MORIAH - 2005

Question one

a) Differentiate $y = \tan^{-1}(3x)$ **2**

b) Find the exact value of:

i) $\log_2(\log_2(\sqrt{2}))$ **2**

ii) $\sin(\tan^{-1}(\frac{2}{3}))$ **3**

c) A is the point (-4,7) and B is the point (2,-5). Find the coordinates of the point P, that divides the interval AB externally in the ratio 5:2 **2**

e) Solve the inequality $\dfrac{4}{|2x-3|} \geq \dfrac{2}{3}$ **3**

Question two (Begin a new book.)

a) Differentiate:

i) $e^{\sin(2x)}$ **2**

ii) $\log\left(\dfrac{x^2}{\sqrt{2x-1}}\right)$ **2**

b)

i) Find $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$ **2**

ii) Find the exact value of $\displaystyle\int_0^{\ln 2} \frac{e^x dx}{e^x + 2}$ **3**

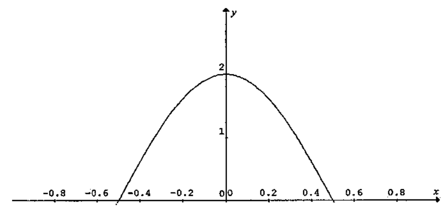
c) State the domain and range of the function $y = 2\cos^{-1}(3-x)$. Draw a neat sketch of this function. **3**

Question three (Begin a new book.)

a) **i)** Express $\sqrt{3}\sin x - \cos x$ in the form $R\sin(x-\alpha)$ where $R \geq 0$ & $0 \leq \alpha \leq \frac{\pi}{2}$. **2**

ii) Hence find the general solution of the equation $\sqrt{3}\sin x - \cos x = 1$. **2**

b) At the entrance to the new Maths Faculty of Moriah College a ceremonial arch is to be erected. The arch which is illustrated below is to be 1 metre wide and 2 metres high.



i) The equation of the arch is $y = 2\cos(kx)$. Explain why $k = \pi$ **2**

ii) Find the area contained by the arch and the horizontal axis. **2**

iii) If the arch was considered to be a parabola,

a) Find the equation of the parabola. **2**

b) Find the difference in area between the two structures (correct to 3 d.p.) **2**

Question four (Begin a new book.)

a) **i)** Draw a neat sketch of the curve $y = e^{3x}$ **1**

ii) Find equation of the tangent to this curve at the point, $P(k, e^{3k})$. **2**

iii) Find the value of k if this tangent passes through the origin. **2**

iv) Hence or otherwise find the values of m for which the equation $e^{3x} = mx$ has two solutions. **2**

b) **i)** If $f(x) = \sqrt{6-x}$, find the equation of $y = f^{-1}(x)$ and write down its domain and range. **3**

ii) Find the point of intersection of the curve and its inverse. **2**

Question five (Begin a new book)

a) Write down the largest natural domain of the function $y = \cos^{-1}(\log_e x)$ **2**

b) Use the method of Mathematical induction to prove that : **4**

$$1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{n}{3}(2n-1)(2n+1)$$

c) **i)** Find the equation of the <u>tangent</u> and <u>normal</u> to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$. **2**

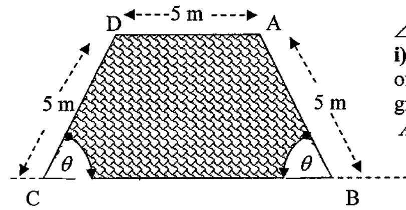
ii) T and G are the points where the tangent and normal meet the axis of the parabola. Find the coordinates of these points. **1**

iii) If S is the focus of the parabola, Show that P, T and G all lie on circle whose centre is S and find the equation of this circle. **3**

Question six (Begin a new book)

a) Find the acute angle between the lines $y = 2x$ and $2y = x$ correct to the nearest minute. **2**

b) The illustration below is part of the cross section of the roof of the new Mathematics faculty.



$\angle ABC = \angle DCB = \theta$.

i) Show that the area of this cross section is given by: **2**
$A = 25\sin\theta(1+\cos\theta)$

ii) Find the value of θ which will make this area a maximum. **4**

c) **i)** Find the derivative of $\sin^{-1}(x-1)$ in simplest form. **2**

ii) Hence show that $\int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{x(2-x)}} = \frac{\pi}{6}$ **2**

Question seven (Begin a new book.)

a) **i)** On the same diagram sketch the curves $y=\sin x+1$ and $y=\cos x$ in the domain $0 \le x \le \frac{\pi}{2}$. **2**

ii) Shade the region bounded by these two curves and the line $x=\frac{\pi}{2}$ **1**

iii) Find the volume formed when this region is rotated about the x-axis. **3**

b) **i)** Draw a neat sketch of the function, $y=\log_e(x-1)$. **1**

ii) This function meets the line $y = 2$ at the point P and the x- axis at the point Q. **2**
Find the coordinates of the points P and Q and label them clearly on your sketch.

iii) If S is the point (0,2), find the co-ordinates of the point R if $OSPR$ is a rectangle. Label the points S and R on your sketch. **1**

iv) Show that the arc PQ, divides the rectangle $OSPR$ into two regions of equal area. **2**

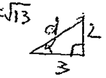
Question ①

a) $\frac{dy}{dx} = \frac{3}{1+9x^2}$

b) i) $\log_2(\sqrt{2}) = \log_2(2)^{\frac{1}{2}} = \frac{1}{2}$

$\therefore \log_2(\log_2\sqrt{2}) = \log_2(\frac{1}{2}) = -1$

ii) Let $\tan^{-1}\frac{2}{3} = \alpha$



$\tan\alpha = \frac{2}{3} \Rightarrow \sin\alpha = \frac{2}{\sqrt{13}}$

$\therefore \sin(\tan^{-1}(\frac{2}{3})) = \frac{2}{\sqrt{13}}$

c) $A(-4,7)$ $B(2,-5)$

$5:-2$

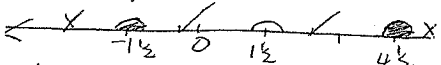
P is $\left(\frac{8+10}{3}, -\frac{39}{3}\right) = (6,-13)$

d) $x \neq 3/2$

consider $\frac{4}{|2x-3|} = \frac{2}{3}$

$6 = |2x-3|$

$x = 4\frac{1}{2}, -1\frac{1}{2}$



test: -10 ✗, 10 ✗, 0 ✓, 2 ✓

$\therefore$ Solution is:

$-1\frac{1}{2} \leq x < 0$

$0 < x \leq 4\frac{1}{2}$

Question ②

a) i) $\frac{dy}{dx} = 2\cos 2x e^{\sin 2x}$

ii) $y = 2\log x - \frac{1}{2}\log(2x-1)$

$\frac{dy}{dx} = \frac{2}{x} - \frac{1}{2x-1}$

$= \frac{3x-2}{x(2x-1)}$

b) i) $\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\sin^{-1}(2x) + C$

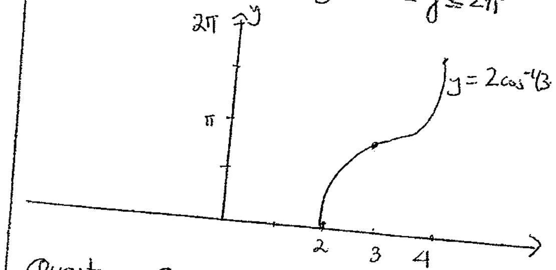
ii) $[\ln(e^x+2)]_0^{\ln 2} = \ln 4 - \ln 3$

$= \ln(4/3)$.

c) $-1 \leq 3-x \leq 1$ $\quad 0 \leq \cos^{-1}(\ ) \leq \pi$

$-4 \leq -x \leq -2$ $\quad \therefore 0 \leq 2\cos^{-1}(\ ) \leq 2\pi$

Dom: $2 \leq x \leq 4$ Range: $0 \leq y \leq 2\pi$



Question ③

a) $R\sin(x-\alpha) = \sqrt{3}\sin x - \cos x$.

$\Rightarrow R\sin x\cos\alpha - R\cos x\sin\alpha = \sqrt{3}\sin x - \cos x$

So, $R\cos\alpha = \sqrt{3}$ & $R\sin\alpha = 1$ $\Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) = 4$, $R = 2$

& $\sin\alpha = \frac{1}{2}$ $\cos\alpha = \frac{\sqrt{3}}{2}$ $\Rightarrow \alpha = \frac{\pi}{6}$

$\therefore \sqrt{3}\sin x - \cos x = 2\sin(x - \pi/6)$.

ii) Equation becomes:–

$2\sin(x-\pi/6) = 1$

$\sin(x-\pi/6) = \frac{1}{2}$

$x - \pi/6 = n\pi + (-1)^n \pi/6$.

$x = n\pi + (-1)^n \pi/6 + \pi/6$

$x = n\pi$ (n odd)

$x = n\pi + \pi/3$ (n even)

b) i) When $x=0$ $y = 2\cos(K.0) = 2$.

$x = \pi$ then $2\cos(\pi x) = 0$ when

$\pi x = \pm\frac{\pi}{2}$ $\therefore x = \pm\frac{1}{2}$ which fits the intercepts made by the graph.

$\therefore K = \pi$.

ii) $A = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\cos\pi x\, dx = \frac{1}{\pi}\left[2\sin\pi x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= 4/\pi\ u^2$.

iii) Let the equation by $(x-h)^2 = -4a(y-k)$

Vertex $(0,2)$ $\therefore (x)^2 = -4a(y-2)$

passes through $(\frac{1}{2},0)$ $\frac{1}{4} = 8a$ $\therefore a = 1/32$

c) iii) $\therefore$ all 3 points lie on a circle centre S with radius $a(p^2+1)$ whose equation is

$x^2 + (y-a)^2 = a^2(p^2+1)^2$.

Question ⑥

a) m of $y = 2x+2 = m_1$

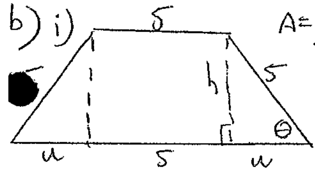
m of $2y = x$ is $\frac{1}{2} = m_2$.

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right| = \left|\frac{2-\frac{1}{2}}{1+2.\frac{1}{2}}\right|$

$= 3/4$

$\therefore \theta =$

b) i)



$A = \frac{1}{2}h(5+5+2u)$

$h/5 = \sin\theta$

$h = 5\sin\theta$

$\frac{u}{5} = \cos\theta$

$u = 5\cos\theta$

$\therefore A = \frac{1}{2}.5\sin\theta(10 + 2(5\cos\theta))$

$= \frac{1}{2}.50\sin\theta(1+\cos\theta)$

$= 25\sin\theta(1+\cos\theta)$.

ii) For max.

$\frac{dA}{d\theta} = 0$, i.e. $\frac{d}{d\theta}(\sin\theta + \sin\theta\cos\theta) = 0$

$\cos\theta + \cos^2\theta - \sin^2\theta = 0$

$\cos\theta + \cos^2\theta - (1-\cos^2\theta) = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}, -1$ $\quad \theta = \pi/3, \pi$

$\frac{d^2A}{d\theta^2} = 4\cos\theta.-\sin\theta - \sin\theta$

$\theta = \frac{\pi}{3}$ $\frac{d^2A}{d\theta^2} = 4.\frac{1}{2}.-\frac{\sqrt{3}}{2} - \frac{1}{2} < 0$

$\therefore$ max.

for maximum area $\theta = \pi/3$.

c) i) $y = \sin^{-1}(x-1)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x-1)^2}} = \frac{1}{\sqrt{1-(x^2-2x+1)}}$

$= \frac{1}{\sqrt{2x-x^2}}$.

ii) $\int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{x(2-x)}} = \int_{\frac{1}{2}}^{1} \frac{dx}{\sqrt{2x-x^2}}$

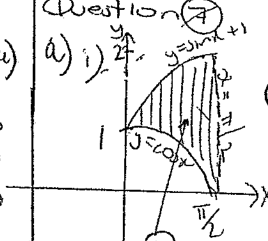
$= \left[\sin^{-1}(x-1)\right]_{\frac{1}{2}}^{1}$

$= \sin^{-1}(0) - \sin^{-1}(-\frac{1}{2})$

$= 0 - -\pi/6 = \pi/6$

Question ⑦

a) i)



②

iii) $V = \pi\int_0^{\frac{\pi}{2}} (\sin x+1)^2 - \cos^2 x\, dx$

$= \pi\int_0^{\frac{\pi}{2}} \sin^2 x + 2\sin x$

$= \pi\int_0^{\frac{\pi}{2}} 1 - \cos 2x + 2\sin x\, dx$

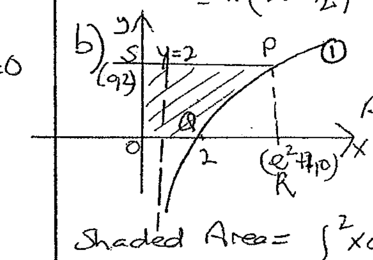
$= \pi\left[x - \frac{1}{2}\sin 2x - 2\cos x\right]_0^{\frac{\pi}{2}}$ ①

$= \pi\left[(\frac{\pi}{2} - 0 - 0) - (0 - 0 - 2)\right]$

$= \pi(2 + \frac{\pi}{2})u^3$ ①

②

b)



$Q(2,0)$ $P(e^2+1, 2)$

R is $(e^2+1, 0)$ ①

Area of OSPR

$= (e^2+1) \times 2$

$= 2e^2 + 2$

Shaded Area $= \int_0^2 x\, dy$

$= \int_0^2 e^y + 1\, dy = (e^y + y)_0^2$

$= e^2 + 2 - e^0$.

$= e^2 + 1u^2$ ①

$= \frac{1}{2}(2e^2+2)$ ①

$= \frac{1}{2}$ Area of rectangle.

$\therefore$ Arc divides rectangle into two equal portions of area e^2+1u^2