Mathematical induction

Solutions Main Menu

68 Mathematical induction is used to prove

$$1^2+3^2+5^2+...+(2n-1)^2=\frac{1}{3}n(2n+1)(2n-1) \text{ for all positive integers } n\geq 1.$$

Which of the following has an incorrect expression for part of the induction proof?

(A) Step 1: To prove the statement true for $n=1$

$\text{LHS}=1^2=1 \quad \text{RHS}=\frac{1}{3}\times 1\times(2\times 1+1)(2\times 1-1)=1$

Result is true for $n=1$

(B) Step 2: Assume the result true for $n=k$

$1^2+3^2+5^2+...+(2k-1)^2=\frac{1}{3}(k+1)(2k+1)(2k-1)$

(C) To prove the result is true for $n=k+1$

$$\begin{aligned}&1^2+3^2+5^2+...+(2k-1)^2+\left(2(k+1)-1\right)^2\\&=\frac{1}{3}(k+1)\left(2(k+1)+1\right)\left(2(k+1)-1\right)\\&=\frac{1}{3}(k+1)\left(2k+3\right)\left(2k+1\right)\end{aligned}$$

(D)

$$\begin{aligned}\text{LHS}&=1^2+3^2+5^2+...+(2k-1)^2+\left(2(k+1)-1\right)^2\\&=\frac{1}{3}k(2k+1)(2k-1)+\left(2k+1\right)^2\\&=\frac{1}{3}(2k+1)\left(k(2k-1)+3\left(2k+1\right)\right)\\&=\frac{1}{3}(2k+1)\left(2k^2-k+6k+3\right)\\&=\frac{1}{3}(2k+1)\left(2k^2+5k+3\right)\\&=\frac{1}{3}(2k+1)(k+1)(2k+3)\\&=\text{RHS}\end{aligned}$$

69 Mathematical induction is used to prove

$$\sum_{r=1}^{n}\frac{r^2}{(2r-1)(2r+1)}=\frac{n(n+1)}{2(2n+1)} \text{ for all positive integers } n.$$

Which of the following has an incorrect expression for part of the induction proof?

(A) $\sum_{r=1}^{n}\frac{r^2}{(2r-1)(2r+1)}=\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+...+\frac{n^2}{(2n-1)(2n+1)}$

Step 1: To prove the statement true for $n=2$

$\text{LHS}=\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}=\frac{1}{3}+\frac{4}{15}=\frac{3}{5} \quad \text{RHS}=\frac{2(3)}{2(5)}=\frac{6}{10}=\frac{3}{5}$

Result is true for $n=2$

(B) Step 2: Assume the result true for $n=k$

$\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+...+\frac{k^2}{(2k-1)(2k+1)}=\frac{k(k+1)}{2(2k+1)}$

(C) To prove the result is true for $n=k+1$

$$\begin{aligned}&\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+...+\frac{(k+1)^2}{(2k+1)(2k+3)}=\frac{(k+1)(k+2)}{2(2k+3)}\\&=\frac{k(k+1)}{2(2k+1)}+\frac{(k+1)^2}{(2k+1)(2k+3)}\end{aligned}$$

(D)

$$\begin{aligned}\text{LHS}&=\frac{k(k+1)}{2(2k+1)}+\frac{(k+1)^2}{(2k+1)(2k+3)}\\&=\frac{k(k+1)(2k+3)+2(k+1)^2}{2(2k+1)(2k+3)}\\&=\frac{(k+1)(2k^2+3k+2k+2)}{2(2k+1)(2k+3)}\\&=\frac{(k+1)(2k+1)(k+2)}{2(2k+1)(2k+3)}\\&=\frac{(k+1)(k+2)}{2(2k+3)}=\text{RHS}\end{aligned}$$

70 Mathematical induction is used to prove

$$\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+...+\frac{2}{n\times(n+1)}=\frac{2n}{n+1} \text{ for all positive integers } n.$$

Which of the following is the correct expression for part of the induction proof?

(A) $\text{LHS}=\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+...+\frac{2}{k\times(k+1)}$
$=\frac{2k}{k+1}+\frac{2}{(k+1)}$

(B) $\text{LHS}=\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+...+\frac{2}{(k+1)}+\frac{2}{(k+2)}$
$=\frac{2k}{k+1}+\frac{2}{(k+2)}$

(C) $\text{LHS}=\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+...+\frac{2}{k\times(k+1)}+\frac{2}{(k+1)(k+2)}$
$=\frac{2k}{k+1}+\frac{2}{(k+1)(k+2)}$

(D) $\text{LHS}=\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+...+\frac{2}{k\times(k+1)}+\frac{2}{(k+1)(k+3)}$
$=\frac{2k}{k+1}+\frac{2}{(k+1)(k+3)}$

71 Victoria made an error proving that $3^{2n}-1$ is divisible by 8 (where n is an integer greater than 0) using mathematical induction. Part of the proof is shown below.

Step 2: Assume the result true for $n=k$

$3^{2k}-1=8P$ where P is an integer. Line 1

Hence $3^{2k}=8P+1$

To prove the result is true for $n=k+1$

$3^{2(k+1)}-1=8Q$ where Q is an integer. Line 2

$\text{LHS}=3^{2(k+1)}-1$

$=3^{2k}\times 3^2-1$

$=(8P+1)\times 3^2-1$ Line 3

$=72P+1-1$ Line 4

$=72P$

$=8(9P)$

$=8Q$

$=\text{RHS}$

Which line did Victoria make an error?

(A) Line 1
(B) Line 2
(C) Line 3
(D) Line 4

72 Mathematical induction is used to prove:

$(n)^3+(n+1)^3+(n+2)^3$ is divisible by 9 for all positive whole numbers n

Which of the following has an incorrect expression for part of the induction proof?

(A) Step 1: To prove the statement true for $n=1$

$(1)^3+(1+1)^3+(1+2)^3=1+8+27=36$

Divisible by 9

Result is true for $n=1$

(B) Step 2: Assume the result true for $n=k$

$(k)^3+(k+1)^3+(k+2)^3=9P$ where P is an integer.

Hence $(k+1)^3+(k+2)^3=9P-(k)^3$ - (1)

(C) To prove the result is true for $n=k+1$

$(k+1)^3+(k+2)^3+(k+3)^3=9Q$ where Q is an integer.

(D)

$\text{LHS}=(k+1)^3+(k+2)^3+(k+3)^3$

$=9P-(k)^3+(k+3)^3$ from (1)

$=9P-k^3+(k+3)(k^2-3k+9)$

$=9P-k^3+k^3-3k^2+9k+3k^2-9k+27$

$=9(P+3)$

$=9Q$

$=\text{RHS}$

Mathematical induction		Main Menu
	Solution	**Criteria**
68	Step 2: Assume the result true for $n=k$ $1^2+3^2+5^2+...+(2k-1)^2=\frac{1}{3}k(2k+1)(2k-1)$	1 Mark: B
69	$\sum_{r=1}^{n}\frac{r^2}{(2r-1)(2r+1)}=\frac{1^2}{1\times3}+\frac{2^2}{3\times5}+...+\frac{n^2}{(2n-1)(2n+1)}$ Step 1: To prove the statement true for $n=1$ $\text{LHS}=\frac{1^2}{1\times3}=\frac{1}{3}$ $\quad\text{RHS}=\frac{1(2)}{2(3)}=\frac{1}{3}$ Result is true for $n=1$	1 Mark: A
70	Step 2: Assume the result true for $n=k$ $\frac{2}{1\times2}+\frac{2}{2\times3}+\frac{2}{3\times4}+...+\frac{2}{k\times(k+1)}=\frac{2k}{k+1}$ To prove the result is true for $n=k+1$ $\frac{2}{1\times2}+\frac{2}{2\times3}+\frac{2}{3\times4}+...+\frac{2}{k\times(k+1)}+\frac{2}{(k+1)(k+2)}=\frac{2(k+1)}{(k+2)}$ $\text{LHS}=\frac{2}{1\times2}+\frac{2}{2\times3}+\frac{2}{3\times4}+...+\frac{2}{k\times(k+1)}+\frac{2}{(k+1)(k+2)}$ $=\frac{2k}{k+1}+\frac{2}{(k+1)(k+2)}$	1 Mark: C
71	$\text{LHS}=3^{2(k+1)}-1$ $=3^{2k+2}-1$ $=3^{2k}\times3^2-1$ $=(8P+1)\times3^2-1$ $=72P+9-1$ $=72P+8$ $=8(9P+1)$ $=8Q=\text{RHS}$ Line 4 contains the error	1 Mark: D
72	$\text{LHS}=(k+1)^3+(k+2)^3+(k+3)^3$ $=9P-(k)^3+(k+3)^3$ from (1) $=9P-k^3+(k+3)(k^2+6k+9)$ $=9P-k^3+k^3+6k^2+9k+3k^2+18k+27$ $=9(P+k^2+3k+3)$ $=9Q=\text{RHS}$	1 Mark: D

Binomial theorem		Main Menu
	Solution	**Criteria**
73	$T_6={}^9C_5(2x)^{9-5}(-3y)^5$ $={}^9C_5\times2^4\times x^4(-3)^5y^5$ $={}^9C_5\times2^4\times(-3)^5x^4y^5$	1 Mark: C
74	$T_{r+1}={}^9C_r(x^2)^{9-r}\left(-\frac{2}{x}\right)^r$ $={}^9C_rx^{18-2r}(-2)^rx^{-r}$ $={}^9C_r(-2)^rx^{18-3r}$ Term independent of x $18-3r=0$ $r=6$ $T_7={}^9C_6(-2)^6x^{18-3\times6}={}^9C_6(-2)^6$	1 Mark: B
75	$T_{r+1}={}^{12}C_r(2x^3)^{12-r}\left(-\frac{1}{x}\right)^r$ $={}^{12}C_r\times2^{12-r}\times(-1)^r\times x^{36-3r}\times x^{-r}$ $={}^{12}C_r\times2^{12-r}\times(-1)^r\times x^{36-4r}$ Term with a coefficient of x^8 $36-4r=8$ $4r=28$ $r=7$ $T_8={}^{12}C_7\times2^{12-7}\times(-1)^7\times x^{36-4\times7}$ $=-{}^{12}C_7\times2^5\times x^8$ Coefficient of x^8 is $-{}^{12}C_7\times2^5$	1 Mark: D
76	$T_{r+1}={}^{20}C_r(2x^2)^{20-r}\left(-\frac{1}{x}\right)^r$ $={}^{20}C_r\times2^{20-r}\times(-1)^r\times x^{40-2r}\times x^{-r}$ $={}^{20}C_r\times2^{20-r}\times(-1)^r\times x^{40-3r}$ Term with a coefficient of x^{-5} $40-3r=-5$ $3r=45$ $r=15$ $T_{16}={}^{20}C_{15}\times2^{20-15}\times(-1)^{15}\times x^{40-3\times15}$ $=-{}^{20}C_{15}\times2^5\times x^{-5}$ Coefficient of x^8 is $-{}^{20}C_{15}\times2^5$	1 Mark: D