CRANBROOK SCHOOL

Term 1, 2011

Year 12 Mathematics

Term 1 Examination

Monday 28th March, 2011

Time Allowed: 1.5 hours, plus 5 minutes reading time

Total Marks: 67

There are 3 questions.

Submit your work in three 4 Page booklets.

All necessary working should be shown in every question.

Full marks may not be awarded if work is careless or badly arranged. Board of Studies approved calculators may be used.

A table to standard integrals can be found on the back page

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \ln\left\{x+\sqrt{(x^2-a^2)}\right\}, \quad |x|>|a|$$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \ln\left\{x+\sqrt{(x^2+a^2)}\right\}$$

NOTE : $\ln x = \log_e x; \quad x>0$

D-15

30m

PART A – Series (23 Marks) Marked by GHW

1. If the second term of a geometric series is -18 and the fifth term is -486, find:

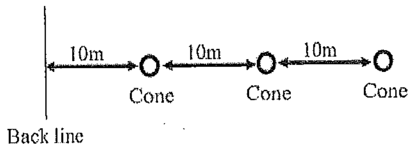
   a) the first term and the common ratio ✓ **2**

   b) S_{10} **2**

2. Mr Sadler is saving for a 2008 Manly Sea Eagles Premiership winning signed jersey. He puts $120 in his account in the first week, but each subsequent week he is only able to save 95% of the previous week's amount as he has become addicted to buying Star Trek memorabilia from the internet.

   If the jersey costs $2387, will he be able to save enough money to afford it? ✓ **2**

3. The coach of a rugby team wants to make his team run 1.1km but he decides this will be more effective if they do it in shuttle runs. This means they start at the back line, run to a cone, run back to the back line, run out further to the next cone, run back to the back line etc. He places cones 10m apart down the pitch.



   a) Write the first 3 terms of the sequence created by this activity **2**

   b) How many cones will they have to run between to cover the distance? ✓ **3**

4. Rob purchased a new home theatre from Harvey Norman for $4500. The terms of his purchase were over 3 years at 6% p.a. compounded monthly with no repayments for the first 12 months.

   a) Write an equation to show the amount owing after 12 months **1**

   b) Show that the amount owing after 15 months is:
   $A_{15} = 4500(1.005)^{15} - M(1 + 1.005 + 1.005^2)$ **2**

   c) Find the size of each monthly repayment **3**

5. From January 1st 1988, Sarah puts $900 into her superannuation account every month. It is compounded quarterly at 12% p.a.. She retires at the end of December 2010.

   a) What is the value of her first $900 when she retires? **1**

   b) How much will her whole investment be worth when she retires? **3**

   c) How much interest will she earn? **2**

PART B – Geometrical Applications of Calculus (20 Marks) — Marked by RABS

1. Consider the function $y = x^3 - 6x^2 + 9x - 2$.

   a) Show that $\frac{dy}{dx} = 3(x-1)(x-3)$ — **1**

   b) Determine what values of x for which $y = x^3 - 6x^2 + 9x - 2$ is decreasing. Include all necessary working. — **2**

2. Consider the function $f(x) = \frac{1}{3}x^3 - x^2 - 3x - 6$.

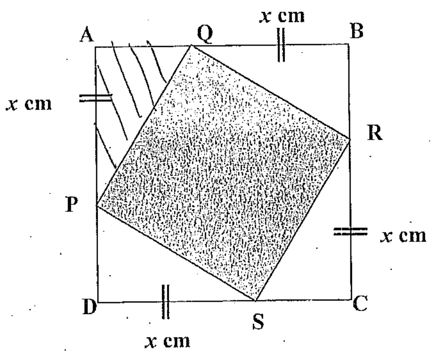
   a) Find any stationary points and determine their nature. — **3**

   b) Find any points of inflexion. — **2**

   c) Hence sketch the graph of $y = \frac{1}{3}x^3 - x^2 - 3x - 6$, showing all critical points. — **4**

   d) What is the maximum value of this function for $-2 < x < 6$? — **2**

3. ABCD is a square of side 12cm. PQRS is another square with vertices touching the sides of ABCD as shown. BQ = RC = DS = AP = x cm.



   a) Find an expression for the area of triangle APQ — **1**

   b) Hence, find an expression for the area PQRS in terms of x. — **2**

   c) Find x such that the area of PQRS is a minimum. — **3**

PART C – Integration, Exponentials (24 Marks) — Marked by CRA

1. Given $y = e^x + e^{-x}$, prove that $\frac{d^2y}{dx^2} = y$ — **2**

2. Find the gradient of the tangent to the curve $y = e^{-2x+1}$ at the point where $x = -\frac{1}{2}$ — **2**

3. Evaluate the following integrals, leaving your answer in exact form where necessary:

   a) $\int \frac{x^5 - 1}{x^2}\, dx$ — **2**

   b) $\int_{-1}^{0} e^{2x-1}\, dx$ — **2**

4. If $f''(x) = 2x - 1$, find $f(x)$ if $f'(0) = 3$ and $f(1) = 2\frac{5}{6}$ — **2**

5. Find the area enclosed by the curve $y = x^3$, the x axis, and the lines $x = -2$ and $x = 2$. — **3**

6. Consider the curves $y = x^2$ and $x = y^2$

   a) Find the points of intersection of the two curves. — **2**

   b) Calculate the area bounded by the given curves — **2**

7. Use Simpsons rule with four sub-intervals to find an approximate value of $\int_0^2 e^{x^2}\, dx$, correct to 2 d.p. — **3**

8. The arc of the parabola $y = x^2 - 4$ between the lines $y = 0$ and $y = -4$ is rotated about the y-axis. Find the volume of the solid of revolution formed. Leave your answer as an exact value. — **4**

Section A

1a) $T_n = ar^{n-1}$
$T_2 = ar = -18$
$T_5 = ar^4 = -486$

$\therefore r^3 = 27$
$r = 3$ ①
$a = -6$ ①

b) $S_{10} = \frac{a(r^n - 1)}{r-1}$

$= \frac{-6(3^{10}-1)}{3-1}$ ①

$= -177144$ ①

2 a. $S_\infty = \frac{a}{1-r}$ ①

$S_\infty = \frac{120}{1-0.95}$
$= \$2400$
$\therefore$ he can afford the jersey ①

3. a) 20, 40, 60 ②

b) $S_n = \frac{n}{2}[2a + (n-1)d]$
$1100 = \frac{n}{2}[40 + (n-1)20]$ ①

$2200 = 40n + 20n^2 - 20n$
$20n^2 + 20n - 2200 = 0$ ①

$n^2 + n - 110 = 0$
$(n+11)(n-10) = 0$
$\therefore n = 10$ ①
as $n \neq 11$ as $n > 0$

4a) $A_{12} = 4500(1.005)^{12}$ ①

b) ①
$A_{13} = 4500(1.005)^{13} - M$
$A_{14} = 4500(1.005)^{14} - M(1 + 1.005)$
$A_{15} = 4500(1.005)^{14} - M(1 + 1.005 + 1.005^2)$ ①

c) $A_{36} = 4500(1.005)^{36} - M\left(\frac{1.005^{24} - 1}{1.005 - 1}\right)$

where $A_{36} = 0$. ①

$\therefore M\left(\frac{1.005^{24} - 1}{1.005 - 1}\right) = 4500(1.005)^{36}$ ①

$M = 4500(1.005)^{36} \times \frac{(1.005 - 1)}{1.005^{24} - 1}$

$= \$211.74$ ①

5 a) $900 \times 1.03^{92} = \$13\,654.23$ ①

b)
$A_{92} = (3 \times 900)(1.03)$
$A_{91} = (3 \times 900)(1.03)^2$
$A_{90} = (3 \times 900)(1.03)^3$ ①
$A_n = (3 \times 900)(1.03)^n$

$\therefore S_{92} = 2700\left(\frac{1.03(1.03^{92} - 1)}{1.03 - 1}\right)$ ①

$= \$1\,313\,685.59$ (2dp). ① ①

c) $I = \$1313685.59 - 900 \times 12 \times 23$
$= \$1065285.59$ ①

YEAR 12 2011 T1 EXAM SOL'NS

PART B

1a $y = x^3 - 6x^2 + 9x - 2$

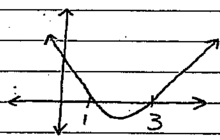
$y' = 3x^2 - 12x + 9$
$= 3(x^2 - 4x + 3)$
$= 3(x-1)(x-3)$ ✓

· SOME WROTE THE QUESTION THEN THE REQUIRED ANSWER. TO GET THE MARKS, SHOW WORKING!

b SOLVE FOR $y' < 0$

$0 > 3(x-1)(x-3)$

QUADRATIC INEQUALITY: DRAW GRAPH.



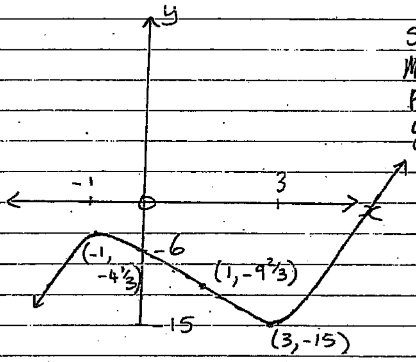
SELECT THE RANGE OF VALUES < 0

$\therefore 1 < x < 3$ ✓✓

· IT WAS IMPORTANT TO SHOW ALL WORKING, AS REQUESTED, TO GET FULL MARKS.

· TOO MANY STUDENTS FAILED TO TEST & CONFIRM WHETHER IT WAS A POINT OF INFLEXION. $f''(x) = 0$ DOESN'T AUTOMATICALLY MEAN THAT IT IS!

C.



SHAPE ✓
MAX/MIN ✓
POI ✓
y-INT ✓

· SHOULD HAVE BEEN ANSWERED BETTER. MAKE SURE GRAPHS ARE LARGE ENOUGH & NEAT.

· A LOT OF STUDENT ADDED A LARGE "KINK" TO HIGHLIGHT THE POI

· FACTORISING A CUBIC TO FIND x-INTS IS GENERALLY TOO HARD. DON'T WASTE TIME ON IT!

2a. $f(x) = \frac{1}{3}x^3 - x^2 - 3x - 6$

$f'(x) = x^2 - 2x - 3$

STAT PT: $0 = x^2 - 2x - 3$
$0 = (x-3)(x+1)$ ✓ $\therefore x = 3$ & -1

DETERMINE NATURE:

$f''(x) = 2x - 2$

$f''(3) = 4$, $f''(3) > 0$ $\therefore (3, -15)$ MIN ✓

$f''(-1) = -4$ $f''(-1) < 0$ $\therefore (-1, -4\frac{1}{3})$ MAX.

· QUITE A FEW DID NOT DETERMINE EACH STAT. PT'S NATURE

· SOME SILLY SUBSTITUTION ERRORS

b. POSS. PT. OF INFLEX. AT $f''(x) = 0$

$0 = 2x - 2$
$\therefore x = 1$ ✓

CHECK!

x	0.9	1	1.1
$f''(x)$	−	0	+

✓

CHANGE IN CONCAVITY

$\therefore$ POI AT $(1, -9\frac{2}{3})$

d CHECK $x = -2$ & $x = 6$

$\left.\begin{array}{l} f(-2) = -6\frac{2}{3} \\ f(6) = 12 \end{array}\right\}$ ✓

$\therefore$ MAX VALUE IS 12 ✓

· NOTE THAT THE QUESTION, STRICTLY SPEAKING, SHOULD SAY $-2 \leq x \leq 6$. ONLY ONE STUDENT RECOGNISED THIS

· MANY STUDENTS DIDN'T BOTHER TO CHECK $f(-2)$ WITHOUT STATING REASONS WHY. BE CAREFUL!

· MAX VALUE ≠ MAX POINT $(12 \neq (6, 12))$

3a Area $= \frac{1}{2} b \times h$
$= \frac{1}{2} x(12 - x)$ ✓

· MANY DID NOT WRITE "IN TERMS OF x"

b Area = TOTAL AREA − 4 TRIANGLES
$= 144 - 4x\left[\frac{1}{2}x(12-x)\right]$ ✓

$= 144 - 2x(12-x)$
$= 144 - 24x + 2x^2$
or
$A = 2x^2 - 24x + 144$ ✓

- MANY STUDENTS FOLLOWED THE HARDER PATH OF FINDING THE HYPOTENUSE OF EACH TRIANGLE THEN THE AREA OF THE SQUARE

(c) LET $y = 2x^2 - 24x + 144$

$y' = 4x - 24$

STAT PTS: $0 = 4x - 24$ ✓

$\therefore x = 6$ ✓

CHECK! $y'' = 4$

$\therefore y'' > 0 \quad \therefore$ MIN! ✓

- MANY STUDENTS DID NOT CHECK THE ANSWER WAS A MIN!

PART C - CRA

	Marks	Comments
(1) If $y = e^{x} + e^{-x}$		
$\frac{dy}{dx} = e^{x} - e^{-x}$	✓	Done well on the whole.
$\frac{d^2y}{dx^2} = e^{x} + e^{-x}$	✓	
$\therefore \frac{d^2y}{dx^2} = y$		
(2) $y = e^{-2x+1}$		
gradient $= m = y' = (-2)e^{-2x+1}$ $= -2e^{-2x+1}$	✓	Some getting differentiation/integration confused. Done well on the whole.
when $x = -\frac{1}{2}$, $y' = -2e^{-2(-1/2)+1}$ $= -2e^{2}$	✓	
(3) (a) $\int \frac{x^5 - 1}{x^2}\,dx$		
$= \int \frac{x^5}{x^2} - \frac{1}{x^2}\,dx$		Had to break up the original fraction into two fractions first.
$= \int x^3 - x^{-2}\,dx$	✓	
$= \frac{x^4}{4} - \frac{x^{-1}}{-1} + c$		
$= \frac{x^4}{4} + \frac{1}{x} + C$	✓	Only a couple forgot "+C" Well done!
(b) $\int_{-1}^{0} e^{2x-1}$		
$= \left[\frac{e^{2x-1}}{2}\right]_{-1}^{0}$	✓	Again, a few confused with integration + differentiation, otherwise done well on the whole.
$= \left(\frac{e^{-1}}{2}\right) - \left(\frac{e^{-3}}{2}\right)$		
$= \frac{1}{2e} - \frac{1}{2e^3}$	✓	

CRA

$f''(x) = 2x - 1$

$f'(x) = \frac{2x^2}{2} - x + c$

$\therefore f'(x) = x^2 - x + c$

If $f'(0) = 3$

$\therefore 3 = 0 - 0 + c$

$\therefore c = 3.$

$\therefore f'(x) = x^2 - x + 3$ ✓

$f(x) = \frac{x^3}{3} - \frac{x^2}{2} + 3x + c$

If $f(1) = 2\frac{5}{6}$

$2\frac{5}{6} = \frac{1}{3} - \frac{1}{2} + 3 + c$

$2\frac{5}{6} = 2\frac{5}{6} + c$

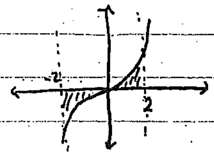
$\therefore c = 0$

$\therefore f(x) = \frac{x^3}{3} - \frac{x^2}{2} + 3x$ ✓

Done well on the whole.

One mark for finding $f'(x)$ and one mark for finding $f(x)$.

$y = x^3$



$\therefore A = 2\int_0^2 x^3\,dx$ ✓

$= 2\left[\frac{x^4}{4}\right]_0^2$ ✓

$= 2\left[\left(\frac{2^4}{4}\right) - \left(\frac{0^4}{4}\right)\right]$

$= 8$ units^2 ✓

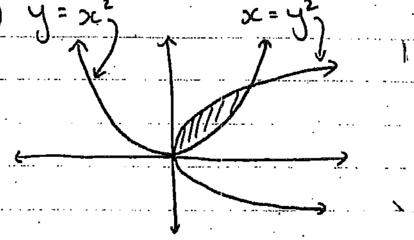
One mark given for recognising x^3 as an odd function.

Many solved $\int_{-2}^{2} x^3\,dx = 0$ units^2 ✗

Remember to write your units in!!

CRA

⑥ $y = x^2$ $\quad x = y^2$



(a) $y = x^2 \quad x = y^2$

$\therefore x = x^4$

$0 = x^4 - x$

$0 = x(x^3 - 1)$ ✓

$\therefore x = 0, 1 \quad \therefore @ (0,0)\ (1,1)$ ✓

(b) $\therefore A = \int_0^1 x^{1/2}\,dx - \int_0^1 x^2\,dx$

$= \int_0^1 x^{1/2} - x^2\,dx$

$= \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$ ✓

$= \left[\frac{2x^{3/2}}{3} - \frac{x^3}{3}\right]_0^1$

$= \left(\frac{2(1)^{3/2}}{3} - \frac{(1)^3}{3}\right) - \left(\frac{2(0)^{3/2}}{3} - \frac{(0)^3}{3}\right)$

$= \left(\frac{2}{3} - \frac{1}{3}\right) - (0)$

$= \frac{1}{3}$ units^2 ✓

Always draw a quick sketch. Many boys had their equation the wrong way around:

ie $\int_0^1 x^2\,dx - \int_0^1 x^{1/2}\,dx$ ✗.

Simple fraction/integer work stumbled many.

Ensure you show all working + steps.

CRA

(7) $\int_0^2 e^{x^2}\,dx$ 4 sub intervals

$= 5$ function values.

$h = \frac{2-0}{4} = 0.5$ ✓

x	0	0.5	1	1.5	2
$f(x)$	1	$e^{0.25}$	e^{1}	$e^{2.25}$	e^{4}
		$\times 4$	$\times 2$	$\times 4$	

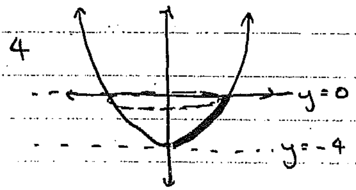
Many boys tried to integrate $\int_0^2 e^{x^2}\,dx$ before using simpsons rule...?

$\therefore = \frac{0.5}{3}\left[1 + 4\left(e^{0.25} + e^{2.25}\right) + 2\left(e^{1}\right) + e^{4}\right]$ ✓

Remember your formula!!

$= 17.35$ ✓

(8) $y = x^2 - 4$



$y = x^2 - 4$

$y + 4 = x^2$

$\therefore x = \sqrt[2]{y+4}$ ✓

Plenty of boys did not recognise the curve is being rotated about the y-axis. Hence the equation must be $x = \ldots$

$\therefore V = \pi \int_{-4}^{0} x^2\,dy$

$= \pi \int_{-4}^{0} y + 4\,dy$ ✓

$= \pi \left[\frac{y^2}{2} + 4y\right]_{-4}^{0}$ ✓

$= \pi\left[\left(\frac{(0)^2}{2} + 4(0)\right) - \left(\frac{(-4)^2}{2} + 4(-4)\right)\right]$

$= \pi\left[0 - (-8)\right]$

$= 8\pi$ units^3 ✓

Ensure you write your units!!