CRANBROOK
SCHOOL

+ Soln's

Year 12 Mathematics Extension 1

HSC Half Yearly, March 2011

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 41

- Attempt Questions 1–6
- You will need 6 booklets

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

BEGIN A NEW BOOKLET (10)

a) Find a monic polynomial equation of degree 3 which has roots 2, -2, 1. **1**

b) i) Solve $P(x) = x^3 - 6x^2 + 11x - 6$ **3**

ii) Sketch this curve showing the intercepts only. **2**

c) If $\alpha\beta\gamma\delta$ are the roots of $2x^4 - 3x^3 + 8x - 1 = 0$, find

i) $\alpha\beta\gamma\delta$ **1**

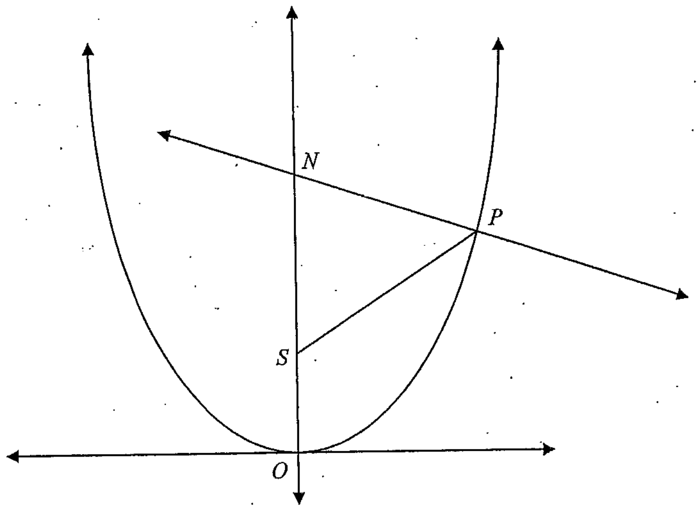
ii) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **1**

iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ **2**

Question 2

BEGIN A NEW BOOKLET (6)

S is the focus of the parabola $x^2 = 4ay$. A point $P(2at, at^2)$ is a variable point on the parabola



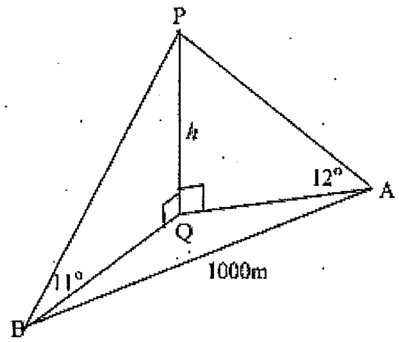
i) Show that the normal at P is given by $x + ty = 2at + at^3$. **3**

ii) If this normal cuts the y axis at N, show that $4OS.SP = PN^2$, where O is the origin. **3**

Question 3

BEGIN A NEW BOOKLET (5)

The angle of elevation of a tower PQ of height h from a point A due east of Q is 12°. From another point B, the bearing of the tower is 051°T and the angle of elevation is 11°. $AB = 1000$m and AB is on the same level as the base Q.



i) With the aid of a diagram show that $\angle AQB = 141°$ **1**

ii) Show that $AQ = h\tan 78°$ and $BQ = h\tan 79°$ **2**

iii) Using the cosine rule in $\triangle AQB$, find h (to the nearest metre) **2**

Question 4

BEGIN A NEW BOOKLET (10)

On the curve $y = x^2 e^{-x}$,

i)	find any maximum or minimum turning points	**3**
ii)	find any points of inflexion	**2**
iii)	find any intercepts	**1**
iv)	find any limits as $x \to \pm\infty$	**2**
v)	Sketch the curve	**2**

Question 5

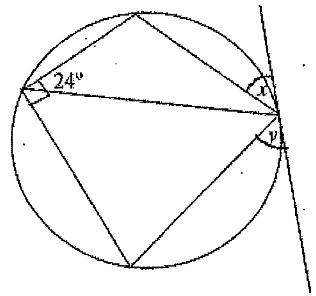
BEGIN A NEW BOOKLET (4)

Prove by Mathematical Induction that $7^n - 3^n$ is divisible by 4 **4**

Question 6

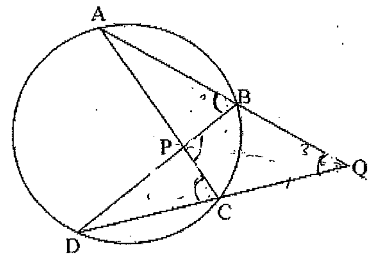
BEGIN A NEW BOOKLET (6)

a)



Find the value of x and y. **3**

b)



Prove that $\angle BPC + \angle BQC = 2\angle ABD$ **3**

(hint: let $\angle ABD = x^o$)

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✓ = 1 MARK

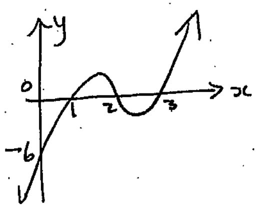
1 a) $P(x) = (x-3)(x+2)(x-1)$ ✓

This is sufficient & Many did a lot of time consuming unnecessary working here. Important to keep the simple rules in mind too!

b) i) $P(1) = 1 - 6 + 11 - 6$
$= 0$

$\therefore x - 1$ is a factor ✓

$\therefore P(x) = x^3 - 6x^2 + 11x - 6$
$= (x-1)(x^2 - 5x + 6)$ ✓
(BY inspection OR use long division)
$= (x-1)(x-2)(x-3)$ ✓

ii) 

c)(i) $\alpha\beta\gamma\delta = \frac{e}{a} = -\frac{1}{2}$ ✓

V. Well answered - showing care with details.

(ii) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} = \frac{-8}{2} = -4$ ✓

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{-4}{-\frac{1}{2}} = 8$ ✓ ✓

2 i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$\therefore$ at $x = 2at$ ✓

$M_T = \frac{2at}{2a} = t$

then $M_N = -\frac{1}{t}$ ✓

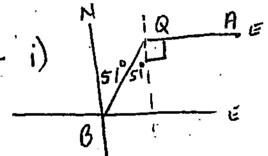
Eq'n of normal
$y - at^2 = -\frac{1}{t}(x - 2at)$
$ty - at^3 = -x + 2at^2$
$\therefore x + ty = 2at + at^3$ ✓

ii) at N $x = 0$ $\therefore ty = 2at + at^3$
$y = 2a + at^2$
$= a(2 + t^2)$ ✓

S is $(0,a)$ the focus
$SP = \sqrt{(2at)^2 + (at^2 - a)^2}$
$= \sqrt{4a^2t^2 + a^2t^4 - 2a^2t^2 + a^2}$
$= a\sqrt{t^4 + 2t^2 + 1}$
$= a\sqrt{(t^2+1)^2}$

$PN = \sqrt{4a^2t^2 + (at^2 - 2a - at^2)^2}$ ✓
$PN^2 = (\sqrt{4a^2(t^2+1)})^2$
LHS $= 4 \times a \times SP = 4a^2(t^2+1)$ ✓
RHS $= PN^2 = 4a^2(t^2+1)$
$\therefore 4OS.SP = PN^2$ ✓

Q2 Here again this was well done when set out clearly and attention given to detail.

Q3 i) 

$\therefore \angle BQA = 141°$ ✓

$\tan 12° = \frac{h}{AQ}$ $\tan 11° = \frac{h}{BQ}$

$AQ = \frac{h}{\tan 12°}$ $\therefore BQ = \frac{h}{\tan 11°}$
$= h\cot 12°$ $= h\cot 11°$
$= h\tan 78°$ $= h\tan 79°$
(or similar) ✓ ✓

(ii) $a^2 = b^2 + c^2 - 2bc\cos A$

$1000^2 = h^2\tan^2 79° + h^2\tan^2 78° - 2h\tan 79° \times h\tan 78° \cos 141°$ ✓
$= h^2(\tan^2 79° + \tan^2 78° - 2\tan 79° \tan 78° \cos 141°)$

$\therefore h = \sqrt{\frac{1000^2}{(\tan^2 79° + \tan^2 78° - 2\tan 79° \tan 78° \cos 141°)}}$ ✓

$= 108$ m (nearest metre)

- generally well done though diagrams need more practice.

Q4 i) $y = x^2 e^{-x}$ $y' = x^2(-e^{-x}) + e^{-x}2x$
$0 = xe^{-x}(-x+2)$ ✓

$\therefore$ st pts are $x = 0$, $y = 0$
$x = 2$, $y = \frac{4}{e^2}$

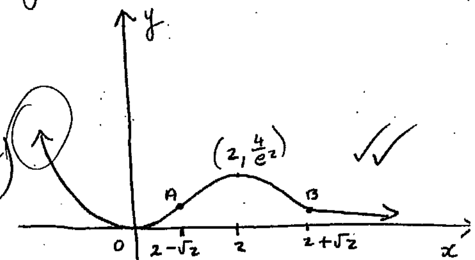
$\therefore$ using table below $(0,0)$ is MIN ✓
$(2, \frac{4}{e^2})$ is MAX ✓

use $y' = e^{-x}(2x - x^2)$

ii) $y'' = e^{-x}(2 - 2x) + (2x - x^2)(-e^{-x})$
$= e^{-x}(2 - 2x - 2x + x^2)$
$0 = e^{-x}(x^2 - 4x + 2)$ ✓
$x = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$

x	0	$2-\sqrt{2}$	2	$2+\sqrt{2}$	4
y''	+	0	−	0	+

$\therefore$ inflexions at A $(2-\sqrt{2}, (2-\sqrt{2})^2 e^{\sqrt{2}-2})$
and B $(2+\sqrt{2}, (2+\sqrt{2})^2 e^{-2-\sqrt{2}})$ ✓
see table for tests



✓✓

iii) $x = 0$, $y = 0$ ✓
iv) as $x \to \infty$ $y \to 0$ as $x \to -\infty$, $y \to +\infty$ ✓✓

5 Prove true for $n=1$

$7^1 - 3^1 = 4$ which is divisible by 4. ✓

Assume true for $n=k$

ie $7^k - 3^k = 4Q$ $(Q \in J^+)$ ✓

$7^k = 4Q + 3^k$

Prove true for $n = k+1$

ie $7^{k+1} - 3^{k+1} = 4P$ $(P \in J^+)$

$LHS = 7^k \times 7 - 3^k \times 3$

$= (4Q + 3^k)7 - 3^k \times 3$ ✓

$= 28Q + 4(3^k)$

$= 4(7Q + 3^k)$

$= 4P$ where $P = 7Q + 3^k$

∴ If true for $n=k$ this is true for $n=k+1$. But it is true for $n=1$

∴ true for 2, 3 .. all integral k. ✓

∴ Result is proven true by M I.

Q5 handled very well by most.

6/ a) $\angle EDB = \angle BAD$

∴ $x = 24°$ Angle between tangent & chord equals ∠ in the alternate segment. ✓

$\angle BDC = 90°$ (OPP ∠'s OF CYCLIC QUAD)

∴ $\angle CDF = 180° - 114°$ (∠'s on st. line) ✓✓

∴ $y = 66°$ //

generally well done

6 b $\angle ABD = \angle ACD = x$ (angles subtended by the same arc.) ✓

$\angle ACQ = \angle QBD = 180 - x$ (Straight ∠'s)

Since PBQC is a quadrilateral ✓

$2(180 - x) + \angle BQC + \angle BPC = 180°$

∴ $\angle BPC + \angle BQC = 2x$

$= 2\angle ABD$ // ✓

Some students tried to find these 2 angles individually – note this was not asked – this highlights the need to RTQ.

Q4 note The method for these more complex curves is largely the same as for the simple polynomials – but yet again it highlights the crucial nature of reading the question answering precisely as asked, and setting out clearly and logically.