Name:

Teacher:

MORIAH COLLEGE

Year 11

MATHEMATICS Extension 1

2003 PRELIMINARY

Date:	Friday, 5th September, 2003
Time Allowed:	1 hour 30 minutes + 5minutes reading
Examiners:	L. Bornstein, E, Apfelbaum, A. Joshua, G. Wagner

INSTRUCTIONS :

- **There are 5 questions.**
- **Answer each question in a separate booklet.**
- **Show all necessary working.**

Question 1 **(Start each question in a new booklet)** **Marks**

a) Given that $f(x+h)=(x+h)^3=x^3+3x^2h+3xh^2+h^3$ find, by first principles, the gradient function of $f(x)=x^3$. Marks will not be awarded for finding the derivative without using a limiting process. (3)

b) Find $\dfrac{dy}{dx}$ for the following functions and simplify your answers giving them without fractional and negative indices.

i) $y=\sqrt[3]{(x^6-4)^2}$ (3)

ii) $y=\dfrac{5x^3-3x+6}{2x^2}$ (3)

c) A function $y=f(x)$ is defined as follows:

$$f(x)=\begin{cases}\dfrac{8}{x} & \text{if } |x|\geq 2\\ x^2 & \text{if } |x|<2\end{cases}$$

i) Find $f(2)$ and $f(-2)$. (2)

ii) Sketch the function $y=f(x)$. (2)

iii) Find the range of the function $y=f(x)$. (1)

iv) For what values of x is the function not differentiable. (1)

Question 2 (Start in a new booklet)

a) Solve the equation $\log_8(x+1)-\log_8(x-1)=\frac{2}{3}$ (2)

b) Find the value(s) of k such that the quadratic equation $3x^2-kx+2=0$ has

i) no real roots. (2)

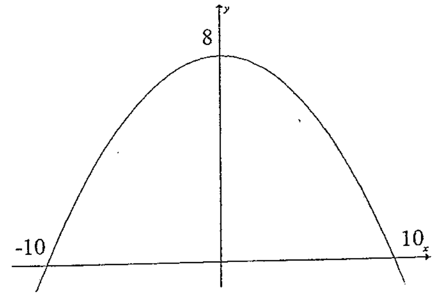
ii) the product of its roots equal to the sum of their reciprocals. (3)

c) Prove, by induction that if n is a positive integer, then 4^n+14 is a multiple of 6. (4)

d) The line $y=mx$ is a tangent to the circle whose centre is the point $(5,0)$ and whose radius is 3 units. Find the possible value(s) of m. (4)

Question 3 (Start in a new booklet)

a) A parabolic tunnel is 20 metres wide at ground level and 8 metres high at its vertex. The parabolic tunnel is shown on the axes below. The axis of symmetry of the tunnel is the y-axis.



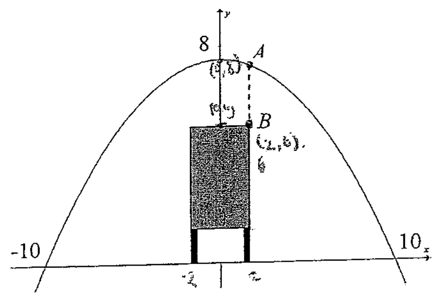
i) Show that the equation of the tunnel is given by (3)

$$y=\frac{-2}{25}\left(x^2-100\right)$$

ii) How high above the floor of the tunnel is the focus of the parabola? (3)

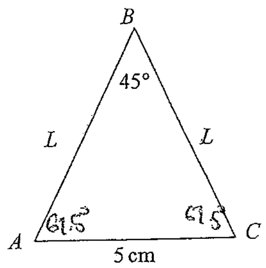
iii) A flat-topped truck passes through the middle of the tunnel. The truck is 4 metres wide and 6 metres high. There is a gap between the top of the truck (which is parallel to the x axis) and the roof of the tunnel. The greatest gap is 2 metres.

Find the least vertical gap AB as shown in the diagram. (2)



Question 3 (Continued)

b) The isosceles triangle shown below has $\angle ABC = 45°$, $AC = 5$ cm and equal sides of length L.



i) Use the cosine rule to find the exact value of L^2 (Give your answer with a rational denominator). (4)

ii) Show that the exact area of the triangle is $\dfrac{25\left(\sqrt{2}+1\right)}{4}\text{cm}^2$. (3)

Question 4 (Start in a new booklet)

a) Sketch the curve $y = 2\cos 2x$ for $-180° \leq x \leq 180°$ (2)

b) For what values of r does an infinite geometric series have a limiting sum? (1)

c) Consider the infinite geometric series below:

$$2\cos 2x + 4\cos^2 2x + 8\cos^3 2x + \ldots\ldots$$

If $-90° \leq x \leq 90°$, then using parts a) and b) or otherwise, find the values of x for which the infinite geometric series has a limiting sum. (5)

d) i) Prove that the point $P(a\cos\theta, a\sin\theta)$ lies on the circle $x^2 + y^2 = a^2$. (2)

Let A be the point $(-a, 0)$ and B be the point $(a, 0)$.

ii) Find the gradient of PA in terms of θ. (2)

iii) Hence show that $\angle APB$ is a right angle. (3)

Question 5 **Start in a new booklet**

a) The equation $(x-3y+5)+k(x+2y)=0$ represents all of the straight lines passing through a fixed point P.

i) For what value of k is one of the lines parallel to the straight line $x+y=2$? (3)

ii) For what value of k does one of the lines pass through the centre of the circle $x^2+y^2-10y+21=0$? (3)

iii) Find the coordinates of the point P. (2)

b) For the function $f(x)=\dfrac{x+\frac{1}{x}}{x-\frac{1}{x}}$ find the derivative and explain why there are no values of x for which the function has a horizontal tangent. (3)

c) Find the equation of the tangent to the curve $y=x^2-4x$ at the point $P(x_1, y_1)$ giving your answer in terms of x_1 only. If this tangent passes through the point $(4,-1)$, find the coordinates of P. (4)

QUESTION 1

(a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$

$= 3x^2$

(b) (i) $y = (x^6 - 4)^{2/3} \longrightarrow \frac{dy}{dx} = \frac{2}{3}(x^6 - 4)^{-1/3} \cdot 6x^5$

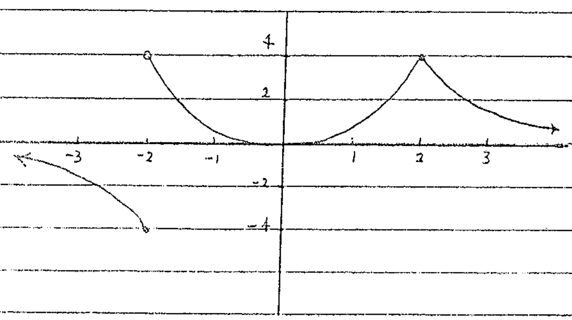
$= \frac{4x^5}{\sqrt[3]{(x^6-4)}}$

(ii) $y = \frac{5}{2}x - \frac{3}{2}x^{-1} + 3x^{-2} \longrightarrow \frac{dy}{dx} = \frac{5}{2} + \frac{3}{2}x^{-2} - 6x^{-3}$

$= \frac{5}{2} + \frac{3}{2x^2} - \frac{6}{x^3}$

(c) (i) $f(2) = 4$, $f(-2) = -4$

(ii)



(iii) Range is $\{y : -4 \le y \le 4\}$.

(iv) $x = -2$ and $x = 2$

QUESTION 2

(a) $\log_8\left(\frac{x+1}{x-1}\right) = \frac{2}{3} \longrightarrow \frac{x+1}{x-1} = 4$

$x + 1 = 4x - 4$

$x = \frac{5}{3}$

(b) (i) $\Delta < 0$ for no real roots

$\Delta = k^2 - 24 \longrightarrow k^2 - 24 < 0$

$-2\sqrt{6} < k < 2\sqrt{6}$

(ii) Let roots be α and β. $\longrightarrow \alpha\beta = \frac{1}{\alpha} + \frac{1}{\beta}$

Now $\alpha + \beta = \frac{k}{3}$ $\quad (\alpha\beta)^2 = \alpha + \beta$

$\frac{4}{9} = \frac{k}{3}$

$\alpha\beta = \frac{2}{3}$ $\quad k = \frac{4}{3}$

(c)

Let $n = 1$

$4^1 + 14 = 18$ which is a multiple of 6.

$\therefore$ True for $n = 1$

Assume true for $n = k \longrightarrow 4^k + 14 = 6Q$ for some integral Q

R.T.P true for $n = k+1 \longrightarrow$ R.T.P $4^{k+1} + 14$ is a multiple of 6

Now $4^{k+1} + 14 = 4(4^k) + 14$

$= 4(6Q - 14) + 14$

$= 24Q - 42$

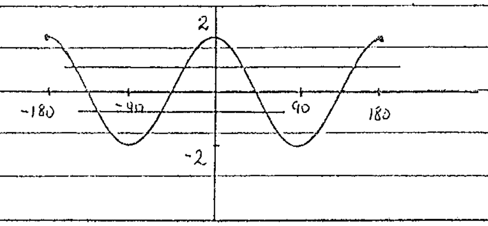
$= 6(4Q - 7)$

$\therefore 4^{k+1} + 14$ is a multiple of 6 since $(4Q - 7)$ is integral.

$\therefore$ Expression is a multiple of 6 when $n = 1$ and $n = k+1$ using the assumption that it is true for $n = k$. Therefore, by the principles of mathematical induction it is true for all values as required.

QUESTION 4

(a)



(b) $-1 < r < 1$

(c) $r = 2\cos 2x \longrightarrow -1 < 2\cos 2x < 1$

Solve $2\cos 2x = -1$ $\quad\quad$ $2\cos 2x = 1$

$\cos 2x = -\frac{1}{2}$ $\quad\quad$ $\cos 2x = \frac{1}{2}$

$2x = \pm 120°, \pm 240°$ $\quad\quad$ $2x = \pm 60°, \pm 300°$

$x = \pm 60°, \pm 120°$ $\quad\quad$ $x = \pm 30°, \pm 150°$

From graph in (a) $-60° < x < -30°$ and $30° < x < 60°$

(d) (i) $LHS = a^2\cos^2\theta + a^2\sin^2\theta$ $\quad\quad$ $RHS = a^2$

$= a^2(\cos^2\theta + \sin^2\theta)$

$= a^2$

$\therefore$ P lies on $x^2 + y^2 = a^2$

(ii) $m_{PA} = \dfrac{a\sin\theta}{a\cos\theta + a} = \dfrac{\sin\theta}{\cos\theta + 1}$

(iii) $m_{PA} \times m_{PB} = \dfrac{\sin\theta}{\cos\theta + 1} \times \dfrac{\sin\theta}{\cos\theta - 1}$

$= \dfrac{\sin^2\theta}{\cos^2\theta - 1}$

$= \dfrac{\sin^2\theta}{-\sin^2\theta}$

$= -1$

$\therefore$ $\angle APB$ is a right angle

QUESTION 5

(a) (i) $x - 3y + 5 + kx + 2ky = 0$

$y(2k - 3) = x(-1 - k) - 5$

$y = \dfrac{(-1-k)x}{(2k-3)} - \dfrac{5}{(2k-3)}$

Need gradients to be equal

$\therefore \dfrac{-1-k}{2k-3} = -1$

$-1 - k = -2k + 3$

$k = 4$

(ii)

centre $= (0, 5)$

sub. in equation $\longrightarrow (-15 + 5) + k(10) = 0$

$10k = 10$

$k = 1$

(iii) Solve $x - 3y + 5 = 0$ $\longrightarrow$ $5y - 5 = 0$

$x + 2y = 0$ $\quad\quad$ $y = 1$

$x = -2$

$\therefore P = (-2, 1)$

(b) $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$ where $x \neq 0, \pm 1$

$f'(x) = \dfrac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2}$

$= \dfrac{-4x}{(x^2-1)^2}$

$f'(x) = 0$ if $x = 0$ but $x \neq 0$ as function not defined at $x = 0$.