Student Name: ....

Teacher: ......

MORIAH WAR MEMORIAL COLLEGE

MATHEMATICS

Advanced

Year 12

TASK 1

Tuesday 4th December, 2007

Directions to Candidates

- Attempt ALL questions
- The marks for each question are clearly indicated on the examination paper.
- All necessary working must be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Time allowed: 1 hour 30 minutes
- *Start each question on a new page*

Question 1 (9 marks)

(a) Factorise $2x^2 + 3x - 2$ **1**

(b) Solve $x^2 + x - 4 = 0$ leaving your answer in simplest surd form **2**

(c) Solve simultaneously $y = x^3 + x^2$ and $y = x + 1$ **2**

(d) Find a primitive function of $2x + 11$ **2**

(e) The formula $S = 12t + 5t^2$ gives the distance travelled S, in metres by an object in t seconds. The speed of the object at any time t is given by $\dfrac{dS}{dt}$. **2**

Find the speed of the object after 2 seconds.

Question 2 (16 marks)

(a) Let $y=(x^2+1)^5$

i. Differentiate y using the chain rule **1**

ii. Use the product rule to find the second derivative and simplify your answer. **2**

(b) Differentiate with respect to x:

i. $\dfrac{4x-1}{x^2-3x}$ **2**

ii. $\sqrt{x}+\dfrac{4}{x^2}$ **2**

(c) Given $f(x)=x^4-4x^3$, in the domain $-1\le x\le 5$

i. Find the coordinates of any stationary points and determine their nature **4**

ii. Find the coordinates of any points of inflexion **1**

iii. Hence, sketch $y=f(x)$, clearly showing all the points found above **2**

iv. Find where the curve $y=f(x)$ is increasing **1**

v. Find the global maximum of $f(x)$ in the restricted domain **1**

Question 3 (12 marks)

(a) If α and β are roots of the quadratic $x^2-6x+3=0$, find:

i. $\alpha+\beta$ **1**

ii. $\alpha\beta$ **1**

iii. $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ **2**

iv. $\alpha^2+\beta^2$ **2**

(b) For what values of a is $f(x)=ax^2+8x+a$

i. Positive definite **2**

ii. Negative definite **1**

(c) Find the values of A, B and C if
$4x^2-3x+7\equiv A(x+4)^2+B(x+4)+C$ **3**

Question 4 (13 marks)

(a) Find the primitive for each of the following functions:

i. $3x^2-2x+1$ **1**

ii. $(3x-1)^3$ **2**

(b) Let $f''(x)=12x-42x^5$. Find $f(x)$ if $f(1)=f'(1)=-2$. **2**

(c) On the last page you will find three number planes. Write your name on the sheet and tear it off before answering this question.

The second number plane contains the graph of $y=f'(x)$

i. Use the graph $y=f'(x)$ and sketch the graph of $y=f''(x)$ on the bottom set of axes. **1**

ii. Explain the significance of Point A and its relation to the primitive function. (Use the space provided next to the graphs). **1**

iii. Hence sketch the primitive function $y=f(x)$ on the top set of axes. Label all stationary points. **4**

iv. Explain in no more than two sentences why it is possible to have more than one correct answer for the sketch of the primitive function $y=f(x)$. (Use the space provided next to the graphs). **2**

(Graphs on last page – tear off and hand in with your solutions)

Question 5 (18 marks)

(a) A Perth company manufactures items that cost \$2 each. All items are sold each month at $\$x$ per item.

i. The company sells $\dfrac{800}{x^2}$ items per month. Show that the total monthly profit (\$P) can be represented by $P=\dfrac{800}{x}-\dfrac{1600}{x^2}$, $(x\neq 0)$. **2**

ii. Find the value of x for which the company could expect to maximise its monthly profit and calculate this maximum profit. **4**

(b) Given the quadratic equation $2x^2-(m+2)x+m=0$

i. Find the value(s) of m that give the equation exactly one real root **2**

ii. Find m, such that the equation has two roots, one twice the other. **3**

(c) Let $Q(x)=-3x^2-12x-7$

i. Find, by completing the square, the maximum value of $Q(x)$ **3**

ii. Hence, or otherwise sketch $y=Q(x)$ **2**

iii. Test your solution of the maximum value from part (i) using calculus and show that you are correct **2**

2 unit.

Question 1.

(a) $2x^2+3x-2 = 2x^2+4x-x-2$

$= 2x(x+2)-1(x+2)$

$= (x+2)(2x-1)$ ✓

$P=-4$

$S=3$

$F: 4, -1$

(b) $x^2+x-4=0 \Rightarrow x = \dfrac{-1 \pm \sqrt{1-4\times(-4)}}{2}$ (1)

$= \dfrac{-1 \pm \sqrt{17}}{2}$ ✓✓

(c) $y = x^3+x^2$ —(1)

$y = x+1$ —(2)

$(1)=(2) \Rightarrow x^3+x^2 = x+1$

$x^2(x+1)-(x+1)=0$

$(x^2-1)(x+1)=0$

$(x+1)^2(x-1)=0$

$\therefore x = \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$ ✓

y ✓ $=$

$\therefore$ Soln's $(-1,0)$ & $(1,2)$

(d) let $\dfrac{dy}{dx} = 2x+11$ ✓

$\therefore y = x^2+11x+C$ ✓ (−1 for +C)

(e) $S = 12t+5t^2$

$\dfrac{dS}{dt} = 12+10t$ ✓

when $t=2 \Rightarrow \dfrac{dS}{dt} = 12+10\times 2 = 32\ ms^{-1}$ ✓

Question 2.

(a) $y = (x^2+1)^5$

(i) $\dfrac{dy}{dx} = 5(x^2+1)^4 \times 2x = 10x(x^2+1)^4$ ✓ 1

(ii) let $u = 10x$, $v = (x^2+1)^4$ ✓

$u' = 10$, $v' = 4(x^2+1)^3 \times 2x$

$= 8x(x^2+1)^3$ ✓ 1

now $\dfrac{d^2y}{dx^2} = vu' + uv'$

$= 10(x^2+1)^4 + 80x^2(x^2+1)^3$

$= 10(x^2+1)^3[x^2+1+8x^2]$

$= 10(x^2+1)^3(9x^2+1)$ ✓ 1

(b) i) let $y = \dfrac{4x-1}{x^2-3x}$ and let $u = 4x-1$, $v = x^2-3x$

$u' = 4$, $v' = 2x-3$ ✓

now $\dfrac{dy}{dx} = \dfrac{vu'-uv'}{v^2} = \dfrac{4(x^2-3x)-(4x-1)(2x-3)}{(x^2-3x)^2}$

$= \dfrac{4x^2-\cancel{12x}-[8x^2-\cancel{12x}-2x+3]}{(x^2-3x)^2}$ ✓

$= \dfrac{-4x^2+2x-3}{(x^2-3x)^2}$ ✓

b ii) let $y = x^{1/2}+4x^{-2}$ ✓

$y' = \frac{1}{2}x^{-1/2} - 8x^{-3}$

$= \dfrac{1}{2\sqrt{x}} - \dfrac{8}{x^3}$ ✓

(c) $f(x) = x^4 - 4x^3 \qquad (-1 \le x \le 5)$

(i) $f'(x) = 4x^3 - 12x^2$

let $f'(x) = 0 \Rightarrow 4x^2(x-3) = 0$

$\therefore x = 0$ and $x = 3$

$f(0) = 0 \qquad f(3) = -27$

now $f''(x) = 12x^2 - 24x$

$= 12x(x-2) \quad \Rightarrow f''(0) = 0, \quad f''(3) = 36 > 0$

Hence $(3, -27)$ is a minimum local turning pt.

Test $(0,0)$

x	-1	0	1
$f'(x)$	\	—	\

4

Hence $(0,0)$ is a H.P.I

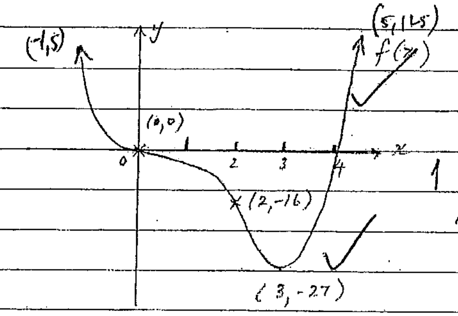
(ii) From (i) $f''(x) = 12x(x-2)$

let $f''(x) = 0 \Rightarrow x = 0$ and $x = 2$

From (i) $(0,0)$ is a H.P.I. $\qquad f(2) = -16$

and $(2, -16)$ is a pt of inflexion.

(iii)



1 shape

1 points (end)

(iv) $f(x)$ is increasing when $5 > x > 3$

(v) $f(-1) = 4 \times -4 = -16$

$f(5) = 4 \times 25 \times 2 = 200 \quad \therefore$ global max is $f(5) = \cancel{200}$ 125

Question 3

(a) $x^2 - 6x + 3 = 0$ has roots α, β

(i) $\alpha + \beta = \frac{-b}{a} = \frac{-(-6)}{1} = 6$

(ii) $\alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{6}{3} = 2$

(iv) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (6)^2 - 2 \times 3$

$= 30$

(b) $f(x) = ax^2 + 8x + a$

(i) Positive definite when $a > 0$ and $\Delta < 0$.

$\therefore (8)^2 - 4 \times a \times a < 0$

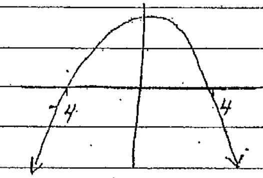
No marks for $a < \pm 4$

$8^2 - 4a^2 < 0$

$(8 - 2a)(8 + 2a) < 0$

$\therefore -4 > a > 4$

but both conditions must hold hence $a > 4$



$a < -4 \quad a > 4$

(ii) Negative definite when $a < 0$, hence from work in part (i) $a < -4$

(c) $4x^2 - 3x + 7 \equiv A(x+4)^2 + B(x+4) + C$

$\equiv A(x^2 + 8x + 16) + Bx + 4B + C$

$\equiv Ax^2 + (8A + B)x + (16A + 4B + C)$

$\therefore$ Equating coefficients $\Rightarrow A = 4$

$B = -35$

$C = 83$

Question 4

$-\frac{1}{2}$ no "c"

(ai) let $y' = 3x^2 - 2x + 1$

$\therefore y = x^3 - x^2 + x + C$ ✓ 1.

(ii) let $\frac{dy}{dx} = (3x-1)^3$

$\therefore y = \frac{1}{4} \times (3x-1)^4 \times \frac{1}{3}$ ✓ 2.

$= \frac{1}{12}(3x-1)^4 + c$ ✓

(b) Given $f''(x) = 12x - 42x^5$

$\therefore f'(x) = 6x^2 - 7x^6 + C$ ✓

now $f'(1) = -2 \Rightarrow -2 = 6 - 7 + C$

$\therefore C = -1$ ✓

$\therefore f'(x) = 6x^2 - 7x^6 - 1$ ✓

hence $f(x) = 2x^3 - x^7 - x + D$ 2.

now $f(1) = -2 \Rightarrow -2 = 2 - 1 - 1 + D$

$-2 = D$ ✓

$\therefore f(x) = 2x^3 - x^7 - x - 2$

(c) See graph page.

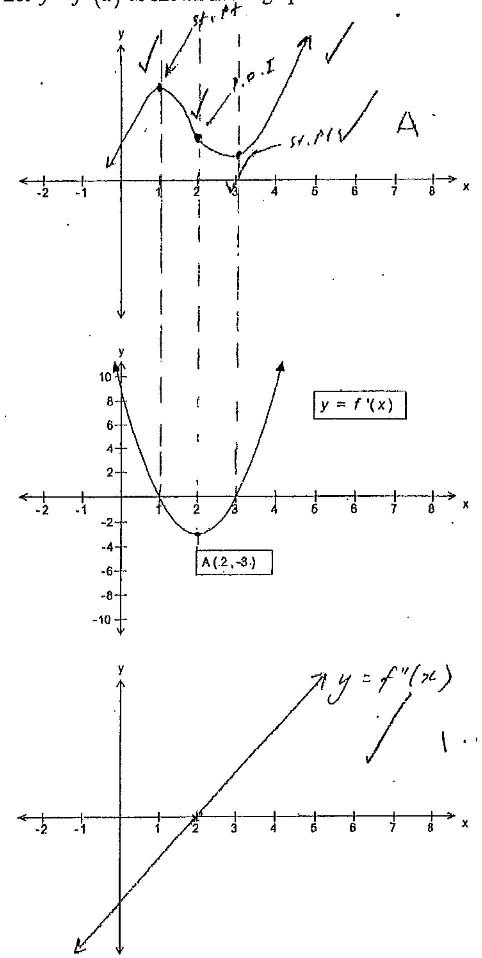
Start each question on a new page

Student Name: Anne Surs

***Remove this page and hand in with your solutions**

(Question 4, Part C continued)

Let $y = f'(x)$ as shown in the graphs below



4c (ii)

It must have been a point of inflexion. ✓ 1.

4c (iv)

Because we don't know the constant term which determines the vertical shift of the graph there can be any number of solutions. ✓ 2.

8.

Question 5.

(a) i) $P = \frac{800}{x^2} \times (x-2)$

$= \frac{800}{x} - \frac{1600}{x^2}$

QED

ii) $\frac{dP}{dx} = \frac{-800}{x^2} + \frac{3200}{x^3}$

let $\frac{dP}{dx} = 0 \Rightarrow \frac{800}{\cancel{x^2}} = \frac{3200}{\cancel{x^3}}$

$8x = 32$

$x = 4$ is a st. pt

x	3	4	5
$\frac{dP}{dx}$	/	—	\

∴ $x = 4$ gives P a max value.

∴ Max mthly profit $= \frac{800}{4} - \frac{1600}{(4)^2}$

$= 200 - 100$

$= \$100$

(b) Given $2x^2 - (m+2)x + m = 0$

(i) exactly one real root $\Rightarrow \Delta = 0$

$\therefore b^2 - 4ac = 0$

$(m+2)^2 - 4 \times 2 \times m = 0$

$m^2 + 4m + 4 - 8m = 0$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$\therefore m = 2$

5b cont.

(ii) let roots be α and 2α

Sum: $\alpha + 2\alpha = \frac{m+2}{2}$

$6\alpha = m + 2$ — (1)

Product: $\alpha \times 2\alpha = \frac{m}{2}$

$4\alpha^2 = m$ — (2)

(2) in (1) ∴ $6\alpha = 4\alpha^2 + 2$

$4\alpha^2 - 6\alpha + 2 = 0$

$2\alpha^2 - 3\alpha + 1 = 0$ P: 2

$2\alpha^2 - 2\alpha - \alpha + 1 = 0$ S: −3

$2\alpha(\alpha - 1) - 1(\alpha - 1) = 0$ F: −2, −1

$(2\alpha - 1)(\alpha - 1) = 0$

$\therefore \alpha = \frac{1}{2}$ and $\alpha = 1$

Hence $m = 1$ and $m = 4$

(c) $Q(x) = -3x^2 - 12x - 7$

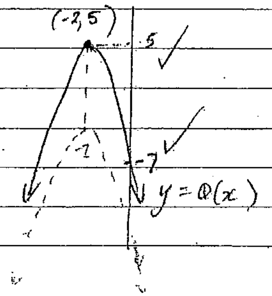
i) $Q(x) = -3[x^2 + 4x] - 7$

$= -3[(x+2)^2 - 4] - 7$

$= -3(x+2)^2 + 12 - 7$

$= -3(x+2)^2 + 5$ ∴ max value is 5.

ii)



(iii) $Q'(x) = -6x - 12$

when $Q'(x) = 0$

$\therefore -6x = 12$

$x = -2$

$\therefore Q(-2) = -3 \times 4 - 12 \times -2 - 7$

$= -12 + 24 - 7$

$= 12 - 7$

$= 5$

Test for max