MORIAH COLLEGE MATHEMATICS DEPARTMENT

Year 11 Extension 1 Preliminary Exam Revision

Question 1

(a) If $\frac{1}{a+b}, \frac{1}{b+c}, \frac{1}{c+a}$ form an arithmetic sequence, prove that b^2, a^2, c^2 form an arithmetic sequence.

(b) On a number plane, indicate the region specified by y $y \geq |x+1|$ and $y \leq 1$

(c) The interval PQ has end points $P(-3,4)$ and $Q(11,12)$. Find the co-ordinates of J which divides PQ externally in the ratio 3:2

(d) Let $f(x) = 3x^2 - x$. Use the definition $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$ to find the derivative of $f(x)$ at the point where $x = a$

(e) If $a + b = 1$ show that $(a^2 - b^2)^2 + ab = a^3 + b^3$

Solve for x: i) $\frac{3x}{x-2} \leq x+6$

ii) $x + 2y = 5$
$-x^2 + 2xy = 3$

Question 2

(a) Find $\frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$ 2

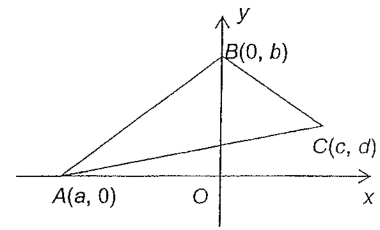
b) Prove by mathematical induction that: $10^n + 3 \times 4^{n+2} + 5$ is a multiple of 9

c) Prove by mathematical induction that:

$$\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \dots\dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

d) In the diagram below, N is the midpoint of AB and P is the midpoint of AC

Show that $NP \parallel BC$



Question 3

a)Write down the derivatives of:

i) $y = x^4 + 3x^2 - 5\sqrt{x} + \dfrac{4}{x^2}$

ii) $y = \left(x + \dfrac{1}{x}\right)^3$

iii) $y = (x^2 - 3x)(x+1)^3$

iv) $y = (2x-5)^2\sqrt{2x-5}$

v) $y = \dfrac{x^4 - 3x^2 + 8}{x^2}$

vi) $y = \dfrac{4x-1}{2x^2+1}$

b) Find the gradient of the tangent to the semicircle $y = \sqrt{25 - x^2}$ at the point P (3, 4). The tangent to the semi-circle at Q (0,5) meets the tangent at P in the point R. Find the coordinates of R.

c) i) Find the derivative of $y = \dfrac{1}{1+x^2}$

ii) Determine the equation of the normal to the curve at the point (1, ½) on it.

iii) At what point does this tangent cross the x axis?

iv) Find the coordinates of the point on the curve where the tangent is parallel to the x axis.

d) The tangent at P (1, ½) on the curve $y = \dfrac{x}{x+1}$ meets the x axis at T.

i) Find the coordinates of T

ii) A line is drawn from P perpendicular to the x axis meeting it at G. Show that the origin bisects the interval TG.

e) If $y = x(2x-1)^7$, find the equation of the normal to the curve at the point (1, 1).

Question 4

a) A spherical balloon is expanding so that the radius is increasing at the rate of 0·5 cm/s. When the radius is 10 cm, what is the rate of increase of

i) the surface area ii) the volume?

b) The temperature of a cube is falling so that the volume decreases at a constant rate of 0·06 cm^3/s. Find at what rate the edge is contracting when the edge is 10 cm.

c) i) Prove that the area A cm^2 of an equilateral triangle of side x cm is given by

$$A = \frac{\sqrt{3}x^2}{4}$$

ii) The sides of an equilateral triangle are increasing at the rate of $\frac{1}{6}$ cm/s. At what rate is the area increasing at the instant when the sides are 12 cm?

d) The volume of a cube is increasing at the arte of 0·001 cm^3 /s. Find how fast the *surface area* is changing when the side is 5 cm

e) A horizontal trough 10m long has a cross-section in the shape of a right-angled isosceles triangle. If water is poured in at the rate of $8m^3$/min, at what rate is the water level rising when the depth of the water is 2m?

Q1 a) $\frac{1}{b+c} - \frac{1}{a+b} = \frac{1}{c+a} - \frac{1}{b+c}$

$$\frac{a+b-b-c}{(b+c)(a+b)} = \frac{b+c-c-a}{(c+a)(b+c)}$$

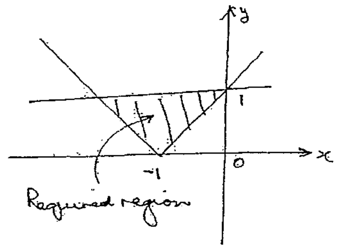
$$\frac{a-c}{a+b} = \frac{b-a}{c+a}$$

$\therefore\ a^2-c^2 = b^2-a^2$

OR $\quad a^2-b^2 = c^2-a^2$

$\therefore\ b^2, a^2, c^2$ are in AP since there is a common difference between terms.

b)



c) $P(-3, 4) \quad Q(11, 12)$

$3:-2$

$x = \frac{-2\times-3+3\times11}{3+-2} \qquad y = \frac{-2\times4+3\times12}{3+-2}$

$= 39 \qquad\qquad = 28$

$\therefore$ J is $(39, 28)$

d) $f(a+h) - f(a) = 3(a+h)^2 - (a+h) - [3a^2-a]$

$= 6ah + 3h^2 - h$

$= h(6a+3h-1)$

$\therefore f'(a) = \lim_{h\to0} \frac{\cancel{h}(6a+3h-1)}{\cancel{h}}$

$= 6a-1$

e) LHS $= (a^2-b^2)^2 + ab$

$= [(a+b)(a-b)]^2 + ab$

$= (a-b)^2 + ab$

$= a^2 - 2ab + b^2 + ab$

$= a^2 - ab + b^2$

RHS $= (a+b)(a^2-ab+b^2)$

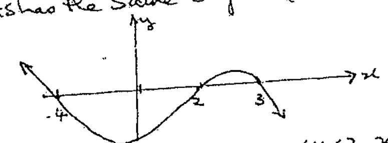
$= a^2 - ab + b^2$

$=$ LHS

f) i) $\frac{3x}{x-2} - (x+6) \leq 0$

$\frac{12-x-x^2}{x-2} \leq 0$

LHS has the same sign as $(12-x-x^2)(x-2)$:



$\therefore$ required solution is $-4 \leq x < 2,\ x \geq 3$

ii) $x + 2y = 5$ (1)

$-x^2 + 2xy = 3$ (2)

from (1) $2y = 5-x$ $\therefore$ $-x^2 + x(5-x) = 3$

$2x^2 - 5x + 3 = 0$

$x = 1$ or $1\frac{1}{2}$

Q2 a) $\frac{d}{dx}\left(\frac{x^2}{3x-1}\right) = \frac{2x(3x-1) - x(3)}{(3x-1)^2}$

$= \frac{3x^2-2x}{(3x-1)^2}$

b) If $n=1$, $10 + 3\times4^3 + 5 = 207$

$= 9\times23$

$\therefore$ the proposition is true for $n=1$

Assume $10^k + 3\times4^{k+2} + 5 = 9m$ where $m, k \in Z^+$

We wish to prove $10^{k+1} + 3\times4^{k+3} + 5 = 9p \quad p \in Z^+$

I shall adopt a difference method here, which is to prove that the difference is a multiple of 9.

this is often easier

$10^{k+1} - 10^k + 3\times4^{k+3} - 3\times4^{k+2} = 10^k(10-1) + 3\times4^{k+2}\times(4-1)$

$= 9\times(10^k + 4^{k+2})$

$\therefore$ the difference is a multiple of 9, and so the proposition for $n=k+1$ will be a multiple of 9 if it is true for $n=k$.

But it is true for $n=1$

$\therefore$ the proposition is true for all $n \in Z^+$ by math. induction.

If $n=1$ LHS $= \frac{1}{1\times2\times3}$ RHS $= \frac{1\times 4}{4\times 2\times 3}$

$= \frac{1}{6}$ $\qquad = \frac{1}{6}$

$\therefore$ Proposition is true for $n=1$.

Assume $\frac{1}{1\times2\times3} + \cdots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$ where $k \in Z^+$

We wish to prove $\frac{1}{1\times2\times3} + \cdots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

LHS $= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$ by assumption

$= \frac{k(k+3)(k+3) + 4}{4(k+1)(k+2)(k+3)}$

$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

$= \frac{\cancel{(k+1)}(k^2+5k+4)}{4\cancel{(k+1)}(k+2)(k+3)}$

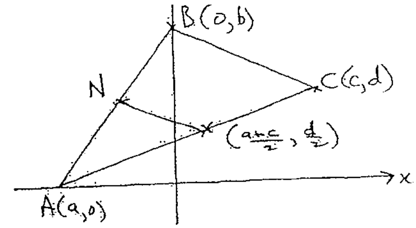
$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

$=$ RHS

$\therefore$ Proposition is true for $n=k+1$ if it is true for $n=k$. But it is true for $n=1$ $\therefore$ it is true for all $n \in Z^+$ by math. induction.

d) N is $\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

P is $\left(\frac{a+c}{2}, \frac{0+d}{2}\right) = \left(\frac{a+c}{2}, \frac{d}{2}\right)$



Gradient BC $= \frac{d-b}{c}$

Gradient NP $= \dfrac{\frac{d}{2}-\frac{b}{2}}{\frac{a+c}{2}-\frac{a}{2}}$

$= \dfrac{\frac{d-b}{2}}{\frac{c}{2}}$

$= \frac{d-b}{c}$

$=$ gradient BC

$\therefore$ NP $\parallel$ BC

Q3 a) i) $\frac{dy}{dx} = 4x^3+6x - \frac{5}{2\sqrt{x}} - \frac{8}{x^3}$

ii) $\frac{dy}{dx} = 3\left(x+\frac{1}{x}\right)^2 \times \left(1-\frac{1}{x^2}\right)$

$= 3\left(x+\frac{1}{x}\right)^3\left(1-\frac{1}{x}\right)$

iii) $\frac{dy}{dx} = (2x-3)(x+1)^3 + (x^2-3x)\times 3(x+1)^2\times 1$

$= (x+1)^2\left[(2x-3)(x+1) + 3x^2-9x\right]$

$= (x+1)^2\left[5x^2-10x-3\right]$

iv) $y = (2x-5)^{\frac{5}{2}}$

$\frac{dy}{dx} = \frac{5}{2}(2x-5)^{\frac{3}{2}}\times 2$

$= 5(2x-5)\sqrt{2x-5}$

v) $y = \frac{x^4}{x^2} - \frac{3x^2}{x^2} + \frac{8}{x^2}$

$= x^2 - 3 + 8x^{-2}$

$\therefore \frac{dy}{dx} = 2x - \frac{16}{x^3}$

vi) $\frac{dy}{dx} = \frac{4(2x^2+1) - (4x-1)\times 4x}{(2x^2+1)^2}$

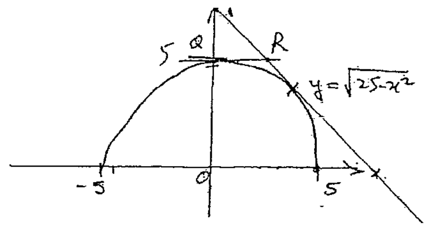
$= \frac{8x^2-4x+5}{(2x^2+1)^2}$

$$\frac{dy}{dx} = \ldots$$

$$= -\frac{x}{\sqrt{25-x^2}}$$

$$= -\frac{3}{\sqrt{16}}$$

$$= -\frac{3}{4} \quad \text{at } (3,4)$$



$\therefore$ Tgt at P is $y-4=-\frac{3}{4}(x-3)$

$$4y-16=-3x+9$$

OR $\boxed{3x+4y-25=0}$

the tgt at Q is $y=5$. $\therefore$ the point of intersection R is when

$$3x+20-25=0$$

$$x=\frac{5}{3}$$

$\therefore$ R is $(1\frac{2}{3}, 5)$

c) i) $\frac{dy}{dx} = \frac{0-2x}{(1+x^2)^2}$

$$= \frac{-2x}{(1+x^2)^2}$$

ii) At $x=1$, $\frac{dy}{dx} = \frac{-2}{4}$

$$= -\frac{1}{2}$$

$\therefore$ Normal at $(1, \frac{1}{2})$ is

$$y-\frac{1}{2} = 2(x-1)$$

$$2x-y-1\frac{1}{2}=0$$

or $\boxed{4x-2y-3=0}$

iii) For x intercept $y=0$ $\therefore$ $4x=3$

$$x=3/4$$

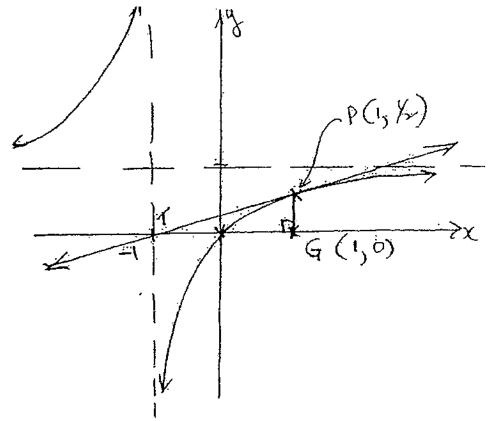
iv) For $\frac{dy}{dx}=0$ $\quad \frac{-2x}{(1+x^2)^2}=0$

$\therefore$ $-2x=0$ or $x=0$ $\quad \therefore$ point is $(0,1)$

a) i) $\frac{dy}{dx} = \frac{\ldots}{(x+1)^2}$

$$= \frac{1}{(x+1)^2}$$

$$= \frac{1}{4} \quad \text{at } x=1$$



$\therefore$ tgt at $(1, \frac{1}{2})$ is

$$y-\frac{1}{2} = \frac{1}{4}(x-1)$$

$$4y-2 = x-1$$

$\boxed{x-4y+1=0}$

For x intercept $y=0$ $\therefore$ $x+1=0$

$\boxed{x=-1}$

ii) Since P is $(1, \frac{1}{2})$ G is $(1,0)$ But T is $(-1,0)$ $\therefore$ O is the midpoint of TG.

e) $\frac{dy}{dx} = (2x-1)^7 + x\left[7(2x-1)^6 \times 2\right]$

$$= (2x-1)^7 + 14x(2x-1)^6$$

$$= (2x-1)^6\left[2x-1+14x\right]$$

$$= (2x-1)^6(16x-1)$$

At $x=1$, $\frac{dy}{dx} = 1^6 \times 15$

$$= 15$$

$\therefore$ the equation of the normal is

$$y-1 = -\frac{1}{15}(x-1)$$

$$15y-15 = -x+1$$

$\boxed{x+15y-16=0}$

a) $\frac{dr}{dt} = 0.5$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$= 8\pi r \times 0.5$$

$$= 40\pi \text{ when } r = 10$$

$\therefore$ Surface area is increasing at 40π cm²/s when $r = 10$

ii) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$= 4\pi r^2 \times 0.5$$

$$= 200\pi \text{ when } r = 10$$

$\therefore$ Volume is increasing at 200π cm³/s when $r = 10$

b) Let the edge be x cm $\therefore V = x^3$ and $\frac{dV}{dt} = -0.06$

Now $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$$-0.06 = 3x^2 \times \frac{dx}{dt}$$

When $x = 10$:

$$-0.06 = 300\frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = -0.0002$$

$\therefore$ the edge is decreasing at 0.0002 cm/s when $x = 10$ cm

$$= \frac{1}{2} \times x \times x \times \sin 60°$$

$$= \frac{1}{2}x^2 \frac{\sqrt{3}}{2}$$

$$= x^2 \frac{\sqrt{3}}{4}$$

$\frac{dx}{dt} = \frac{1}{6}$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

$$= 2x \cdot \frac{\sqrt{3}}{4} \times \frac{1}{6}$$

$$\therefore \frac{dA}{dt} = \frac{24\sqrt{3}}{4} \times \frac{1}{6} \quad \text{when } x = 12$$

$$= \sqrt{3}$$

$\therefore$ Area is increasing at $\sqrt{3}$ cm²/s.

d) Let the edge be x cm.

$\therefore V = x^3 \quad A = 6x^2$

$\frac{dV}{dt} = 0.001$

Now $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$$\therefore 0.001 = 3x^2 \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = \frac{0.001}{3x^2}$$

But $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

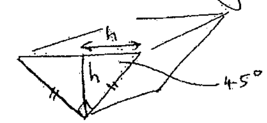
$$= 12x \times \frac{dx}{dt}$$

$$= 12x \times \frac{0.001}{3x^2}$$

$$= \frac{0.004}{x}$$

$$= 0.0008 \text{ when } x = 5$$

$\therefore$ Surface area is increasing at 0.0008 cm²/s

e) 

If the height of the triangle is h, the base is $2h$

$\therefore$ Area of $\Delta = \frac{1}{2} \times 2h \times h$

$\therefore A = h^2$

$\therefore V = A \times 10$

$\therefore V = 10h^2$

$$\therefore \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\frac{dV}{dt} = 20h \times \frac{dh}{dt}$$

When $\frac{dV}{dt} = 8$ and $h = 2$,

$$8 = 20 \times 2 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{8}{40}$$

$$= \frac{1}{5}$$

$\therefore$ the water level is rising at $\frac{1}{5}$ m/min