International Grammar School

Year 12
Mathematics Extension 1
HSC Assessment Task 2
2011

General Instructions

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- All necessary working should be shown in every question

Note: Any time you have remaining should be spent revising your answers.

Total marks – 60

- Attempt Questions 1 – 5
- All questions are of equal value
- Start each question in a new writing booklet
- Write your name on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your name and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Total Marks – 60
Attempt Questions 1–5
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) *Marks*

(a) Solve for x: $\frac{2}{x-1} \leq 3$ *3*

(b) (i) Find the exact values of the gradients of the tangents to the curve $y = \ln x$ at the points where $x = 1$ and $x = 5$. *1*

(ii) Find the acute angle between these tangents to the nearest degree. *2*

(c) The point $P(4,1)$ divides the interval AB internally in the ratio 3:1. If A is the point $(1,-2)$, find the coordinates of B. *2*

d) The letters of the word SQUARE are arranged in a circle. In how many arrangements are all three vowels next to each other? *2*

(e) Show that $\sin 2\theta \cot\theta - \cos 2\theta = 1$. *2*

End of Question One.

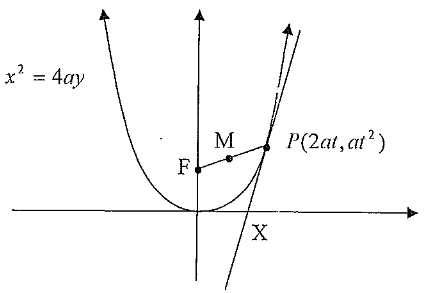
Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Use the substitution $u = x^2 - 4$ to evaluate $\int \frac{2x}{\sqrt{x^2-4}}dx$ 3

(b) Use the substitution $u = 1+\log_e x$ to evaluate 4

$$\int_1^e \frac{dx}{x\sqrt{1+\log_e x}}$$

(c)



$P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$ with focus F. The tangent to the parabola t P cuts the x axis at X. M is the midpoint of PF.

(i) Show that the tangent to the parabola at P has equation $tx - y - at^2 = 0$. 2

(ii) Show that MX is parallel to the y axis. 3

End of Question Two.

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

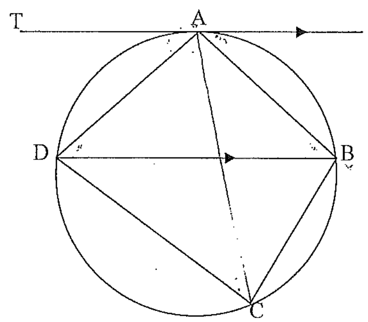
(a) Two of the roots of $x^3 + kx^2 + 4 = 0$ are equal. Find all three roots and the value of *k*. 3

(b) (i) Factorise $3x^3 + 3x^2 - x - 1$. 2

(ii) Hence solve: 3

$3\tan^3\theta + 3\tan^2\theta - \tan\theta - 1 = 0$, $0^\circ \le \theta \le 180^\circ$.

(c)



In the diagram above, ABCD is a cyclic quadrilateral. TA is a tangent to the circle at A. TA || DB.
Copy the diagram onto your paper.

(i) Show that $\angle ACD = \angle ADB$. 2

(ii) Hence show that AC bisects $\angle BCD$. 2

End of Question Three.

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Use Mathematical Induction to prove that $3^{2n}+7$ is divisible by 8 for all positive integers $n \geq 1$, 3

(b) Use Mathematical Induction to prove that $\frac{2^n}{n!} < 1$ for all positive integer values of $n \geq 4$. 4

(c) (i) Use change of base to show that $\frac{1}{\log_a b} = \log_b a$. 2

(ii) Hence evaluate: 3

$$\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\ldots+\frac{1}{\log_{100} 100!}$$

End of Question Four.

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider $y = f(x)$ where $f(x) = \log_e x - x^2$.

(i) Find the coordinates of any stationary points and any points of inflexion and determine their nature. 3

(ii) By considering the domain of $y = f(x)$ and the second derivative, explain why $y = f(x)$ has no solutions for x=0 and y=0. 2

(iii) Describe the behaviour of $y = f(x)$ as $x \to 0$ and $x \to \infty$. 2

(iv) Sketch the curve of $y = f(x)$. 3

(b) Find the value of $\frac{dy}{dx}$ at $x=1$ for $y = 10^x$ 2

End of Examination

(a) Solve $\frac{2}{x-1} \le 3. \quad x \ne 1$

$(x-1)^2 \cdot \frac{2}{(x-1)} < 3(x-1)^2$

$2(x-1) < 3(x-1)^2$

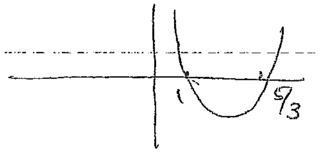
$3(x-1)^2 - 2(x-1) > 0$

$(x-1)[3(x-1)-2] > 0$

$(x-1)(3x-3-2) > 0$

$(x-1)(3x-5) > 0$

$x < 1, \; x \geq \frac{5}{3}.$



b) $y = \ln x$

$y' = \frac{1}{x}$

i) at $x = 1 \quad m_1 = 1$

at $x = 5 \quad m_2 = \frac{1}{5}$

ii) $\tan\theta = \left| \frac{1 - \frac{1}{5}}{1 + \frac{1}{5}} \right|$

$= \left| \frac{\frac{4}{5}}{\frac{6}{5}} \right|$

$= \frac{2}{3}$

$\theta = 33°41'$

$= 34°$

(c) A $(1,-2)$ B (x, y) Q1

$3 : 1$

P $(4, 1)$

$4 = \frac{1+3x}{4}$

$16 = 1 + 3x$

$3x = 15$

$x = 5$

$1 = \frac{-2+3y}{4}$

$4 = -2 + 3y$

$3y = 6$

$y = 2$

$\therefore$ B $(5, 2)$

d) SQUARE

UAE SQR $n = 4$

Total arrangements $= 3! \times 3!$

$= 36$

3.

e ii) $\operatorname{cosec} 2\theta - \cot 2\theta = \tan\theta$

LHS $\cdot$ $\operatorname{cosec} 2\theta - \cot 2\theta = \dfrac{1}{\sin 2\theta} - \dfrac{1}{\tan 2\theta}$

$= \dfrac{1}{2\cos\theta\sin\theta} - \dfrac{(1 - \tan^2\theta)}{2\tan\theta}$

$= \dfrac{1}{2\cos\theta\sin\theta} - \dfrac{\left(1 - \dfrac{\sin^2\theta}{\cos^2\theta}\right)}{\dfrac{2\sin\theta}{\cos\theta}}$ ✓

$= \dfrac{1}{2\cos\theta\sin\theta} - \dfrac{(\cos^2\theta - \sin^2\theta)}{2\cos\theta\sin\theta}$

$= \dfrac{1 - \cos^2\theta + \sin^2\theta}{2\cos\theta\sin\theta}$ ✓

$= \dfrac{1 - (1-\sin^2\theta) + \sin^2\theta}{2\cos\theta\sin\theta}$

$= \dfrac{2\sin^2\theta}{2\cos\theta\sin\theta}$ ✓

$= \tan\theta$.

4.

ua.

a) $\displaystyle\int \frac{2x\,dx}{\sqrt{x^2-4}}$

$u = x^2 - 4$

$du = 2x\,dx$.

$= \displaystyle\int \frac{du}{\sqrt{u}}$ ✓

$= \displaystyle\int u^{-1/2}.du$

$= 2u^{1/2} + C$ ✓

$= 2\sqrt{x^2-4} + C$ ✓

b) $\displaystyle\int_1^e \frac{dx}{x\sqrt{1+\log_e x}}$ ✓

$u = 1 + \ln x$

$du = \dfrac{dx}{x}$

$x = 1 \quad u = 1$

$x = e \quad u = 2$

$= \displaystyle\int_1^2 \frac{du}{\sqrt{u}}$ ✓

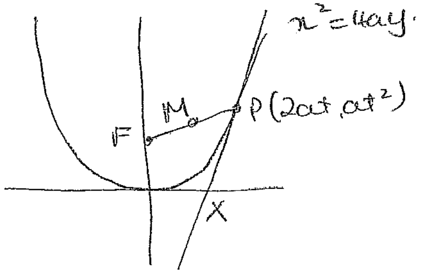
$= \displaystyle\int_1^2 u^{-1/2}.du$

$= \left[2u\right]_1^2$ ✓

$= \left[2\sqrt{u}\right]_1^2$

$= 2\sqrt{2} - 2$ ✓

c)



i) $x^2=4ay$

$y=\frac{x^2}{4a}$

$y'=\frac{2x}{4a}$

at P, $m_T=\frac{4at}{4a}=t$

Eqⁿ: $y-at^2=t(x-2at)$

$=tx-2at^2$

$tx-y-at^2=0$

ii). M midpt $S(0,a)$ $P(2at,at^2)$.

$x=\frac{2at}{2}$ $y=\frac{a+at^2}{2}$ $M\left(at,\frac{a+at^2}{2}\right)$

$=at$

$X: y=0.$ $tx=at^2$

$x=at.$

$X(at,0).$

ie since M and X have same x coord, $x=at$, they lie on the vertical line $x=at$ which is ∥ to y axis

(a) $x^3+kx^2+4=0.$

roots $\alpha,\alpha,\beta.$

$2\alpha+\beta=-k.$

$\alpha^2+\alpha\beta+\alpha\beta=0$

$\alpha^2+2\alpha\beta=0.$

$\alpha(\alpha+2\beta)=0.$

$\alpha\neq0$ $\beta=-\frac{\alpha}{2}.$

$\alpha^2\beta=-4.$

$\alpha^2\cdot-\frac{\alpha}{2}=-4$

$\alpha^3=8$

$\alpha=2.$

$\beta=-\frac{2}{2}=-1.$

so $2\alpha+\beta=-k.$

$4-1=-k$

$k=-3.$

so roots: 2, 2, −1

$k=-3.$

(b) i) $3x^3 + 3x^2 - x - 1 = P(x)$.

try $x = -1$.

$P(-1) = 0$

$\therefore (x+1)$ factor

$$\begin{array}{r} 3x^2 - 1 \\ x+1 \overline{) 3x^3 + 3x^2 - x - 1} \\ \underline{3x^3 + 3x^2} \\ 0 - x - 1 \\ \underline{-x - 1} \\ 0 \end{array}$$

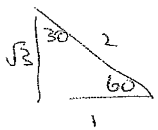
$P(x) = (x+1)(3x^2 - 1)$

$= (x+1)(\sqrt{3}x + 1)(\sqrt{3}x - 1)$. ✓✓

ii) $3\tan^3\theta + 3\tan^2\theta - \tan\theta - 1 = 0 \quad 0° \le \theta \le 180°$

from above.

$\tan\theta = -1, \pm\frac{1}{\sqrt{3}}$.

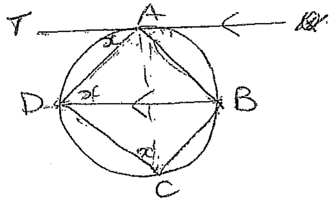


$\tan\theta = -1$ Q1, Q2 $\qquad \tan\theta = \pm\frac{1}{\sqrt{3}}$

$\theta = 135°$ $\qquad \theta = 30°, 150°$

$\therefore \theta = 30°, 135°, 150°$. ✓✓✓

Q3 (c)



i) $\angle ACD = \angle ADB$

$\angle TAD = \angle ACD$ (angle between tangent & chord = angle in opp segment) ✓

and $\angle TAD = \angle ADB$ (alternate angles $TA \parallel DB$)

$\therefore \angle ACD = \angle ADB$ ✓

ii) AC bisects $\angle BCD$.

$\angle ADB = \angle ACB$ (angles in same segment) ✓

$= \angle ACD$ (above)

$\therefore$ AC bisects $\angle BCD$ ✓

Q4

(a) $3^{2n}+7=8A \quad A\in \mathbb{Z}, n\geq 1.$

1. prove true $n=1$.

$3^{2}+7=9+7$

$=16$

$=8\times 2$

$\therefore$ true $n=1$.

2. Assume true $n=k$.

ie $3^{2k}+7=8B \quad B\in\mathbb{Z}$

3. Hence prove true $n=k+1$.

ie $3^{2k+2}+7=8C \quad C\in\mathbb{Z}$.

now $3^{2k+2}+7 = 3^2.3^{2k}+7$

$= 3^2\left[3^{2k}+7\right]-56$

$= 3^2\left[8B\right]-8\times 7$ using assumption in 2)

$= 8\left[9B-7\right]$

$= 8C$ since $9, B, 7 \in \mathbb{Z}$

so since true for $n=1$, from steps 2 + 3 also true for $n=2$ and so on for all $n\geq 1$.

Q4

(b) $\frac{2^n}{n!} < 1$ ie $\frac{2^n}{n!}-1<0 \quad n\geq 4$

1. Prove true $n=4$

$\frac{2^4}{4!}-1 = \frac{16}{24}-1$

$= \frac{2}{3}-1$

$= -\frac{1}{3} < 0$

$\therefore$ true $n=4$.

2. Assume true $n=k$

ie $\frac{2^k}{k!}-1<0$

3. Hence prove true $n=k+1$

ie $\frac{2^{k+1}}{(k+1)!}-1<0$

now $\frac{2^{k+1}}{(k+1)!}-1 = \frac{2(2^k)}{(k+1)k!}-1$

$= \frac{2}{(k+1)}\left[\frac{2^k}{k!}-1\right]+\frac{2}{(k+1)}-1$

now $\frac{2^k}{k!}-1<0$ using assumption in 2.

and $\frac{2}{(k+1)}-1<0$ since $k\geq 4$

$\therefore \frac{2}{(k+1)}\left[\frac{2^k}{k!}-1\right]+\frac{2}{(k+1)}-1 \leq 0$

so since true for $n=4$, from 2 and 3 also true for $n=5$ and so on for all $n\geq 4$.

Q4(c).

i) $\frac{1}{\log_a b} = (\log_a b)^{-1}$

$= \left(\frac{\ln b}{\ln a}\right)^{-1}$

$= \left(\frac{\ln a}{\ln b}\right)$

$= \log_b a.$

ii) $\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{100} 100!}$

$= \log_{100!} 2 + \log_{100!} 3 + \log_{100!} 4 + \cdots + \log_{100!} 100$

$= \log_{100!} (2 \times 3 \times 4 \times \cdots \times 100)$

$= \log_{100!} 100!$

$= 1.$

Q5 (a) SP. $y = \ln x - x^2$

i) $y' = \frac{1}{x} - 2x$

$y' = 0$

$\frac{1}{x} = 2x$

$x^2 = \frac{1}{2}$

$x = \pm \frac{1}{\sqrt{2}}$ $\quad x > 0 \quad \therefore x = \frac{1}{\sqrt{2}}.$

$y'' = -\frac{1}{x^2} - 2.$

at $x = \frac{1}{\sqrt{2}}$, $y'' < 0$ $\frown$ .. max.

$x = \frac{1}{\sqrt{2}}$ $\quad y = \ln \frac{1}{\sqrt{2}} - \frac{1}{2}$

$= -\frac{1}{2}(\ln 2 + 1)$ $\quad$ Max $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}(\ln 2 + 1)\right)$.

ii) POI at $y'' = 0$ & concavity changes.

ie $-\frac{1}{x^2} = 2$

so no POI so always concave down.

iii) $x > 0$ $\therefore$ no y intercepts

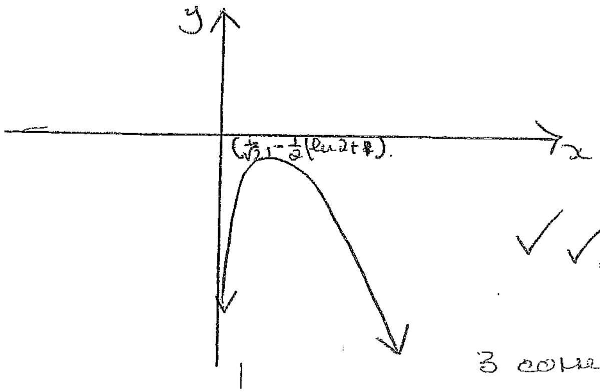
no x intercept since max at $y = -\frac{1}{2}(\ln 2 + 1)$, at always concave down ie Range $y \le -\frac{1}{2}(\ln 2 + 1)$

iv) $x \to 0$ $\quad \ln x \to -\infty \quad -x^2 \to 0$

$\therefore y = \ln x - x^2 \to -\infty.$

$x \to +\infty$ $\quad \ln x \to +\infty \quad -x^2 \to -\infty$

$\therefore y = \ln x - x^2 \to -\infty.$



3 correct
2 TP + behaviour
1 TP.

<u>note</u> if graph drawn contradictory to working, marks not given

b) $y = 10^x$ find $\frac{dy}{dx}$ $x = 1$.

$\log_{10} y = x$

$x = \frac{\ln y}{\ln 10}$

$\frac{dx}{dy} = \frac{1}{\ln 10} \times \frac{1}{y}$

$= \frac{1}{\ln 10 \times y}$

$\frac{dy}{dx} = y \ln 10.$ ✓

$x = 1 \quad y = 10^1 = 10$

$\frac{dy}{dx} = 10 \ln 10.$ ✓