MORIAH COLLEGE

Year 12

MATHEMATICS ADVANCED

ASSESSMENT TASK 3 - 2008

Time Allowed:	1 hour 30 minutes, plus 5 mins reading time
Examiners:	VS, DL, BO, GO
Date:	12th June, 2008
Time:	9:00 am – 10:35 am

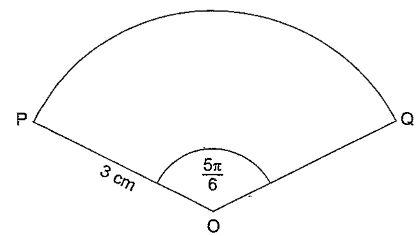
Instructions:

- **Show all necessary working**
- **Marks may be deducted for careless or badly arranged work**
- **Start each question in a new booklet or on a new page**
- **ALL QUESTIONS ARE OF EQUAL VALUE**
- **Approved calculators may be used**

Question 1 (12 marks)

a) Convert 135° to radians in exact form **1**

b) In the diagram below, PQ is the arc of a circle with centre O. **1**
The radius $OP = 3\text{cm}$ and the angle POQ is $\frac{5\pi}{6}$ radians.



Find the exact length of the arc PQ.

c) Find in simplest form the exact value of:

$$\int_0^{\ln 7} e^{-x}\,dx$$ **2**

d) Differentiate the following functions:

i) $y = 2e^{-x}$ **2**

ii) $y = \ln(x^2+1)$ **2**

iii) $\left(e^{2x}+5\right)^3$ **2**

iv) $x^3 \cos x$ **2**

Question 2 (12 marks)

a) Differentiate

i) $e^{\sin x}$ **2**

ii) $x\ln(x+1)$ **2**

iii) $\dfrac{\sin(ax+1)}{\cos(ax+1)}$ **2**

b) Evaluate this indefinite integrals

i) $\displaystyle\int \frac{6}{\operatorname{cosec} 2x}\,dx$ **2**

ii) $\displaystyle\int \sec^2 6x\,dx$ **1**

c) Using a substitution of $u = e^x$, solve $e^{2x} - e^x - 1 = 0$ to show that the only possible solution is approximately $0\cdot481$. **3**

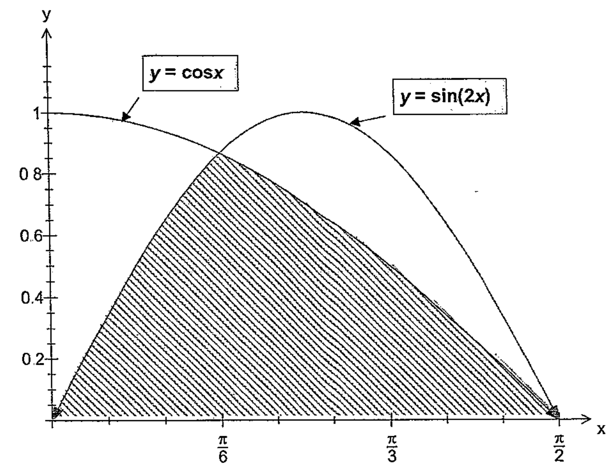
Question 3 (12 marks)

a) Sketch $y = 5\sin 4x$ for $0 \le x \le \pi$ **3**

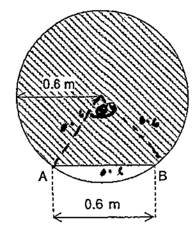
b) The diagram shows the graphs of the functions $y = \cos x$ and $y = \sin 2x$ between $x = 0$ and $x = \dfrac{\pi}{2}$.

i) Show that the two graphs intersect at $x = \dfrac{\pi}{6}$ and $x = \dfrac{\pi}{2}$. **2**

ii) Hence, calculate the area bounded by the curves and the x-axis **3**



c) A table top is in the shape of a circle with a small segment removed as shown. The circle has centre O and radius 0.6 metres. The length of the straight edge AB is also 0.6 metres.



i) Explain why $\angle AOB = \frac{\pi}{3}$ radians 1

ii) Find the area of the table top to 2 decimal places **3**

Question 4 (12 marks)

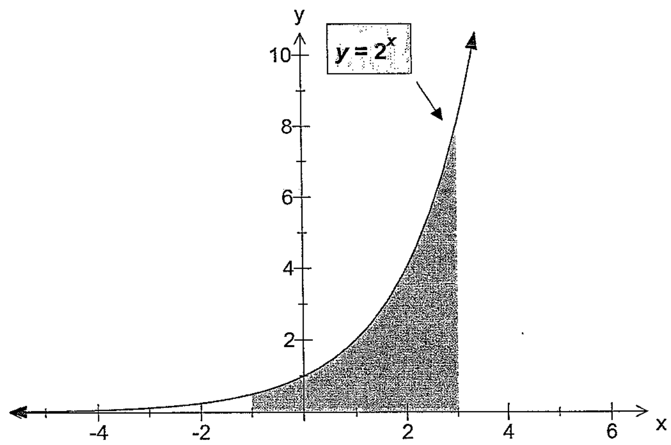
a) Show by using $y = a^x$ that $\int a^x dx = \frac{a^x}{\ln a} + C$ **2**

b) Consider the function $y = 2^x$

i) Copy and complete the table below into your work booklet. **1**

x	-1	0	1	2	3
2^x					

ii) Using Simpson's rule with these five function values, find an estimate for the area shaded in the diagram below. **2**



iii) Find the exact value of $\int_{-1}^{3} 2^x dx$ and calculate your percentage error from using Simpson's rule in part (ii) to 2 decimal places **3**

c) Evaluate:

i) $\lim_{x \to 0} \frac{\sin 4x}{x}$ **1**

ii) Find the diameter of the moon to the nearest kilometre if the distance of its centre is 382 500 km from the earth. The angle subtended by the moon at the earth is 0°30'. **3**

Question 5 (12 marks)

a)

i) Draw a neat sketch of the curve $y = e^{-x} + 1$ **2**

ii) The region contained by the curve at the ordinates $x = 0$ and $x = 2$ is rotated about the x-axis. Find the volume formed correct to 2 d.p. **3**

b) A baker cuts a sector out of a pastry circle with radius 5cm and with an angle of $\frac{2\pi}{3}$ subtended at the centre. This sector is then curved around to form a waffle cone.

i) Find the external surface area of the cone **2**

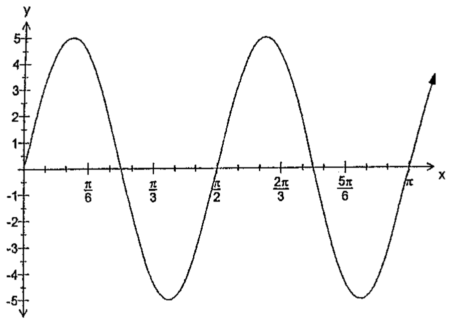
ii) Show that the radius of the open circular base of the cone is $\frac{5}{3}$ cm. **2**

iii) By finding the exact height of the cone, find the capacity of the cone in exact form. (NB: Volume of a cone = $\frac{1}{3}A \times h$) **3**

Solutions to Task 3, Mathematics Advanced, 2008

Question 1 (12 marks)	Mark Allocation
a) $135°\times\frac{\pi}{180}=\frac{3\pi}{4}$	1 mark
b) $l=r\theta=3\times\frac{5\pi}{6}=\frac{5\pi}{2}$ cm	1 mark
c) $\int_0^{\ln 7} e^{-x}dx=\left[-e^{-x}\right]_0^{\ln 7}$ $=\left(-e^{-\ln 7}+e^{0}\right)$ $=1-e^{\ln(7^{-1})}$ $=1-\frac{1}{7}=\frac{6}{7}$	1 mark (integration) 1 mark (simplified answer)
d)	
i) $y=2e^{-x}$ $y'=-2e^{-x}$	2 marks (minus 1 for any errors)
ii) $y=\ln(x^2+1)$ $y'=\frac{2x}{x^2+1}$	2 marks (minus 1 for any errors)
iii) $y=\left(e^{2x}+5\right)^3$ let $\frac{dy}{dx}=3\left(e^{2x}+5\right)^2\times\left(2e^{2x}\right)$ $=6e^{2x}\left(e^{2x}+5\right)^2$	2 marks (chain rule) (minus 1 for any errors)
iv) $y=x^3\cos x$ let $\frac{dy}{dx}=3x^2\cos x-x^3\sin x$ $=x^2\left(3\cos x-x\sin x\right)$	2 marks (chain rule) (minus 1 for any errors)

Question 2 (12 marks)	Mark Allocation
a) Differentiate	
i) Let $y=e^{\sin x}$ $y'=\cos x\times e^{\sin x}$	2 marks (minus 1 for any errors)
ii) Let $y=x\ln(x+1)$ $\frac{dy}{dx}=\ln(x+1)+\frac{x}{x+1}$	1 mark (differentiation of u and v) 1 mark (correct use of prod. rule)
iii) $y=\frac{\sin(ax+1)}{\cos(ax+1)}$ $=\tan(ax+1)$ $y'=\sec^2(ax+1)\times a$ $=a\sec^2(ax+1)$	1 mark (conversion to tan and differentiating / or quotient rule splits) 1 mark (constant multiplier) *[no deduction for answer left as the reciprocal of cos function]*
b) Evaluate this indefinite integrals	
i) $I=\int\frac{6}{\operatorname{cosec}2x}dx$ $=\int 6\sin 2x dx$ $=6\times-\frac{1}{2}\cos 2x+C$ $=-3\cos 2x+C$	1 mark (conversion to sin and integration) 1 mark (negative sign) *[minus 1 for forgetting + C on this question]*
ii) $I=\int\sec^2 6x\,dx$ $=\frac{1}{6}\tan 6x+C$	1 mark *[no deduction for forgetting + C on this question]*
c) Using a substitution of $u=e^x$, solve $e^{2x}-e^x-1=0$ $u^2-u-1=0$ $u=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$ $=\frac{1\pm\sqrt{5}}{2}$ Hence: $u=e^x=\frac{1+\sqrt{5}}{2}$ or $e^x=\frac{1-\sqrt{5}}{2}$ $e^x=1.618$ or $e^x=-0.618$ $\ln\left(e^x\right)=\ln(1.618)$ no soln $x=0.481$	1 mark (quadratic form answer) 1 mark (resubstitution of u as e^x) 1 mark (final answer, with discounting of second soln) *[minus 1 for not discounting second soln]*

Question 3 (12 marks)	Mark Allocation
a) Sketch $y = 5\sin 4x$ for $0 \le x \le \pi$ Period $= \frac{2\pi}{4} = \frac{\pi}{2}$ Amplitude = 5 units  	1 mark (for correct period) 1 mark (for correct amplitude) 1 mark (for correct curve)
b) i) By substituting $x = \frac{\pi}{6}$, $y_1 = \cos x = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $y_2 = \sin 2x = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ therefore is a soln. By substituting $x = \frac{\pi}{2}$, $y_1 = \cos x = \cos\frac{\pi}{2} = 0$ $y_2 = \sin 2x = \sin\pi = 0$ therefore is a soln.	1 mark (correct substitution) 1 mark (correct substitution)

Question 3 continued

ii) Area = $= \int_0^{\frac{\pi}{6}} \sin 2x\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx$ $= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}} + \left[\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $= -\frac{1}{2}\left(\cos\frac{\pi}{3} - \cos 0\right) + \left(\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right)$ $= -\frac{1}{2}\left(\frac{1}{2} - 1\right) + \left(1 - \frac{1}{2}\right)$ $= \frac{1}{4} + \frac{1}{2}$ $= \frac{3}{4}$ sq units	1 mark (correct set up of integral) 1 mark (correct integration) 1 mark (final answer)
c)	
i) Triangle is equilateral, all sides 0.6m	1 mark
ii) Area of segment = $= \frac{1}{2}r^2(\theta - \sin\theta)$ $= \frac{1}{2}(0.6)^2\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$ $= 0.0326$ Hence area of table = circle – segment = $= \pi(0.6)^2 - 0.0326...$ $= 1.098...$ $= 1.10$ sq units (2dp)	1 mark (area of minor segment) 1 mark (set up with subtraction from whole circle area) 1 mark (final answer to 2dp) *[minus ½ mark for incorrect rounding]*

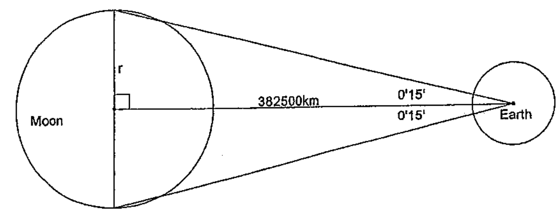
Question 4 (12 marks)	Mark Allocation
a) Given $y = a^x$ $\ln y = \ln a^x$ $\ln y = x\ln a$ $y = e^{x\ln a}$ then $\int a^x dx = \int e^{x\ln a} dx = \frac{e^{x\ln a}}{\ln a} + C$ $= \frac{a^x}{\ln a} + C$	1 mark (manipulation of y) 1 mark (clearly shown) *[Note award marks if shown by differentiating and working backwards, but every step must be shown clearly]*
b)	
i)	

x	-1	0	1	2	3
2^x	½	1	2	4	8
Factor	1	4	2	4	1
F*2^x	½	4	4	16	8
Sum:					32½

1 mark (no errors)
1 mark (total sum of 32.5)

ii) Area $\approx \frac{1}{3} \times 32\frac{1}{2} \approx 10\frac{5}{6} \approx 10.8\dot{3}$ sq.units	1 mark (in exact form)
iii) $\int_{-1}^{3} 2^x dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{3} = \left(\frac{2^3}{\ln 2} - \frac{2^{-1}}{\ln 2}\right)$ $= \left(\frac{8 - \frac{1}{2}}{\ln 2}\right) = \left(\frac{15}{2\ln 2}\right) = 10.82021...$ Percentage error = $\frac{(10.8\dot{3} - 10.82021...)}{10.82021...} \times 100 = 0.12\%$	1 mark (integration) 1 mark (exact) 1 mark (% error to 2dp) *[Do not subtract a mark for rounding error here]*
c) Evaluate:	
i) $\lim_{x\to 0} \frac{\sin 4x}{x} = 4\lim_{x\to 0} \frac{\sin 4x}{4x} = 4$	1 mark

Question 4 continued

ii) 	1 mark (clear diagram)
$\tan(0°15') = \frac{r}{382500}$ $\tan\left(0°15' \times \frac{\pi}{180}\right) = \frac{r}{382500}$	1 mark (conversion to rad)
$r = 382500 \times \tan(0.0043633...)$ $= 382500 \times 0.0043633...$ $= 1668.97...$ $= 1669$ km $D = 2r = 3338$ km	1 mark (correct diameter)

Question 5 (12 marks)	Mark Allocation
a)	
i) Draw a neat sketch of the curve $y=e^{-x}+1$	1 mark (correct negative exp shape) 1 mark (shift vertical up one unit)
ii) Vol $=\pi\int_0^2 y^2dx$ $=\pi\int_0^2\left(e^{-x}+1\right)^2dx$ $=\pi\int_0^2\left(e^{-2x}+2e^{-x}+1\right)dx$ $=\pi\left[-\frac{e^{-2x}}{2}-2e^{-x}+x\right]_0^2$ $=\pi\left[-\frac{1}{2e^{2x}}-\frac{2}{e^{x}}+x\right]_0^2$ $=13.26$ cubic units	1 mark (correct squaring of function) 1 mark (correct integration) 1 mark (correct final answer)
b)	
i) $A_{sector}=SA_{cone}=\frac{1}{2}r^2\theta$ $=\frac{1}{2}(5)^2\times\frac{2\pi}{3}$ $=\frac{25\pi}{3}$ sq units	1 mark (matching area of sector to area of cone) 1 mark (correct answer)
ii) Circumference of the base of the cone will be the same as the arc length of the sector cut out, hence $r_{sector}\theta=2\pi r_{cone}$ $5\times\frac{2\pi}{3}=2\pi\times r_{cone}$ $r_{cone}=\frac{5}{3}$	1 mark (statement explaining first part or simply equation as shown) 1 mark (substitution clearly shown)
iii) By Pythagoras, using slant height of the cone as the radius of the original sector $h_{cone}=\sqrt{5^2-\left(\frac{5}{3}\right)^2}$ $=10\sqrt{2}$ units Then volume:	1 mark (slant height as 5) 1 mark (pythag and height of cone)

$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\times\left(\frac{5}{3}\right)^2\times10\sqrt{2}$ $=\frac{250\pi\sqrt{3}}{81}$ cubic units	1 mark (final exact answer)