MORIAH - PRE-TRIAL - 2004



**Question 1 ( Start in a new booklet)**

| | | |
|---|---|---|
| a. | Find the acute angle between the lines $y=\frac{x}{3}$ and $x+\sqrt{3}y+1=0$.<br>Give your answer in degrees to the nearest minute. | 2 |
| b. | The area of a minor segment of a circle with an angle at the centre of $30^{\circ}$ is $5\cdot 20\,cm^2$.<br>i) Find the radius of the circle correct to 1 decimal place.<br>ii) Find the arc length of this minor segment correct to 3 significant figures. | 3<br>1 |
| c. | Differentiate the following with respect to $x$.<br>i) $y=\log_e\sqrt{\cos x}$<br>ii) $y=x\tan^2 x$ | 1<br>1 |
| d. | Find a primitive function for the following functions.<br>i) $\frac{e^{3x}+e^x-5}{e^{2x}}$<br>ii) $\frac{4x-6}{x^2-3x-7}$ | 2<br>1 |
| e. | Find $\lim_{\theta\to 0}\frac{\tan\frac{\theta}{3}}{\theta}$ | 1 |

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| | **Question 2 ( Start in a new booklet)** | |
| a. | i) Express $2\cos\theta-\sin\theta$ in the form $A\cos(\theta+\alpha)$ for $0\le\alpha\le\frac{\pi}{2}$ and $A>0$<br>ii) Hence solve the equation $2\cos\theta=1+\sin\theta$ for $0\le\theta\le 2\pi$.<br>Give answers to 2 decimal places | 2<br>3 |
| b. | i) Find expressions in surd form for $\sin 15^{\circ}$ and $\cos 15^{\circ}$ using using $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$ expansions respectively and appropriate values of $\alpha$ and $\beta$<br>ii) Prove that $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$<br>iii) Hence find the exact value of $\tan 7\frac{1}{2}^{\circ}$ in simplest form | 2<br>2<br>3 |

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## Question 3 ( Start in a new booklet)

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| a. | i) State the domain and range for the function $y = 2\cos^{-1} 3x$ | 1 |
| | ii) Sketch the graph of the function above. | 1 |
| b. | Find the equation of the tangent to the curve $y = 2\cos^{-1} 3x$ at the point where $x = \frac{1}{6}$ | 3 |
| c | i) Show that $y = \sin^{-1}(\cos x)$ is an even function | 1 |
| | ii) Differentiate $y = \sin^{-1}(\cos x)$ with respect to $x$ for $0 < x < \pi$ | 2 |
| | iii) Sketch the above curve for the domain $-\pi < x < \pi$ | 1 |
| d. | i) Show that the sum $a - ap + ap^2 - ap^3 + ap^4 ....... + ap^{2k} = \frac{a\left[1 + p^{2k+1}\right]}{1+p}$ | 2 |
| | ii) Hence find the sum $3^n - 3^{n+1} + 3^{n+2} - 3^{n+3} + ............. + 3^{3n}$ | 1 |

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## Question 4 ( Start in a new booklet)

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|---|---|---|
| a. | Find the exact value of $\sin(2\tan^{-1}\frac{1}{\sqrt{3}})$ | 1 |
| b. | Find the general solution of $\tan 2x = -\frac{1}{\sqrt{3}}$, giving answer in radian measure | 2 |
| c. | i) Sketch the curve $y = 3\cos 2x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and shade the region bounded by the curve and the x-axis | 2 |
| | ii) The region in part i) is rotated about the x-axis to form a solid. Find the volume of this solid | 4 |
| d. | Find $\frac{d}{dx}\{\log_e(\sin^{-1} x)\}$ and hence show that $\int_{1/2}^{\sqrt{3}/2} \frac{dx}{\sin^{-1} x\sqrt{1-x^2}} = \log_e 2$ | 3 |

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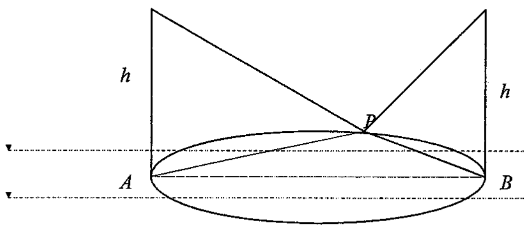
## Question 5 ( Start in a new booklet)

**a.** A special playing field is circular, with radius $r$. Two vertical posts of equal height $h$ are situated at either end, $A$ and $B$, of a diameter of the circle. From a point $P$, on the circumference, the angles of elevation to the tops of the posts are $\alpha$ and $\beta$.




**( recall that the angle in a semi-circle is a right angle )**

i) Copy this diagram and mark all relevant information on it. **1**

ii) Write down expressions for the straight line distances $PA$ and $PB$ in terms of $h$ and the angles $\alpha$ and $\beta$. **1**

iii) Hence show that $h = \dfrac{2r}{\sqrt{\cot^2\alpha + \cot^2\beta}}$ **2**

**b.** $P(6p,3p^2)$, $Q(6q,3q^2)$ are variable points on the parabola $x^2 = 12y$. Given that $pq = 2$, show that the locus of the mid-point of the chord $PQ$ is a parabola whose focus is $\left(0,\dfrac{-9}{2}\right)$ **5**

**c.** Evaluate the definite integral $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3\,dx}{\sqrt{1-x^4}}$ by means of the substitution $u = x^4$ **3**

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## Question 6 ( Start in a new booklet)

**a.** i) Draw a sketch of the curve $f(x) = (x-3)^2 - 2$ **on the number plane provided** **1**

ii) Find the largest domain not containing $x = 0$ for which this function has an inverse **1**

iii) For this domain, find the inverse function $f^{-1}(x)$, stating its domain and range. **2**

iv) On the same sketch as i), draw the inverse function $f^{-1}(x)$ clearly labelling this graph **2**

v) Find the point of intersection of the function and its inverse **2**

**b.** Use the principle of mathematical induction to prove that,

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \dots\dots\dots\dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

for all positive integers of $n$ **4**

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## Question 7 ( Start in a new booklet)

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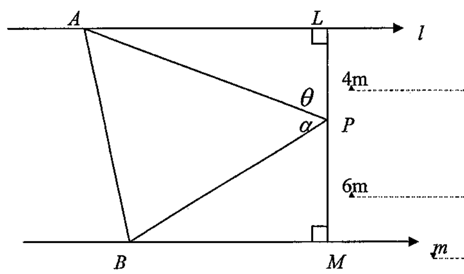
| a. | Let $y = \dfrac{1}{\cos\theta\cos(\theta+\alpha)}$ where $\alpha$ is a constant.<br>Show that $\dfrac{dy}{d\theta} = \dfrac{\sin(2\theta+\alpha)}{\{\cos\theta\cos(\theta+\alpha)\}^2}$ | 3 |
| b. | The diagram shows two parallel lines $l$ and $k$ which are $10m$ apart.<br>The line segment $LM$ is drawn to meet $l$ and $k$ at right angles.<br>$P$ is a fixed point on $LM$, with $PL = 4m$ and $PM = 6m$.<br>$A$ and $B$ are two variable points to the left of $LM$, with $A$ on $l$ and $B$ on $k$.<br>$\angle APB = \alpha$ is fixed. $\angle APL = \theta$ varies with $A$ and $B$.<br> | |
| | i) Show that $AP = \dfrac{4}{\cos\theta}$ | 1 |
| | ii) Show that $BP = \dfrac{-6}{\cos(\theta+\alpha)}$. Note that $\dfrac{\pi}{2} < \theta + \alpha < \pi$ | 1 |
| | iii) Let $A(\theta)$ be the area of triangle $APB$. Show that<br>$A(\theta) = \dfrac{-12\sin\alpha}{\cos\theta\cos(\theta+\alpha)}$ | 2 |
| | iv) Using the result in a), or otherwise, find the value of $\theta$ in terms of $\alpha$ for which $A(\theta)$ has a stationary point.<br>Recall that $\angle APB = \alpha$ is fixed. | 2 |
| | v) For $\alpha = \dfrac{\pi}{6}$, state whether this stationary point is a maximum or a minimum. Justify your answer. | 3 |

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# Solutions

**1** a) $m_1 = \frac{1}{3}$  $m_2 = -\frac{1}{\sqrt{3}}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{1}{3} + \frac{1}{\sqrt{3}}}{1 - \frac{1}{3}\times\frac{1}{\sqrt{3}}}\right| = \left|\frac{\sqrt{3}+3}{3\sqrt{3}-1}\right|$

$= 1.1277$

$\theta = 48°26'$

b) $\theta = \frac{\pi}{6}$

i) $A = \frac{1}{2}r^2(\theta - \sin\theta)$

$5.2 = \frac{1}{2}r^2\left(\frac{\pi}{6} - \sin\frac{\pi}{6}\right)$

$r^2 = \frac{5.2 \times 2}{0.023598775}$

$r = 21$

ii) $l = r\theta$

$= 21 \times \frac{\pi}{6}$

$= 11.0$ cm

c) i) $y = \ln\sqrt{\cos x}$

$= \frac{1}{2}\ln\cos x$

$\frac{dy}{dx} = -\frac{1}{2}\tan x \quad \left(-\frac{1}{2}\frac{\sin x}{\cos x}\right)$

ii) $y = x\tan^2 x$

$\frac{dy}{dx} = 2x\tan x\sec^2 x + \tan^2 x.$

d) i) $\int e^x + e^{-x} - 5e^{-2x}\,dx$

$= e^x - e^{-x} + \frac{5}{2}e^{-2x} + c$

ii) $2\int\frac{2x-3}{x^2-3x-7}\,dx = 2\ln(x^2-3x-7) + c$

e) $\frac{1}{3}\lim_{\theta\to 0}\frac{\tan\frac{\theta}{3}}{\frac{\theta}{3}} = \frac{1}{3}$

**2** a) i)

$A\cos(\theta + \alpha)$

$= A\cos\theta\cos\alpha - A\sin\theta\sin\alpha$

$= 2\cos\theta - \sin\theta$

$\therefore A\cos\alpha = 2,\ A\sin\alpha = 1 \Rightarrow A = \sqrt{5}$ (½) (1)

$\tan\alpha = \frac{1}{2}$

$\alpha = 0.4636$ (½)

$= \sqrt{5}\cos(\theta + 0.4636)$

ii) $\sqrt{5}\cos(\theta + 0.4636) = 1$

$\cos(\theta + 0.4636) = \frac{1}{\sqrt{5}}$ (1)

$\theta + 0.4636 = 1.1071,\ 5.1760$

$\theta = 0.6435,\ 4.7124$ (1)

$= 0.64,\ 4.71$

b) i) $\sin 15 = \sin(45 - 30)$

$= \sin 45°\cos 30° - \cos 45°\sin 30$

$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$ (1)

$\cos 15 = \cos 45°\cos 30° + \sin 45°\sin 30$

$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$ (1)

ii) $RHS = \frac{1-\cos\theta}{\sin\theta} = \frac{1 - \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}}$

$= \frac{1+t^2-1+t^2}{2t} = \frac{2t^2}{2t}$ (2)

$= t = LHS.$

iii) $\tan 7\frac{1}{2} = \frac{1-\cos 15}{\sin 15}$ (1)

$= \frac{1 - \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$ (1)

$= \frac{2\sqrt{6}-3-\sqrt{3}+2\sqrt{2}-\sqrt{3}-1}{2}$

$= \sqrt{6} - 2 + \sqrt{2} - \sqrt{3}$ (1)



**3** a) i) D:

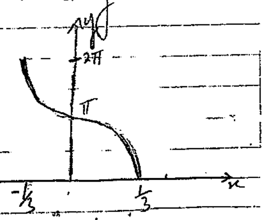
$-1 \le 3x \le 1$

$-\frac{1}{3} \le x \le \frac{1}{3}$

R:

$0 \le \cos^{-1}3x \le \pi$

$0 \le y \le 2\pi$



b) $y = 2\cos^{-1}3x$

$y' = -\frac{2\times 3}{\sqrt{1-9x^2}}$ (1) $= -\frac{12}{\sqrt{3}} = -4\sqrt{3}$ (1)

$= -6.928$ at $x = \frac{1}{6}$

$y = \frac{2\pi}{3}$ (½)

$y - \frac{2\pi}{3} = -6.928\left(x - \frac{1}{6}\right)$

$y = -4\sqrt{3}x + \frac{2}{3}\sqrt{3} + \frac{2\pi}{3}$ (½)

c) i) function is even if

$f(-x) = f(x)$

$f(-x) = \sin^{-1}(\cos -x)$

$= \sin^{-1}(\cos x)$ (1)

$= f(x)$

$\therefore$ even function

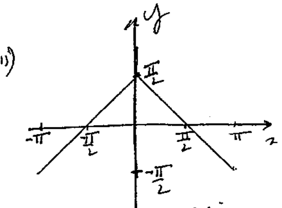
ii) $y = \sin^{-1}\cos x$

$\frac{dy}{dx} = \frac{-\sin x}{\sqrt{1-\cos^2 x}}$

$= \frac{-\sin x}{|\sin x|}$ (1)

$= -1$ for $0 < x < \pi$ (1)

iii)



d) i) G.P. $r = -p$ (½)

$a = a$  $n = 2k+1$ (½)

$S_{2k+1} = \frac{a(1-r^n)}{1-r}$

$= \frac{a(1-(-p)^{2k+1})}{1+p}$ (1)

$= \frac{a(1+p^{2k+1})}{1+p}$

Note $(-1)^{2k+1} = -1$

ii) $a = 3^n$

$N = 2n+1$  $r = -3$

$S_N = \frac{3^n(1+3^{2n+1})}{1+3}$

(½) $= \frac{3^n}{4}(1+3^{2n+1})$ (½)

Let $\tan^{-1}\sqrt{3} = \beta$

4 a) $\sin 2\tan^{-1}\sqrt{3} = \sin 2\beta$


$= 2\sin\beta\cos\beta$

$= 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$.

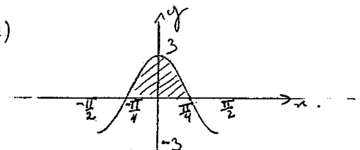
b) $\tan 2x = -\frac{1}{\sqrt{3}}$

$2x = n\pi - \frac{\pi}{6}$

$x = n\frac{\pi}{2} - \frac{\pi}{12}$ when $n$ is an integer.

c) $y = 3\cos 2x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$

i)




ii) $V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y^2\,dx \qquad \cos 4x = 2\cos^2 2x - 1$

$= \pi \int 9\cos^2 2x\,dx$

$= \frac{9\pi}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 + \cos 4x\,dx$

$= \frac{9\pi}{2} \cdot 2 \left[ x + \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{4}}$

$= \frac{9\pi}{2} \cdot 2 \left[ \frac{\pi}{4} + \sin\frac{\pi}{4} \right]$

$= \frac{9\pi^2}{4} u^3$.

d) $\frac{d}{dx} \log_e(\sin^{-1}x) = \frac{1}{\sin^{-1}x\sqrt{1-x^2}}$

$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{d}{dx} \ln(\sin^{-1}x)\,dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sin^{-1}x\sqrt{1-x^2}}\,dx$

$\therefore \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sin^{-1}x\sqrt{1-x^2}}$

$= \left[ \ln \sin^{-1}x \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$

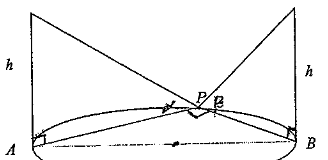
$= \ln \sin^{-1}\frac{\sqrt{3}}{2} - \ln \sin^{-1}\frac{1}{2}$

$= \ln\frac{\pi}{3} - \ln\frac{\pi}{6} = \ln\left(\frac{\pi}{3} \div \frac{\pi}{6}\right)$

$= \ln 2$.

5 a)

i)




ii) $PA = h\cot\alpha$

$PB = h\cot\beta$

iii) Since $\angle APB = 90°$

$h^2\cot^2\alpha + h^2\cot^2\beta = 4r^2$

$h^2(\cot^2\alpha + \cot^2\beta) = 4r^2$

$h = \frac{2r}{\sqrt{\cot^2\alpha + \cot^2\beta}} \qquad h > 0$

b) see next page

c) $\int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3\,dx}{\sqrt{1-x^4}} \qquad u = x^4, \quad du = 4x^3\,dx$

$= \frac{1}{2} \int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-u}}\,du \qquad x = 0 \quad u = 0; \quad x = \frac{1}{\sqrt{2}} \quad u = \frac{1}{4}$

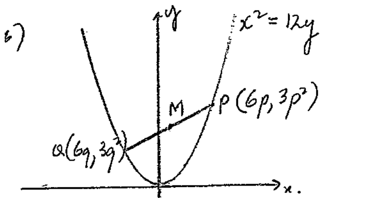
$= \frac{1}{2} \int_0^{\frac{1}{4}} (1-u)^{-\frac{1}{2}}\,du$

$= \frac{-2}{2} \left[ (1-u)^{\frac{1}{2}} \right]_0^{\frac{1}{4}}$

$= -\sqrt{\frac{3}{4}} + 1$



b)




$M \begin{cases} x = 3p + 3q = 3(p+q) \\ y = \frac{3p^2 + 3q^2}{2} = \frac{3}{2}(p^2 + q^2) \end{cases}$

$\therefore \quad y = \frac{3}{2}\left((p+q)^2 - 2pq\right)$

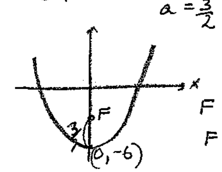
$y = \frac{3}{2}\left(\left(\frac{x}{3}\right)^2 - 2 \times 2\right)$

$\frac{2y}{3} + 4 = \frac{x^2}{9}$

$\therefore \quad x^2 = 9\left(\frac{2y}{3} + 4\right)$

$= 6(y + 6)$

$V(0, -6) \qquad 4a = 6, \quad a = \frac{3}{2}$




$F\left(0, -4\frac{1}{2}\right)$

$F\left(0, -\frac{9}{2}\right)$

6 a) see next page.

b) Prove statement true for $n = 1$

$LHS = \frac{1}{1 \times 3} = \frac{1}{3}$

$RHS = \frac{1}{2+1} = \frac{1}{3} = LHS$.

Assume statement true for $n = k$

$\sum_{n=1}^{k} \frac{1}{(2n-1)(2n+1)} = \frac{k}{2k+1}$

Prove statement true for $n = k+1$

ie $\sum_{n=1}^{k+1} \frac{1}{(2k-1)(2k+1)} = \frac{k+1}{2k+3}$

$LHS = \sum_{n=1}^{k} \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$ using assumption

$= \frac{k(2k+3) + 1}{(2k+1)(2k+3)}$

$= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{k+1}{2k+3} = RHS$.

If the statement is true for $n = k$ we have proved it true for $n = k+1$. However, we proved the statement true for $n = 1$, so the statement is true for $n = 2, 3, 4 \ldots$ for all positive integers.

(4)

6a) $x > 3$
$y > -2$ (1)

ii) $x \geq 3$

iii) $y = (x-3)^2 - 2$

$x = (y-3)^2 - 2$ inverse

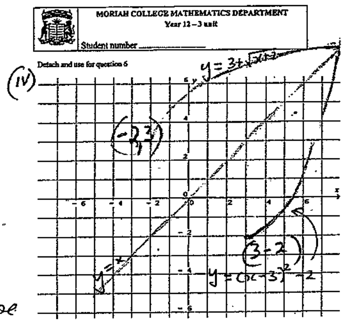
$(y-3)^2 = x+2$ (1)

$y = 3 + \sqrt{x+2}$ is inverse fn.

D: $x \geq -2$

R: $y \geq 3$ (1)

(iv)




v) The function and its inverse intersect on the line $y = x$

$x = 3 + \sqrt{x+2}$

$x - 3 = \sqrt{x+2}$

$(x-3)^2 = x+2$

$x^2 - 7x + 7 = 0$

$x = \frac{7 + \sqrt{49-28}}{2}$

$= 5.79.$

$= \left(\frac{7+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}\right)$ (2)



7

a) $y = \frac{1}{\cos\theta \cos(\theta+\alpha)}$

$= [\cos\theta\cos(\theta+\alpha)]^{-1}$

$\frac{dy}{dx} = -[\cos\theta\cos(\theta+\alpha)]^{-2}[-\cos\theta\sin(\theta+\alpha) - \cos(\theta+\alpha)\sin\theta]$

$= \frac{\sin(\theta+\alpha)\cos\theta + \cos(\theta+\alpha)\sin\theta}{[\cos\theta\cos(\theta+\alpha)]^2}$

$= \frac{\sin(\theta+\alpha+\theta)}{\{\cos\theta\cos(\theta+\alpha)\}^2}$

$= \frac{\sin(2\theta+\alpha)}{\{\cos\theta\cos(\theta+\alpha)\}^2}.$

b(i) $\cos\theta = \frac{4}{AP}$

$\Rightarrow AP = \frac{4}{\cos\theta}$

ii) $\angle BPM = 180 - (\theta+\alpha)$

$\cos(180 - (\theta+\alpha)) = -\cos(\theta+\alpha)$

$\cos\angle BPM = \frac{6}{BP}$

$BP = -\frac{6}{\cos(\theta+\alpha)}.$

iii) Area $\triangle APB = \frac{1}{2}ab\sin\theta$

$A(\theta) = \frac{1}{2} AP.PB \sin\alpha$

$= \frac{1}{2}\cdot\frac{4}{\cos\theta} \times \frac{-6}{\cos(\theta+\alpha)}\sin\alpha$

$= \frac{-12\sin\alpha}{\cos\theta\cos(\theta+\alpha)}$

iv) S.P. when

$A'(\theta) = \frac{-12\sin\alpha\sin(2\theta+\alpha)}{\{\cos\theta\cos(\theta+\alpha)\}^2} = 0$

$\alpha$ is fixed so $\sin\alpha$ is a const

so S.P. when $\sin(2\theta+\alpha) = 0$

when $2\theta + \alpha = \pi$

$\theta = \frac{\pi - \alpha}{2}$

v) $= \frac{5\pi}{12}$ when $\alpha = \frac{\pi}{6}$

Note. denominator of $A'(\theta)$ is always positive, so we consider the numerator only to work out if Max or Min.

| $\theta$ | $\frac{\pi}{3}$ | $\frac{5\pi}{12}$ | $\frac{\pi}{2}$ |
|---|---|---|---|
| $A'(\theta)$ | $-12\times\frac{1}{2}\frac{1}{2}$ | 0 | $-12\times\frac{1}{2}(-\frac{1}{2})$ |
| | $-ve$ | 0 | $+ve$ |

$\sin\left(\frac{2\pi}{3}+\frac{\pi}{6}\right)$ \ — /

$\therefore$ Min.