MORIAH - 2001

Question 1.(start a new booklet) Marks

(a) Differentiate with respect to x: 3

i) $\cos^{-1}[e^{2x}]$

ii) $\sec^2\dfrac{x}{2}$

(b) Evaluate in exact simplified form: 4

i) $\displaystyle\int_0^4 \frac{dx}{16+x^2}$

ii) $\displaystyle\int_0^4 \frac{x\,dx}{16+x^2}$

(c) Solve the following inequality for x: $\dfrac{x+4}{x-2} \geq 7$ 3

(d) Find the coordinates of the point P which divides the interval AB externally in the ratio 4:1, if A is (1,4) and B is (3,-6). 2

Question 2.(start a new booklet) Marks

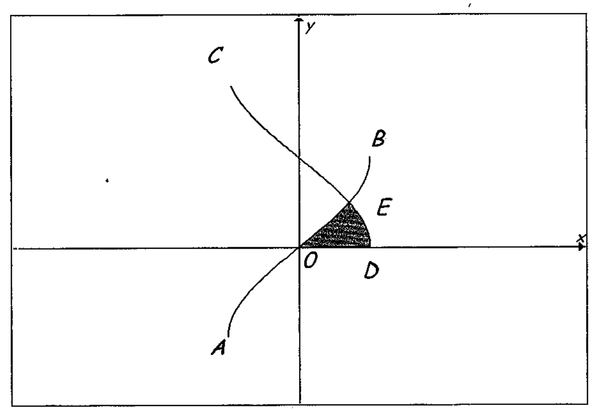
(a) The two graphs below represent $y=\sin^{-1}x$ and $y=\cos^{-1}x$. 8

i) State the coordinates of A, B, C and D

ii) Show that $f(x)=\sin^{-1}x+\cos^{-1}x$ is a constant.

iii) Find the value of the constant and show that the coordinates of E are $\left(\dfrac{1}{\sqrt{2}}, \dfrac{\pi}{4}\right)$.

iv) Find the area of the shaded region.



(b) By using the substitution, $x=\sin\theta$, find $\displaystyle\int_0^{\frac{1}{2}}(1-x^2)^{\frac{-3}{2}}\,dx$, in exact form. 4

Question 3.(start a new booklet) Marks

(a) The angle between $y = mx + 4$ and the line $x = 2y$ is 45°. 3

Find the exact value/s of m.

(b) Evaluate $\lim_{x \to 0} \dfrac{x + \tan\frac{x}{2}}{\sin\frac{x}{2}}$ 3

(c) 6

i) Write $\sqrt{3}\sin 2\alpha - \cos 2\alpha$ in the form $R\sin(2\alpha - \beta)$ where $R > 0$ and $0 < \beta < \dfrac{\pi}{2}$.

ii) Show that $y = \sqrt{3}\sin\alpha\cos\alpha - \cos^2\alpha$ can be expressed in the form $y = -\dfrac{1}{2} + \sin(2\alpha - \beta)$.

iii) Find the General Solutions of the equation $y = -\dfrac{1}{2}$

Question 4.(start a new booklet) Marks

(a) If $t = \tan\dfrac{x}{2}$ $(0^0 \le x < 180^0)$, prove that: 8

i) $\sin x = \dfrac{2t}{1+t^2}$

ii) $\cos x = \dfrac{1-t^2}{1+t^2}$

iii) Hence solve the equation $3\sin x - 4\cos x = 4$, to the nearest minute, for values of x, in the interval $0^0 \le x \le 180^0$

(b) i) State the domain and range of $y = 3\sin^{-1}(1-x)$ 4

ii) Find the gradient of $y = 3\sin^{-1}(1-x)$ when $x = 1$

iii) Sketch $y = 3\sin^{-1}(1-x)$

Question 5.(start a new booklet) Marks

(a) The acceleration of a particle moving in a straight line is given by $\ddot{x} = \frac{-900}{x^3}$, 6
where x is the displacement from the origin after t seconds. Initially the particle is 10m to the right of the origin with velocity
3 ms^{-1}.

i) Find the equation for the velocity of the particle in terms of displacement.

ii) Find the time when the particle is 100 m from the origin.

(b) i) Prove that the equation of the normal to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ is given by $x + py - ap^3 - 2ap = 0$. 4

ii) This normal meets the y axis at T. Show that the locus of the midpoint of PT is a parabola and find it's vertex and focus..

(c) Find $\frac{d}{dx}(2x\tan^{-1}x - \ln(1+x^2))$ and hence find $\int_0^1 \tan^{-1}x\, dx$ 2

Question 6.(start a new booklet) Marks

(a) A golfer projects his ball with a velocity of 20 ms^{-1} to hit the hole on a green that is 1 metre above the ground, 20 metres from the point of projection. 8

i) Find the equations of motion.

ii) Find the two possible angles of projection. Take g=10ms^{-1}.

iii) At what velocity will the ball hit the green? (correct to 3 significant figures)

(b) A sphere is expanding so that the surface area is increasing at the rate of $0\cdot 05cm^2s^{-1}$. Find the rate of increase of the volume, at the instant when the radius of the sphere is 10cm. 4

Question 7.(start a new booklet) Marks

(a) A particle moves along the x axis, in Simple Harmonic Motion, after being initially at rest at the origin. It's acceleration after t seconds is given by $\ddot{x} = 160 - 4x$ where x cm is its displacement after t seconds. 9

i)Find the centre and period of the motion.

ii) Show that $v^2 = 4(80x - x^2)$.

iii) Hence or otherwise, determine the interval in which the particle is moving.

iv) Find the distance travelled before the velocity is a maximum and find the maximum velocity.

v) From the information above, draw a neat sketch of the displacement/time graph and hence write down the equation of the displacement of the particle in terms of time.

(b) Prove by the method of Mathematical Induction that: 3

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2}).....(1-\frac{1}{n^2}) = \frac{n+1}{2n} \quad \text{for } n \geq 2.$$

i) $\frac{d}{dx}\cos^{-1}[e^{2x}]$

$= \frac{-2e^{2x}}{\sqrt{1-e^{4x}}}$

ii) $\frac{d}{dx}\sec^2\frac{x}{2} = \frac{2}{2}\left(\sec\frac{x}{2}\right)\sec\frac{x}{2}\tan\frac{x}{2}$

$= \sec^2\frac{x}{2}\tan\frac{x}{2}$

b) (i) $\left[\frac{1}{4}\tan^{-1}\frac{x}{4}\right]_0^4 = \frac{\pi}{16}$

(ii) $\int_0^4 \frac{x\,dx}{16+x^2} = \frac{1}{2}\left[\ln(16+x^2)\right]_0^4$

$= \frac{1}{2}\left[\ln 32 - \ln 16\right]$

$= \frac{1}{2}\ln 2.$

c) $\frac{x+4}{x-2} \geq 7.$ $x \neq 2.$

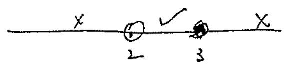
Consider the equation:

$x+4 = 7x-14$

$6x = 18$

$x = 3.$

Test solutions on number line:



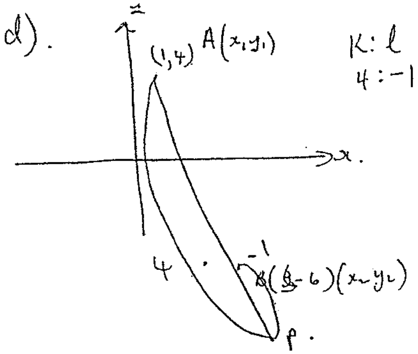
Test $x = 0$ ✗

$x = 2\frac{1}{2}$ ✓

$x = 4$ ✗ $4 \not\geq 7$

$\therefore$ $\boxed{2 < x \leq 3}$

d).



K : l

4 : -1

$P = \left(\frac{kx_2+lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right)$

$\left(\frac{12-1}{3}, \frac{4\cdot -6-4}{3}\right)$

$= \left(\frac{11}{3}, \frac{-28}{3}\right)$

2 a) (i) $A\left(-1, -\frac{\pi}{2}\right)$ $B\left(1, \frac{\pi}{2}\right)$

$C(-1, \pi)$ $D(1, 0)$

(ii) $\frac{d}{dx}\sin^{-1}x + \cos^{-1}x$

$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0.$

Since the differential is zero, then the original fn must be a constant.

(iii) Let $x = 1.$

then $\sin^{-1}1 + \cos^{-1}1$

$\frac{\pi}{2} + 0 = \frac{\pi}{2}.$

WTS $\sin^{-1}x = \cos^{-1}x.$

this occurs when $\alpha = \frac{\pi}{4}$

$\therefore$ and $\therefore$ $x = \frac{1}{\sqrt{2}}.$

(iv) Area: if $y = \cos^{-1}x$

then $x = \cos y.$

and $x = \sin y.$

$\therefore A = \int_0^{\pi/4} \cos y - \sin y\,dy$

$= \left[\sin y + \cos y\right]_0^{\pi/4}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)$

$= \boxed{\sqrt{2} - 1.\ u^2.}$

$\int_0 (1-x^2)^{-\frac{3}{2}}\,dx$ $x = \sin\theta$

$dx = \cos\theta\,d\theta.$

$x = \frac{1}{2},\ \theta = \frac{\pi}{6}.$

$x = 0,\ \theta = 0.$

$\therefore \int_0^{\pi/6} \frac{\cos\theta\,d\theta}{\left(\sqrt{1-\sin^2\theta}\right)^3}$

$= \int_0^{\pi/6} \sec^2\theta\,d\theta.$

$= \left[\tan\theta\right]_0^{\pi/6}$

$= \frac{1}{\sqrt{3}}$

3/ $y = mx + 4$ $y = \frac{x}{2}.$

$\tan 45^\circ = \left|\frac{m-\frac{1}{2}}{1+\frac{m}{2}}\right| = \left|\frac{\frac{2m-1}{2}}{\frac{2+m}{2}}\right|$

$\therefore 2+m = 2m-1$

$3 = m$

$2+m = -2m+1$

$3m = -1$

$m = -\frac{1}{3}.$

b). $\lim_{x\to 0} \frac{x}{\sin\frac{x}{2}} \times \frac{2}{2} + \frac{\tan\frac{x}{2}}{\sin\frac{x}{2}}$

$= 2 + 1$

$= 3.$

c). $R\sin(2\alpha - \beta)$

$= R\sin 2\alpha\cos\beta - R\cos 2\alpha\sin\beta.$

$\sqrt{3}\sin 2\alpha - \cos 2\alpha.$

Comparing co-efficients

$\sqrt{3} = R\cos\beta.$

$1 = R\sin\beta.$

$\therefore \tan\beta = \frac{1}{\sqrt{3}}$ $\therefore \beta = \frac{\pi}{6}.$

$\therefore R = 2.$

$= 2\sin\left(2\alpha - \frac{\pi}{6}\right)$

(ii) Now

$\sqrt{3}\sin 2\alpha - \cos 2\alpha$

$= 2\sqrt{3}\sin\alpha\cos\alpha - 2\cos^2\alpha + 1$

$\therefore 2\sqrt{3}\sin\alpha\cos\alpha - 2\cos^2\alpha + 1 = 2\sin(2\alpha$

$\underbrace{\sqrt{3}\sin\alpha\cos\alpha - \cos^2\alpha + \frac{1}{2}}_{y} = \sin\left(2\alpha - \frac{\pi}{6}\right)$

$\therefore y = -\frac{1}{2} + \sin\left(2\alpha - \frac{\pi}{6}\right)$

$\therefore -\frac{1}{2} = \sin\left(2\alpha - \frac{\pi}{6}\right) - \frac{1}{2}$

$\therefore \sin\left(2\alpha - \frac{\pi}{6}\right) = 0.$

$2\alpha - \frac{\pi}{6} = n\pi + (-1)^n \times 0$

$\therefore 2\alpha = n\pi + \frac{\pi}{6}$

$\alpha = \frac{n\pi}{2} + \frac{\pi}{12}.$

4a

i) if $t = \tan\frac{x}{2}$

RTP $\sin x = \frac{2t}{1+t^2}$

$\sin x = \sin\left(\frac{x}{2} + \frac{x}{2}\right)$

$= 2\sin\frac{x}{2}\cos\frac{x}{2}$

$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}$

$= \frac{\frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} + \frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}}$

$= \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2} + 1}$

$= \frac{2t}{t^2+1}$

$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = \frac{-900}{x^3}$

$\therefore \frac{v^2}{2} = \frac{450}{x^2} + C$

$x=10,\ v=3:$

$\frac{9}{2} = \frac{450}{100} + C$

$\therefore C = 0$

$\therefore v^2 = \frac{900}{x^2}$

$v \neq 0$ for this equation,

may take $v>0$ since $v>0$ at $t=0$

$\therefore \boxed{v = \frac{30}{x}}$ 2

ii) $\therefore \frac{dx}{dt} = \frac{30}{x}$

$\frac{dt}{dx} = \frac{x}{30}$

$v \neq 0$ $\int_0^t dt = \frac{1}{30}\int_{10}^{100} x\,dx$

$t = \frac{1}{30}\left[\frac{x^2}{2}\right]_{10}^{100}$

$= \frac{1}{60}\left[100^2 - 100\right]$

$= 165\text{ s}$

$\boxed{\text{Time taken is } 165\text{s}}$ 2

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$= \frac{2ap}{2a}$ at $P(p)$

$= p$

$\therefore$ the normal at P has equation

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

or $\boxed{x + py - ap^3 - 2ap = 0}$ as required. 1

ii) At $x=0$, $py = ap^3 + 2ap$

$y = ap^2 + 2a \quad (p \neq 0)$

$\therefore T$ is $(0, ap^2+2a)$ 1

Let $M(x,y)$ be the midpoint of PT

Then $x = \frac{2ap}{2}$ $\quad y = \frac{ap^2+2a+ap^2}{2}$

$= ap$ $\quad = ap^2 + a$

$\therefore p = \frac{x}{a} \Rightarrow y = a\left(\frac{x}{a}\right)^2 + a$ 1

or $\boxed{y = \frac{x^2}{a} + a}$ 1

This is a quadratic fn so its graph is a parabola.

Rearranging:

$x^2 = a(y-a)$

$\boxed{\therefore \text{the vertex is at } (0, a)}$ 1

The focal length A is given by

$4A = a$

$A = \frac{a}{4}$

$\boxed{\therefore \text{the focus is at } \left(0, a + \frac{a}{4}\right)}$ 1

RTP $\cos x = \frac{1-t^2}{1+t^2}$

$\cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right)$

$= \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$= \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}$

$= \frac{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}$

$= \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$

$= \frac{1-t^2}{1+t^2}$

(iii) $3\cdot\frac{2t}{1+t^2} - 4\frac{(1-t^2)}{1+t^2} = 4$

$6t - 4 + \cancel{4t^2} = 4 + \cancel{4t^2}$

$6t = 8$

$t = \frac{4}{3}$

$\therefore \tan\frac{x}{2} = \frac{4}{3}$

$\frac{x}{2} = 53^\circ 8'$

$\boxed{x = 106^\circ 16'}$

Test $\boxed{x = 180^\circ}$

$3(0) - 4(-1) = 4 \therefore$ Sol^n!

(i) $y = 3\sin^{-1}(1-x)$

$\frac{y}{3} = \sin^{-1}(1-x)$

D: $-1 \le 1-x \le 1$

$-2 \le -x \le 0$

$\therefore \boxed{0 \le x \le 2}$

R: $-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$

$\therefore -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

ii). $y = 3\sin^{-1}(1-x)$

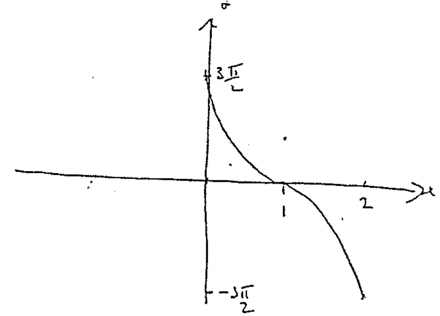
$y' = \frac{3(-1)}{\sqrt{1-(1-x)^2}}$

$= \frac{-3}{\sqrt{1-(1-2x+x^2)}}$

$= \frac{-3}{\sqrt{2x - x^2}}$

$x=1$, $y' = \frac{-3}{1}$

$= -3$.



dx [...] $1+x^2$ $1+x^2$

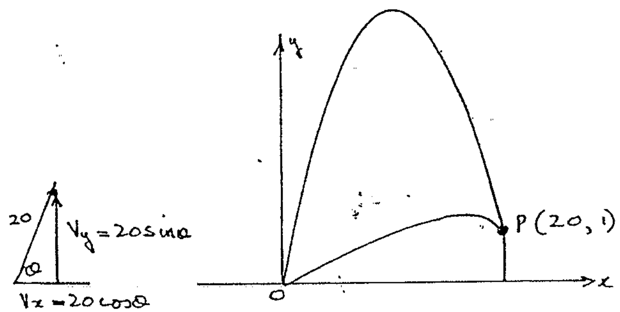
$$\boxed{= 2\tan^{-1}x}$$ 1

$$\therefore \int_0^1 2\tan^{-1}x\,dx = \left[2x\tan^{-1}x - \ln(1+x^2)\right]_0^1$$

$$= 2\frac{\pi}{4} - \ln 2 - (0-0)$$

$$\therefore \boxed{\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \frac{1}{2}\ln 2}$$ 1

a)



Take axes at the point of Projection. Let the initial ∠ of projection be θ

<u>Horizontally</u>

$\ddot{x} = 0$ ①

$\dot{x} = V_x$ since acceln $= 0$

$\dot{x} = 20\cos\theta$ ②

$x = 20\cos\theta t + C_1$

$t=0, x=0$: $0 = 0 + C_1$

$\therefore x = 20\cos\theta t$ ③

<u>Vertically</u>

$\ddot{y} = -10$ ④

$\dot{y} = -10t + C_2$

At $t=0$, $\dot{y} = V_y = 20\sin\theta$

$\therefore 20\sin\theta = C_2$

$\therefore \dot{y} = 20\sin\theta - 10t$ ⑤

$\therefore y = 20\sin\theta t - 5t^2 + C_3$

At $t=0$, $y=0$:

$0 = 0 + C_3$

$\therefore y = 20\sin\theta t - 5t^2$ ⑥ 3

At $x = 20$, $t = \frac{1}{\cos\theta}$. Since also $y=1$, substitute in ⑥ gives

$1 = 20\tan\theta - 5\sec^2\theta$

$1 = 20\tan\theta - 5(1+\tan^2\theta)$

or $5\tan^2\theta - 20\tan\theta + 6 = 0$ (2)

10

$= 2 \pm \sqrt{2.8}$

$\therefore \boxed{\theta = 74°46' \text{ or } 18°5'}$ (1) 3

For $\theta = 74°46'$

$t = \frac{1}{\cos\theta}$

$= 3.80589\ldots$

$\dot{y} = 20\sin\theta - 10t$

$= -18.7616\ldots$

$\dot{x} = 20\cos\theta$

$= 5.25501\ldots$

$\dot{x}^2 + \dot{y}^2 = 379.6127\ldots$

Speed $= \sqrt{\dot{x}^2+\dot{y}^2}$

$= 19.483\ldots$

$\boxed{|v| = 19.5 \text{ m/s}}$ (3 sig. fig)

For $\theta = 18°5'$

$t = \frac{1}{\cos\theta}$

$= 1.05196\ldots$

$\dot{y} = 20\sin\theta - 10t$

$= -4.3116\ldots$

$\dot{x} = 20\cos\theta$

$= 19.01212\ldots$

$\dot{x}^2 + \dot{y}^2 = 380.0506\ldots$

$\therefore$ Speed $= \sqrt{\dot{x}^2+\dot{y}^2}$

$= 19.494\ldots$

$\boxed{|v| = 19.5 \text{ m/s}}$ (3 sig. fig) 2

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b) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ ① 1

Now: $A = 4\pi r^2$

$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$ 1

$0.05 = 8\pi r \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{0.05}{8\pi r}$ 1

Sub in ①

$\frac{dV}{dt} = 4\pi r^2 \times \frac{0.05}{8\pi r}$

$\therefore \boxed{\frac{dV}{dt} = 0.25 \text{ cm}^3/\text{s}}$ 1

1. $\ddot{x} = 160 - 4x$

(i) $\ddot{x} = -4(x-40)$.

$\therefore$ Centre is 40 cm   1

$n = 2$

$\therefore$ period $= \frac{2\pi}{2} = \pi$ secs   1

(ii) $\frac{1}{2}v^2 = \int 160 - 4x \, dx$

$= 160x - 2x^2 + C$

$t=0$, $v=0$, $x=0$ $\therefore C = 0$.

$\therefore v^2 = 2(160x - 2x^2)$

$= 4(80x - x^2)$.   2.

(iii) $= 4(x)(80-x)$.

$v = 0$ when $x = 0, 80$.

$\therefore$ the particle moves between 0 and 80.   1

(iv) Velocity is a max at the centre $x = 40$.

$\therefore$ The particle moves 40 cm.

$\therefore v^2 = 4(40)(80-40)$

$= 6400$

$\therefore v = + 80$ m/s.   2

ie $=$ 80m/s to the right.

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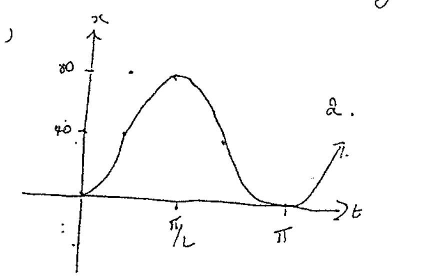

2.

$\therefore x = 40 - 40 \quad 2t$

7(b).

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n} \qquad n \geq 2$$

Prove for $n = 2$.

LHS $1 - \frac{1}{4} = \frac{3}{4}$.

RHS $\frac{2+1}{4} = \frac{3}{4}$ $\therefore$ true for $n=2$.

Assume $n=k$

ie. $\left(1-\frac{1}{(2)^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$.

Prove true for $n = k+1$.

WANT: $\underbrace{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)}\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$

LHS: from assumption: $\left(\frac{k+1}{2k}\right)\left(1 - \frac{1}{(k+1)^2}\right)$

$= \left(\frac{\cancel{k+1}}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^{\cancel{2}}}\right)$

$= \frac{k^2+2k}{2k(k+1)}$

$= \frac{\cancel{k}(k+2)}{\cancel{k}\,2(k+1)}$

$= \frac{k+2}{2k+2}$.

which is the required expression.

$\therefore$ true for $n=k+1$, provided true for $n=k$.

And since true for $n = 2, 3 \ldots$

$\therefore$ true for all integers $n \geq 2$.

hat the Speed is the Same

all is struck with speed V and passes through the point V, 1). there are two angles of projection, the final speed at P is the same, regardless of which gle is chosen.

roof: In the standard way we get the Cartesian equation

$$y = \tan\theta\, x - \frac{1}{2}g\frac{x^2}{V^2}\sec^2\theta$$

$x=V, y=1$: $\quad 1 = V\tan\theta - \frac{g}{2}\sec^2\theta \quad$ (1). Also $V = V\cos\theta\, t$

$\therefore t = \sec\theta$

$\Rightarrow g\tan^2\theta - 2V\tan\theta + 2 + g = 0$

$$\tan\theta = \frac{2V \pm \sqrt{4V^2 - 8g - 4g^2}}{2g}$$

this has two solutions if $V^2 > 2g + g^2$

Now for both these values, $\dot{x} = V\cos\theta \quad \dot{y} = V\sin\theta - gt$

$= V\sin\theta - g\sec\theta$

$\therefore$ final speed $|v|^2 = \dot{x}^2 + \dot{y}^2$

$= V^2\cos^2\theta + (V^2\sin^2\theta - 2Vg\tan\theta + g^2\sec^2\theta)$

$= V^2 - 2gV\tan\theta + g^2\sec^2\theta$

$= V^2 - 2g$ from (1)

which is independent of θ. In our problem, this gives $\sqrt{V^2 - 2g} = 19.49$ m/s

$$\tan\theta = \frac{20 \pm \sqrt{280}}{10}$$

$$= 2 \pm \sqrt{2.8}$$

$$\therefore \boxed{\theta = 74°46' \text{ or } 18°5'} \quad (1) \quad 3$$

For $\theta = 74°46'$

$t = \frac{1}{\cos\theta}$

$= 3.80589\cdots$

$\dot{y} = 20\sin\theta - 10t$

$= -18.7616\cdots$

$\dot{x} = 20\cos\theta$

$= 5.25501\cdots$

$\dot{x}^2+\dot{y}^2 = 379.6127\cdots$

Speed $= \sqrt{\dot{x}^2+\dot{y}^2}$

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For $\theta = 18°5'$

$t = \frac{1}{\cos\theta}$

$= 1.05196\cdots$

$\dot{y} = 20\sin\theta - 10t$

$= -4.3116\cdots$

$\dot{x} = 20\cos\theta$

$= 19.01212\cdots$

$\dot{x}^2+\dot{y}^2 = 380.0506\cdots$

$\therefore$ Speed $= \sqrt{\dot{x}^2+\dot{y}^2}$

$= 19.494\cdots$

$\boxed{|v| = 19.5 \text{ m/s}}$ (3 sig. fig) 2

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b) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad (1) \quad 1$$

Now $A = 4\pi r^2$

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} \quad 1$$

$$0.05 = 8\pi r \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{0.05}{8\pi r} \quad 1$$

Sub in (1)

$$\frac{dV}{dt} = 4\pi r^2 \times \frac{0.05}{8\pi r}$$

$$\boxed{\therefore \frac{dV}{dt} = 0.25 \text{ cm}^3/\text{s}} \quad 1$$

To Teachers

Q6 a) iii)

I thought the question as stated was a little unfair, in that they would have to calculate the speed for both angles. As it happens, the speed is the same in both cases (which will only happen if V is the same as the x value of the final point – see below) so I gave the marks for either calculation.

Proof that the Speed is the same

A ball is struck with speed V and passes through the point $P(V, 1)$. there are two angles of projection, the final speed at P is the same, regardless of which angle is chosen.

Proof: In the standard way we get the Cartesian equation

$$y = \tan\theta\, x - \tfrac{1}{2}g\frac{x^2}{V^2}\sec^2\theta$$

Sub $x = V, y = 1$: $\quad 1 = V\tan\theta - \frac{g}{2}\sec^2\theta \quad (1)$ Also $V = V\cos\theta\, t$ $\therefore t = \sec\theta$

$$\Rightarrow g\tan^2\theta - 2V\tan\theta + 2 + g = 0$$

$$\tan\theta = \frac{2V \pm \sqrt{4V^2 - 8g - 4g^2}}{2g}$$

This has two solutions if $V^2 > 2g + g^2$

Now for both these values, $\dot{x} = V\cos\theta \quad \dot{y} = V\sin\theta - gt$

$= V\sin\theta - g\sec\theta$

$\therefore$ final speed $|v|^2 = \dot{x}^2 + \dot{y}^2$

$= V^2\cos^2\theta + (V^2\sin^2\theta - 2Vg\tan\theta + g^2\sec^2\theta)$

$= V^2 - 2gV\tan\theta + g^2\sec^2\theta$

$= V^2 - 2g$ from (1)

which is independent of θ. In our problem, this gives $\sqrt{V^2 - 2g} = 19.49$ m/s