1}{4} - \frac{\ln(1+\sqrt{2})}{2}\right]$$

$$A = \frac{4+4\sqrt{2}}{4} - \frac{2+2\sqrt{2}}{4} + \frac{\ln(1+\sqrt{2})}{2}$$

$$A = \frac{4+4\sqrt{2}-2-2\sqrt{2}}{4} + \frac{\ln(1+\sqrt{2})}{2} \quad A = \frac{2+2\sqrt{2}}{4} + \frac{\ln(1+\sqrt{2})}{2}$$

$$A = \frac{1+\sqrt{2}}{2} + \frac{\ln(1+\sqrt{2})}{2} = \frac{1}{2}\left[1+\sqrt{2}+\ln(1+\sqrt{2})\right]$$

as required