Student Number: ....

MORIAH COLLEGE MATHEMATICS DEPARTMENT

2004

PRELIMINARY COURSE

MATHEMATICS Extension 1

Examiners: **Mr G Druery**
Mr J Taylor
Mr G Wagner

Instructions

Reading time – 5 minutes

Working time – $1\frac{1}{2}$ hours

- Attempt ALL questions.
- Show all necessary working, marks may be deducted for careless or untidy work.
- Board-approved calculators may be used.
- Additional paper is available.

QUESTION 1 *(Start a new page)* **MARKS**

a) Solve $(x-3)^2=(x+1)^2$ **2**

b) Calculate to the nearest degree the acute angle that the line $2x-3y+5=0$ makes with the y-axis. **2**

c) Without using a calculator prove that $\frac{1}{7-\sqrt{37}}>\frac{1}{\sqrt{37}-5}$. **2**

d) Solve for x: $\frac{2-3x}{x^2-2}\geq 1$. **2**

e) Solve for x: $3x+3=|1-2x|$ **2**

QUESTION 2 *(Start a new page)*

a) Differentiate the following with respect to x, giving your answers in simplest form.

i) $y=\sqrt[6]{4x^3-4}$

ii) $y=(3x^2-1)\sqrt{4x+3}$ **4**

b) Verify that $x=\frac{1}{3}$ and $x=2$ satisfy the equation $7-3x=\frac{2}{x}$

i) On the same set of axes, sketch the graphs of $y=7-3x$ and $y=\frac{2}{x}$

ii) Hence or otherwise, write down all values of x for which $7-3x<\frac{2}{x}$ **3**

c) Find the value(s) of x where $\frac{d}{dx}\left[\frac{x+2}{\sqrt{x-2}}\right]=0$ **3**

QUESTION 3 *(Start a new page)*

a) Prove the trigonometric identity: $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$, where θ is acute. 2

b) Point A has coordinates $(10,2)$ and point B has coordinates $(-2,6)$.
The point M divides AB externally in the ratio $4:3$. Find the coordinates of point M. 2

c) Gas is escaping from a spherical balloon such that its radius is decreasing at a rate of $0\cdot1$ cm/s. Find the rate of decrease of the volume at the instant when there is $\dfrac{3\pi}{4}$ cm^3 of gas in the balloon? Give your answer correct to 2 decimal places. 3

d) The first three terms of an AP are $3, a, b$ while the first three terms of a GP are $3, b, a$.
Find two possible values for a and b where $a \neq b$. 3

QUESTION 4 *(Start a new page)*

a) i) Sketch the function $f(x) = \sqrt{2-x}$
ii) Find $f^{-1}(2)$ 2

b) The point A lies on the line $3x+2y=24$. From A, a perpendicular is dropped to the x axis, meeting the x axis at B.

i) If $B=(a,0)$, find the coordinates of A in terms of a?

ii) The triangle bounded by the line AB, the line $3x+2y=24$ and the x axis has an area of 27 units^2. Find the coordinates of A? 4

c) Prove by induction $\sum n(n+2) = \dfrac{1}{6}n(n+1)(2n+7)$
for all positive integers $n \geq 1$ 4

QUESTION 5 *(Start a new page)* **MARKS**

a) The tangent line to the curve $y = x^4$ meets the curve at the point of contact $P(a,b)$.

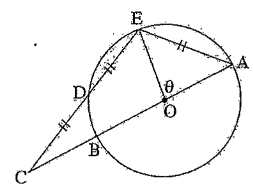
i) Show that the tangent at point P is given by the equation
$4a^3x - y - 4a^4 + b = 0$ 2

ii) Explain briefly why $b = a^4$ 1

iii) If the tangent cuts the x-axis at $x = \dfrac{3}{2}$, then show that the equation of this tangent is $32x - y - 48 = 0$ 2

b) In the figure, AB is a diameter of a unit circle with centre O. 5

$CD = DE = AE = x$ and $BC = BO = 1$.
Angle $AOE = \theta$.



i) Using ΔOAE show that $\cos\theta = \dfrac{2-x^2}{2}$

ii) Using ΔOCE and the identity $\cos(180° - \theta) = -\cos\theta$, 5

show that $\cos\theta = \dfrac{4x^2-5}{4}$.

iii) Find the value of x.

iv) Hence, find the exact value of the area of triangle CEA.

Q1.

a. $(x-3)^2-(x+1)^2=0$

$(x-3+x+1)(x-3-x-1)=0$ (1)

$(2x-2)\cdot -4=0$

$x=1$ (1)

b. $\tan\theta=\frac{2}{3}$

$\theta \doteqdot 34^\circ$ (1)

$\therefore$ angle with y axis is $90-34=56$ (1)

c. $\frac{1}{7-\sqrt{37}}=\frac{7+\sqrt{37}}{49-37}=\frac{7+\sqrt{37}}{12}$ (1)

$\frac{1}{\sqrt{37}-5}=\frac{\sqrt{37}+5}{37-25}=\frac{\sqrt{37}+5}{12}$ (1)

So $\frac{1}{7-\sqrt{37}}>\frac{1}{\sqrt{37}-5}$

d. $x\neq\pm\sqrt{2}$ ($\frac12$)

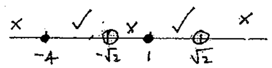
Let $\frac{2-3x}{x^2-2}=1$

$2-3x=x^2-2$

$x^2+3x-4=0$

$(x+4)(x-1)=0$

$x=1$ or $x=-4$ ($\frac12$)



Test $x=0$ $\quad \frac{2}{-2}\geq 1$ × F

Test $x=-5$ $\quad \frac{2+15}{23}\geq 1$ × F ($\frac12$)

Test $x=2$ $\quad \frac{-4}{2}\geq 1$ × F

Test $x=1.2$ $\quad \frac{2-3.6}{1.44-2}\geq 1$ ✓ T

Test $x=-3$ $\quad \frac{11}{7}\geq 1$ ✓ T

Solution is

$-4\leq x<-\sqrt{2}$ OR $1\leq x<\sqrt{2}$ (1)

e) $1-2x=3x+3$ OR $1-2x=-3x-3$

$-5x=2$ $\qquad x=-4$

$x=-\frac{2}{5}$ ($\frac12$) OR $x=-4$ ($\frac12$)

Test $x=-\frac{2}{5}$

$-\frac{6}{5}+3=|1+\frac{4}{5}|$

$1\frac{4}{5}=1\frac{4}{5}$ ✓ T ($\frac12$)

Test $x=-4$

$-12+3=|1+8|$

$-9=9$ × F ($\frac12$)

$\therefore$ Only solution is $x=-\frac{2}{5}$

Q2

a) i) $y=(4x^3-4)^{1/6}$ ($\frac12$)

$\frac{dy}{dx}=\frac{1}{6}(12x^2)(4x^3-4)^{-5/6}$ (1)

$=\frac{2x^2}{(4x^3-4)^{5/6}}$ ($\frac12$)

ii) let $u=3x^2-1$, $v=(4x+3)^{1/2}$

$\frac{du}{dx}=6x$, $\frac{dv}{dx}=\frac{1}{2}\times4\times(4x+3)^{-1/2}$

$=\frac{2}{\sqrt{4x+3}}$ (1)

$\frac{dy}{dx}=6x\sqrt{4x+3}+\frac{2(3x^2-1)}{\sqrt{4x+3}}$ (1)

$=\frac{6x(4x+3)+6x^2-2}{\sqrt{4x+3}}$

$=\frac{30x^2+18x-2}{\sqrt{4x+3}}$

c) $u=x+2$, $v=(x-2)^{1/2}$

$\frac{du}{dx}=1$, $\frac{dv}{dx}=\frac{1}{2}(x-2)^{-1/2}$ ($\frac12$)

$=\frac{1}{2\sqrt{x-2}}$

$\frac{d}{dx}\left[\frac{x+2}{\sqrt{x-2}}\right]=\frac{(x-2)^{1/2}-\frac{x+2}{2\sqrt{x-2}}}{x-2}$ (1)

$=\frac{2(x-2)-x-2}{2(x-2)^{3/2}}$

$=\frac{x-6}{2(x-2)^{3/2}}$ ($\frac12$)

Now, expression equals zero where $x-6=0$

$\therefore x=6$ (1)

b) when $x=\frac{1}{3}$, LHS $=7-3(\frac{1}{3})$

$=6$

RHS $=2/\frac{1}{3}$

$=6$ ($\frac12$)

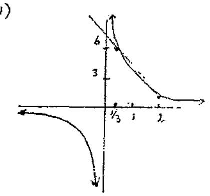
when $x=2$, LHS $=7-3(2)$

$=1$

RHS $=2/2$

$=1$ ($\frac12$)

i)



(1)

ii) $0<x<\frac{1}{3}$ OR $x>2$ (1)

Q3

a) $LHS = \sqrt{\frac{1-\sin\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}}$

$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$ (1)

$= \sqrt{\left(\frac{1-\sin\theta}{\cos\theta}\right)^2}$

$= \frac{1-\sin\theta}{\cos\theta}$ if θ is acute (2)

$= \sec\theta - \tan\theta$ if θ is acute

b) $x_1 = 10$

$x_2 = -2$

$y_1 = 2$

$y_2 = 6$

$k:l = -4:3$

$m = \left(\frac{kx_2 + lx_1}{k+l}, \frac{ky_2 + ly_1}{k+l}\right)$

$= \left(\frac{8+30}{-1}, \frac{-24+6}{-1}\right)$

$= (-38, 18)$ (2)

c) $\frac{dr}{dt} = -0.1$ cm/s

$V = \frac{4}{3}\pi r^3$ (1)

$\frac{dV}{dr} = 4\pi r^2$ (1)

Also, when $V = \frac{3\pi}{4}$

$\frac{4\pi}{3} r^3 = \frac{3\pi}{4}$

$r^3 = \frac{9}{16}$

$r = \sqrt[3]{\frac{9}{16}}$ (1)

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$= 4\pi r^2 \cdot -0.1$

$= -0.4\pi r^2$

when $r = \sqrt[3]{9/16}$, $\frac{dV}{dt} = -0.4\pi \times \left(\frac{9}{16}\right)^{2/3}$

$= -0.856..$

$\doteqdot -0.86$ cm^3/s (1)

d) $a - 3 = b - a$

$2a = b + 3$

$a = \frac{b+3}{2}$ .. ① $\left(\frac{1}{2}\right)$

$\frac{b}{3} = \frac{a}{b}$

$3a = b^2$ ② $\left(\frac{1}{2}\right)$

$b^2 = \frac{3b+9}{2}$

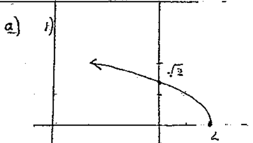
$2b^2 - 3b - 9 = 0$

$(2b+3)(b-3) = 0$

$\therefore b = 3, a = 3$ but $a \neq b$

$\therefore b = -3/2$ and $a = \frac{3}{4}$ (2)

Q4 a) i)



(1)

ii) $y = \sqrt{2-x}$ where $x \leq 2, y \geq 0$

swap x and y for inverse

$x = \sqrt{2-y}$

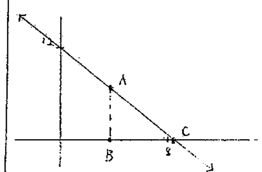
$x^2 = 2-y$ $\left(\frac{1}{2}\right)$

$y = 2-x^2$ where $x \geq 0, y \leq 2$

Now $f^{-1}(2) = 2 - 2^2$

$= -2$ $\left(\frac{1}{2}\right)$

b) i)



(1)

If $x = a$, $3a + 2y = 24$

$2y = 24 - 3a$

$y = \frac{24-3a}{2}$ (1)

$\therefore A$ is $\left(a, \frac{24-3a}{2}\right)$

ii) Area $\Delta ABC = 27$ u^2

$\frac{1}{2} \times \frac{24-3a}{2} \times (8-a) = 27$ (1)

$\frac{3(8-a)^2}{4} = 27$

$(8-a)^2 = 36$

$8 - a = \pm 6$

$\therefore a = 2$ or $a = 14$ (1)

But $a < 8 \rightarrow \therefore a = 2$

$\therefore A = (2, 9)$

c) $n=1 \rightarrow LHS = 1(1+2) = 3$

$RHS = \frac{1}{6}(1)(1+1)(2+7) = 3$

$\therefore$ Statement is true for $n = 1$ (1

Assume $\sum k(k+2) = \frac{1}{6}k(k+1)(2k+7)$ $\left(\frac{1}{2}\right)$

R.T.P true for $n = k+1$

ie

R.T.P $\left(\frac{1}{2}\right)$

$\sum (k+1)(k+3) = \frac{1}{6}(k+1)(k+2)(2k+9$

LHS $= 1.3 + 2.4 + 3.5 + \ldots + k(k+2) + (k+1)(k+$

$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$ $\left(\frac{1}{2}\right)$

(by assumption

$= \frac{1}{6}(k+1)\left[k(2k+7) + 6(k+3)\right]$

$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$

$= \frac{1}{6}(k+1)(k+2)(2k+9)$ (1)

$= RHS$

(1

Statement is true for $n = k+1$ assuming it is true for $n = k$. Since statement is true for $n = 1$, then by principles of mathematical induction it is true for all positive integers $n \geq 1$.

Q5 a) i. $y' = 4x^3$

At $x = a$ $\quad y' = 4a^3$ (1)

Eq of tangent is

$y - b = 4a^3(x - a)$

$y - b = 4a^3 x - 4a^4$

$4a^3 x - y - 4a^4 + b = 0$ (1)

ii. P lies on the curve so its co-ordinates satisfy the equation of the curve. (1)

iii. $4a^3 x - 0 - 4a^4 + b = 0$

$\dfrac{4a^4 - b}{4a^3} = \dfrac{3}{2}$

$8a^4 - 2b = 12a^3$

$2b = 8a^4 - 12a^3$

$b = 4a^4 - 6a^3$

but $b = a^4$

So $a^4 = 4a^4 - 6a^3$

$6a^3 = 3a^4$

$3a^4 - 6a^3 = 0$

$3a^3(a - 2) = 0$ (1)

$\therefore a = 2$ or $a = 0$

$a = 0$ does not give tangent through $x = 3/2$

So $a = 2$ $\quad b = 16$

Eq of tangent is

$32x - y - 64 + 16 = 0$

$32x - y - 48 = 0$ (1)

b) i) $x^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \cos\theta$

$x^2 = 2 - 2\cos\theta$

$2\cos\theta = 2 - x^2$

$\cos\theta = \dfrac{2 - x^2}{2}$ (1)

ii) $\angle COE = 180 - \theta$

$(2x)^2 = 1^2 + 2^2 - 2 \times 1 \times 2 \cos(180 - \theta)$

$4x^2 = 5 + 4\cos\theta$

$4\cos\theta = 4x^2 - 5$

$\cos\theta = \dfrac{4x^2 - 5}{4}$ (1)

iii) $\dfrac{2 - x^2}{2} = \dfrac{4x^2 - 5}{4}$

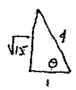
$8 - 4x^2 = 8x^2 - 10$

$12x^2 = 18$

$x^2 = \dfrac{3}{2}$

$x = \dfrac{\sqrt{3}}{\sqrt{2}}$ (1)

iv) $\cos\theta = \dfrac{2 - 3/2}{2} = \dfrac{1}{4}$



$\sin\theta = \dfrac{\sqrt{15}}{4}$

(1)

Area $\Delta AOE = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin\theta = \dfrac{\sqrt{15}}{8}$

Area $\Delta EOC = \frac{1}{2} \cdot 1 \cdot 2 \cdot (\sin 180 - \theta)$

$= \frac{1}{2} \cdot 1 \cdot 2 \cdot \dfrac{\sqrt{15}}{4} = \dfrac{\sqrt{15}}{4}$

$\therefore$ Area of $\Delta CEA = \dfrac{3\sqrt{15}}{8}$ (1)