	MORIAH COLLEGE MATHEMATICS DEPARTMENT **Year 12 – 3 unit** Motion, Simple harmonic, Projectiles Name: ....

QUESTION 1. (17 marks)

Marks

A projectile is launched from level horizontal ground at an angle of 30° to the horizontal at a speed of 200 m/s. The equations of motion in the horizontal and vertical directions are respectively:

$\ddot{x} = 0$ and $\ddot{y} = -10$ (taking acceleration due to gravity as 10 m/s^2).

(a) Derive expressions for $\dot{x}, \dot{y}, x, y$. **4**

(b) For how long is the projectile airborne? **2**

(c) Calculate the greatest height reached by the projectile. **2**

(d) At what distance from its launch site does it strike the ground again. **2**

(e) How high is the projectile after it has travelled 2000 metres horizontally? **2**

(f) At what angle, and at what speed, is the projectile travelling after 15 seconds? **3**

(g) Find the Cartesian equation of the trajectory of the projectile. **2**

QUESTION 2. (9 marks)

(a) A particle moves such that, when its displacement is x metres from an origin, its velocity is given by: $v = \sqrt{8x+1}$. Initially the particle is at the origin. **4**

Show that $x = 2t^2 + t$, where t seconds is the time taken to reach the position x metres from the origin.

(b) A particle moves with a constant acceleration of 9 m/s^2. Given that the velocity is 12 m/s when the particle is 6 metres from the origin, find: **5**

(i) an expression for velocity in terms of displacement.

(ii) the velocity when $x = 0$.

QUESTION 3. (14 marks)

Marks

(a) A spring, hanging vertically from a fixed point, has an object attached to its lower end. The object is then 20 cm below the top of the spring. The object is then pulled down so that it is then 25 cm from the top of the spring, and it is then released. The object reaches the original position in 1 second. Assuming the motion is simple harmonic, find: **6**

(i) the period of the motion.

(ii) the acceleration acting on the object at the instant of release.

(iii) the speed of the object at the instant it reaches the equilibrium position.

(b) A particle moves in a straight line so that its velocity v at a position x is given by: $v^2 = 4(3 + 2x - x^2)$. **8**

(i) Show that $\ddot{x} = -4(x-1)$.

(ii) State the centre of motion.

(iii) What is the amplitude of the motion?

(iv) What is the period of the motion?

(v) What is the maximum speed of the particle?

SUGGESTED SOLUTIONS

QUESTION 1

(a) Initially, $\dot{x} = 200\cos 30° = 100\sqrt{3}$

$\dot{y} = 200\sin 30° = 100$

$x = 0, \quad y = 0$

Consider motion in the x-direction:

$\ddot{x} = 0$

Integrate: $\dot{x} = C$

When $t = 0$, $\dot{x} = 100\sqrt{3}$, $\therefore C = 100\sqrt{3}$

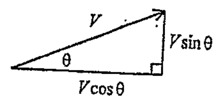
$\therefore \dot{x} = 100\sqrt{3}$ 1

Integrate: $x = 100\sqrt{3}\,t + C'$

When $t = 0, x = 0$, $\therefore C' = 0$

$\therefore x = 100\sqrt{3}\,t$ 1

Note: Projectile, projected at speed V, at an angle θ to the horizontal, has initial components of velocity of:
Horizontal : $V\cos\theta$
Vertical : $V\sin\theta$.



Consider motion in the y-direction:

$\ddot{y} = -10$

Integrate: $\dot{y} = -10t + K$

When $t = 0$, $\dot{y} = 100$, $\therefore K = 100$

$\therefore \dot{y} = 100 - 10t$ 1

Integrate: $y = 100t - 5t^2 + K'$

When $t = 0, y = 0$, $\therefore K' = 0$

$\therefore y = 100t - 5t^2$ 1 Total = 4

Note: Four different symbols for constants of integration have been used : C, C', K, K'. You could use C_1, C_2, C_3, C_4 or other groups.

(b) When $y = 0$, $100t - 5t^2 = 0$ 1

$5t(20 - t) = 0$

$t = 0, \quad t = 20$

The projectile is airborne for 20 seconds. 1 Total = 2

Note: The projectile is airborne until it strikes the ground, i.e. when $y = 0$.

(c) Greatest height reached when $\dot{y} = 0$

$100 - 10t = 0$

$t = 10$ 1

When $t = 10$, $y = 100 \times 10 - 5 \times 10^2$

$= 500$

Greatest height is 500 metres. 1 Total = 2

(d) When $t = 20$, $x = 100\sqrt{3} \times 20$

$= 2000\sqrt{3}$

Projectile strikes ground $2000\sqrt{3}$ metres from launch site. 2

Note: Projectile strikes ground after 20 seconds [see (b)].

(e) When $x = 2000$, $2000 = 100\sqrt{3}\,t$

$t = \frac{20}{\sqrt{3}}$ 1

$y = 100 \times \frac{20}{\sqrt{3}} - 5 \times \frac{400}{3}$

$y \approx 488$

Height is 488 metres (nearest metre) 1 Total = 2

Note: Substitute $t = \frac{20}{\sqrt{3}}$ into $y = 100 - 5t^2$.

(f) When $t = 15$, $\dot{x} = 100\sqrt{3}$, $\dot{y} = -50$

$v = \sqrt{(\dot{x})^2 + (\dot{y})^2}$ 1

$= \sqrt{30000 + 2500}$

$= 180.3$ (1 d.p.) 1

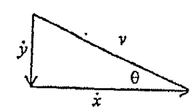
$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right| = \frac{50}{100\sqrt{3}}$

$\theta = 16.1°$ 1

Speed of projectile is 180.3 m/s at an angle of 16.1° to the horizontal, moving downwards.

Total = 3

Note: Velocity diagram:



Note: θ is the acute angle to the horizontal. The negative $\dot{y}$ value indicates it is moving downwards.

(g) $x = 100\sqrt{3}\,t$

$t = \dfrac{x}{100\sqrt{3}}$ 1

$\therefore\ y = 100t - 5t^2$

$y = 100 \times \dfrac{x}{100\sqrt{3}} - 5 \times \dfrac{x^2}{30000}$

$y = \dfrac{x}{\sqrt{3}} - \dfrac{x^2}{6000}$ 1 Total = 2

Note : To find the Cartesian equation, we need to eliminate 't' between the two equations for x and y, thus obtaining an equation in just x, y.

QUESTION 2

(a) $v = \sqrt{8x+1}$

Note : $\int (ax+b)^n dx = \dfrac{1}{n+1} \times \dfrac{1}{a}(ax+b)^{n+1} + C$

$\dfrac{dx}{dt} = \sqrt{8x+1}$

$\dfrac{dt}{dx} = \dfrac{1}{\sqrt{8x+1}}$ 1

$= (8x+1)^{-\frac{1}{2}}$

$t = 2 \times \dfrac{1}{8}(8x+1)^{\frac{1}{2}} + C$ 1

When $t = 0,\ x = 0$

$\therefore\ 0 = \dfrac{1}{4} \times 1 + C$

$C = -\dfrac{1}{4}$ 1

$\therefore\ t = \dfrac{1}{4}\sqrt{8x+1} - \dfrac{1}{4}$

$4t + 1 = \sqrt{8x+1}$

$16t^2 + 8t + 1 = 8x + 1$ *Note :* Square both sides.

$8x = 16t^2 + 8t$

$x = 2t^2 + t$ 1 Total = 4

Alternative solution :

If $x = 2t^2 + t$ is a solution,

$v = ut + 1.$ $v = \sqrt{8x+1}$

$= \sqrt{8(2t^2+t)+1}$

$= \sqrt{16t^2+8t+1}$

$= \sqrt{(4t+1)^2}$

$= ut + 1$

Since both expressions for v are the same,

$x = 2t^2 + t.$

(b) (i) $\ddot{x} = 9$

$\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 9$ 1

$\dfrac{1}{2}v^2 = 9x + C$ 1

When $x = 6,\ v = 12$:

$\dfrac{1}{2} \times 12^2 = 9 \times 6 + C$

$72 = 54 + C$

$C = 18$ 1

$\therefore\ \dfrac{1}{2}v^2 = 9x + 18$

$v^2 = 18x + 36$

$v = \pm\sqrt{18x+36}$ 1 Total = 4

Note : Since we require velocity in terms of displacement, we must use the expression for acceleration: $\ddot{x} = \dfrac{d}{dx}(\frac{1}{2}v^2)$, so that we integrate with respect to x.
[NB. Using $\ddot{x} = \dfrac{dv}{dt}$ would lead to $v = at + C$.]

Note : Substituting data given in the question.

(ii) When $x = 0,\ v = \pm\sqrt{36}$

Velocity is ± 6 metres/second. 1

Note : The particle may have initially been moving in a negative direction, hence two answers are possible. Once it starts to move in a positive direction, it remains in a positive direction, as its acceleration is a positive constant.

QUESTION 3

(a) (i) Period = 4 seconds 1

Note : Period is the time taken to complete one cycle. Data in the question states that it completes $\frac{1}{4}$ cycle in 1 second.

(ii) Period $T = \dfrac{2\pi}{n}$

$4 = \dfrac{2\pi}{n}$

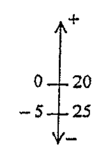
$n = \dfrac{\pi}{2}$ 1

$\ddot{x} = -n^2 x$ *Note :* This is the equation that describes S.H.M.

$\ddot{x} = -\left(\dfrac{\pi}{2}\right)^2 \times -5$ 1

Acceleration is $\dfrac{5\pi^2}{4}$ m/s². 1

Note : At instant of release, $x = -5$: x is the distance from the centre of oscillation (which is the point 20 cm below the top).



Total = 3

(iii) Using $v^2 = n^2\left(a^2 - x^2\right)$, $n = \frac{\pi}{2}$, $a = 5$, $x = 0$

$$v^2 = \frac{\pi^2}{4}(25 - 0) \qquad 1$$

$$v = \pm\frac{5\pi}{2}$$

Note : $v^2 = n^2(a^2 - x^2)$ is a formula you should learn for S.H.M. It applies to oscillations about $x = 0$.
a is the amplitude of motion.

In equilibrium position, speed is $\frac{5\pi}{2}$ cm/s. 1 Total = 2

(b) $v^2 = 4\left(3 + 2x - x^2\right)$

(i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$$= \frac{d}{dx}2\left(3 + 2x - x^2\right) \qquad 1$$

$$= 2(2 - 2x)$$

$$= -4(x - 1) \qquad 1$$

Total = 2

(ii) Centre of motion is $x = 1$ 1

(iii) When $v = 0$, $3 + 2x - x^2 = 0$

$$x^2 - 2x - 3 = 0$$

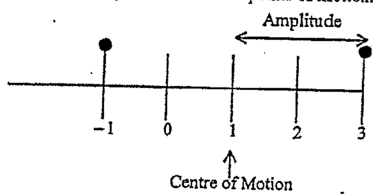
$$(x - 3)(x + 1) = 0$$

$$x = 3 \;,\; x = -1 \qquad 1$$

Amplitude is half the distance between the end points.

Amplitude is 2 units. 1 Total = 2

Note : Velocity is zero at the endpoints of motion.



(iv) $n^2 = 4$, $\therefore n = 2$ $(n > 0)$

Period $= \frac{2\pi}{n} = \frac{2\pi}{2}$

Period $= \pi$ 1

Note : Comparing $\ddot{x} = -4(x - 1)$ with $\ddot{x} = -n^2x$.
n is always taken as positive.

(v) Maximum speed occurs at centre of motion $x = 1$

$$v^2 = 4(3 + 2 - 1) \qquad 1$$

$$v = \pm 4$$

Maximum speed is 4 units / time unit. 1 Total = 2