KINCOPPAL – ROSE BAY
SCHOOL OF THE SACRED HEART

COR UNUM

2011
HIGHER SCHOOL CERTIFICATE
HALF YEARLY EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new page for each question

Total marks – 60

- Attempt Questions 1 – 4
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

Total Marks – 60
Attempt Questions 1-4
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $|4-7i|$. 1

(b) Express in the form $x+iy$ where x and y are real:

(i) $(5+2i)\overline{(3-i)}$ 1

(ii) $\dfrac{2-3i}{5-2i}$ 2

(c) Find the real numbers a and b such that $(a+bi)^2 = 9+40i$ 3

(d) (i) Express $-1+i$ in modulus –argument form. 2

(ii) Hence find the least positive integer n for which $(-1+i)^n$ is real. 1

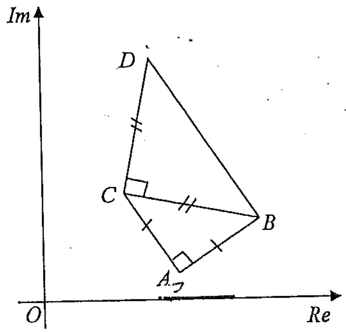
(e) Sketch the region on an argand diagram where: 2

$|z-1| \le \sqrt{2}$ and $0 \le arg\,(z+i) \le \dfrac{\pi}{4}$ both hold.

Question 1 continues on page 3

Question 1 (continued) **Marks**

(f)



In the diagram, the points A, B, C and D represent the complex numbers z_1, z_2, z_3 and z_4 respectively. Both ΔABC and ΔBCD are right angled isosceles triangles as shown.

(i) Show that the complex number z_3 can be written as 1

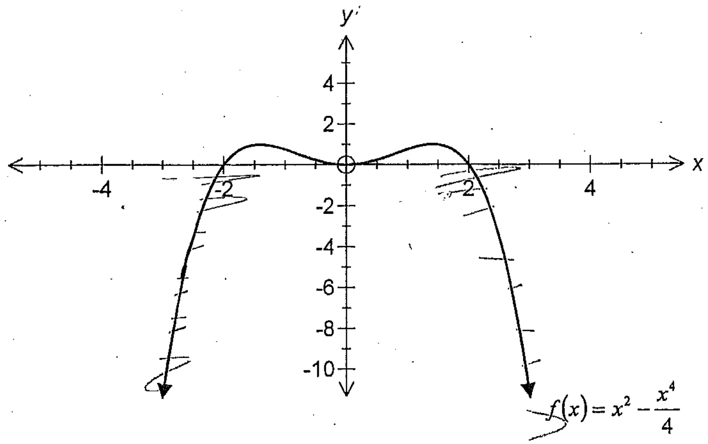
$$z_3 = (1-i)\,z_1 + iz_2.$$

(ii) Hence express the complex number z_4 in terms of z_1 and z_2, giving your answer in simplest form. 2

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The graph above shows the curve $f(x) = x^2 - \frac{x^4}{4}$. The maximum turning points are at $(-\sqrt{2}, 1)$ and $(\sqrt{2}, 1)$.

Draw separate one-third page sketches of the graphs of the following. You must indicate important features and the new coordinates of the points indicated as appropriate.

(i) $y = \frac{1}{f(x)}$ **2**

(ii) $y = [f(x)]^2$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

Question 2 continues on page 5

Kincoppal-Rose Bay, School of the Sacred Heart
HSC Mathematics Extension 2, Half Yearly Examination 2011

Question 2 (continued) **Marks**

(b) Consider the curve given implicitly by the equation $x^2 + 4y^2 = 4$

(i) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{-x}{4y}$ **2**

(ii) Find the equation of the tangent to $x^2 + 4y^2 = 4$ at the point $\left(1, \frac{\sqrt{3}}{2}\right)$ **2**

(iii) Find the points on the curve whose tangents are vertical or horizontal **2**

(iv) Hence sketch the curve showing all important features. **1**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The polynomial equation $z^3 - 7z^2 + 25z - 39 = 0$ has one zero equal to $(2+3i)$. Find the other two zeros. **3**

(b) Consider the polynomial $P(x) = (x+2)^2 Q(x) + R(x)$

(i) Explain why $R(x)$ is a linear polynomial. **1**

(ii) When $P(x)$ and $P'(x)$ are both divided by $(x+2)$, the remainder in each case is 6. Find $R(x)$. **3**

(c) The equation $x^3 + 3px - 1 = 0$, where p is real, has roots α, β and γ.

(i) Show that the monic cubic equation, with coefficients in terms of p, whose roots are α^2, β^2 and γ^2 is $x^3 + 6px^2 + 9p^2x - 1 = 0$. **2**

(ii) Hence or otherwise obtain the monic cubic equation, with coefficients in terms of p, whose roots are $\dfrac{\beta\gamma}{\alpha}$, $\dfrac{\gamma\alpha}{\beta}$ and $\dfrac{\alpha\beta}{\gamma}$. **3**

(d) Given that $\dfrac{16x-43}{(x-3)^2(x+2)}$ can be written as $\dfrac{16x-43}{(x-3)^2(x+2)} = \dfrac{a}{(x-3)^2} + \dfrac{b}{(x-3)} + \dfrac{c}{x+2}$ **3**

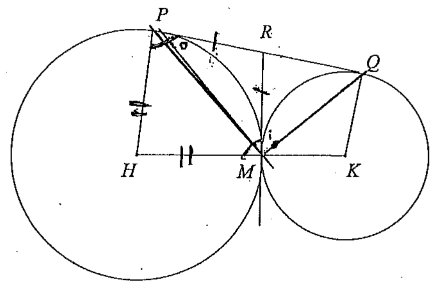
where a, b, c are real numbers, find a, b and c.

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) If w is a complex cube root of unity, show that $1 + w + w^2 = 0$ **1**

(b) Shown are two circles centres H and K which touch at M. PQ and RM are common tangents.



(i) Show that quadrilaterals $HPRM$ and $MRQK$ are cyclic. **2**

(ii) Prove that triangles PRM and MKQ are similar. **2**

(c) If $u_1 = 5$, $u_2 = 11$ and $u_n = 4u_{n-1} - 3u_{n-2}$ for $n \geq 3$. Show by Mathematical Induction that $u_n = 2 + 3^n$ for $n = 1, 2, 3, 4, \ldots$ **4**

(d) If $z = \cos\theta + i\sin\theta$,

(i) Find the complex roots of $z^5 + 1 = 0$. **2**

(ii) Hence express $z^5 + 1 = 0$ as the product of linear and quadratic factors. **2**

(iii) Using your result for part (i), show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$. **2**

End δf paper

Question (1)

a) $|4-7i|$

$= \sqrt{4^2 + (-7)^2}$

$= \sqrt{65}$ ✓ 1/1

b) $(5+2i)(3+i)$

i) $= 15 + 5i + 6i - 2$

$= 13 + 11i$ ✓ 1/1

ii) $\dfrac{2-3i}{5-2i} \times \dfrac{5+2i}{5+2i}$

$= \dfrac{10 + 4i - 15i - 6i^2}{25 - 4i^2}$

$= \dfrac{16 - 11i}{27}$

29 1/2

c) $z = a+ib$

$z^2 = 9+40i$

$z = \sqrt{9+40i}$

$(a+ib)^2 = 9+40i$

$a^2 + 2abi - b^2 = 9+40i$

$a^2 - b^2 = 9$ ✓ $\qquad 2ab = 40$

$a^2 + b^2 = \sqrt{9^2+40^2} \qquad ab = 20$

$a^2 + b^2 = 41 \qquad b = \dfrac{20}{a}$

$2a^2 = 50 \qquad = \pm 4$ ✓

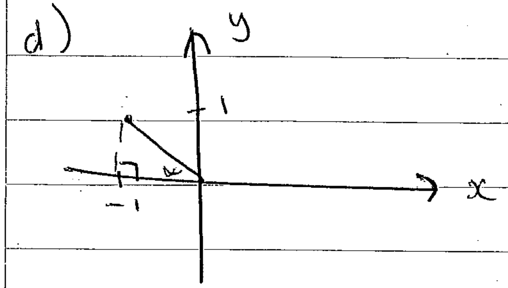
$a^2 = 25$

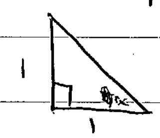
$a = \pm 5$ ✓

3/3

$z = \pm 5 \pm 4i$

d)





$\tan x = 1$

$x = \dfrac{\pi}{4}$

$\therefore \arg\theta = 180 - \dfrac{\pi}{4}$

$\arg\theta = \dfrac{3\pi}{4}$

$c^2 = 1^2 + 1^2$

$c = \sqrt{2}$

i) $\boxed{= \sqrt{2}\ \text{cis}\ \dfrac{3\pi}{4}}$ ✓ 2/2

ii) ~~A = z~~

~~$(-1+i)^3$~~

~~$(-1)^3 + 3(-1)^2(i)$~~

$(-1+i)^4$

$(-1+i)^2(-1+i)^2$

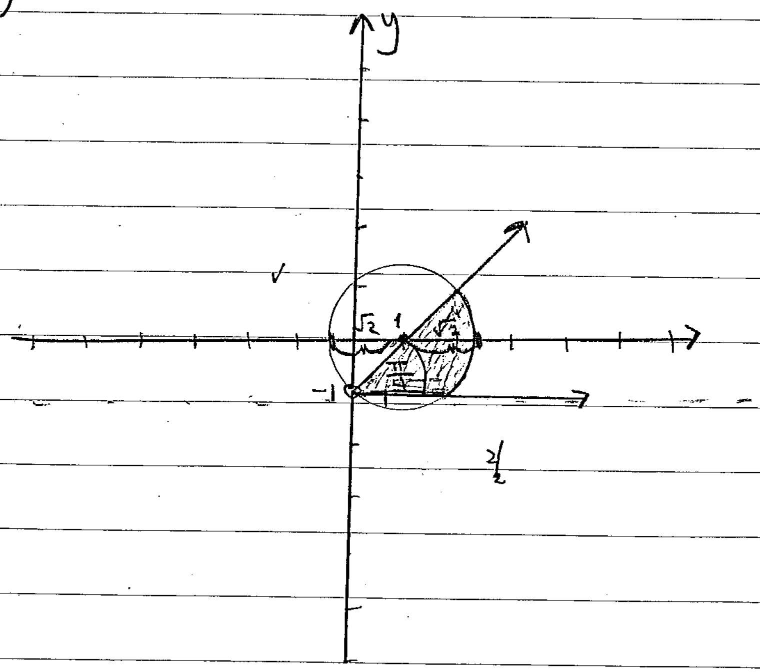
$(1-2i-1)(1-2i-1)$

$(-2i)(-2i)$

4

$\boxed{n=4}$ ✓ 1/1

e)



✓

2/2

$A = z_1$
$B = z_2$
$C = z_3$
$D = z_4$

f) i) $z_2 = z_1 + (z_2 - z_1)$

+ anticlockwise 90° × by i explain!

$\vec{AC} = i \times \vec{AB}$

$(z_3 - z_1) = (z_2 - z_1) \times i$ ✓

~~$z_2 = z_1 \bar{+} (z_3 - z_1)$~~

$z_2 i = z_1 i + z_3 - z_1$

$z_3 = z_2 i - z_1 i + z_1$

$\therefore z_3 = (1-i)z_1 + iz_2$ ✓ 1/1

ii) $z_3 + (z_4 - z_3) = z_4$

$z_4 - z_3 = (z_2 - z_3)i$ ✓

$z_3 + (z_2 - z_3)i = z_4$

careful left out $+iz_2$

$LHS = (1-i)z_1 + i(z_2 - (1-i)z_1 -$ ✓

$= (1-i)z_1 + iz_2 - iz_1(1-i) + z_2$

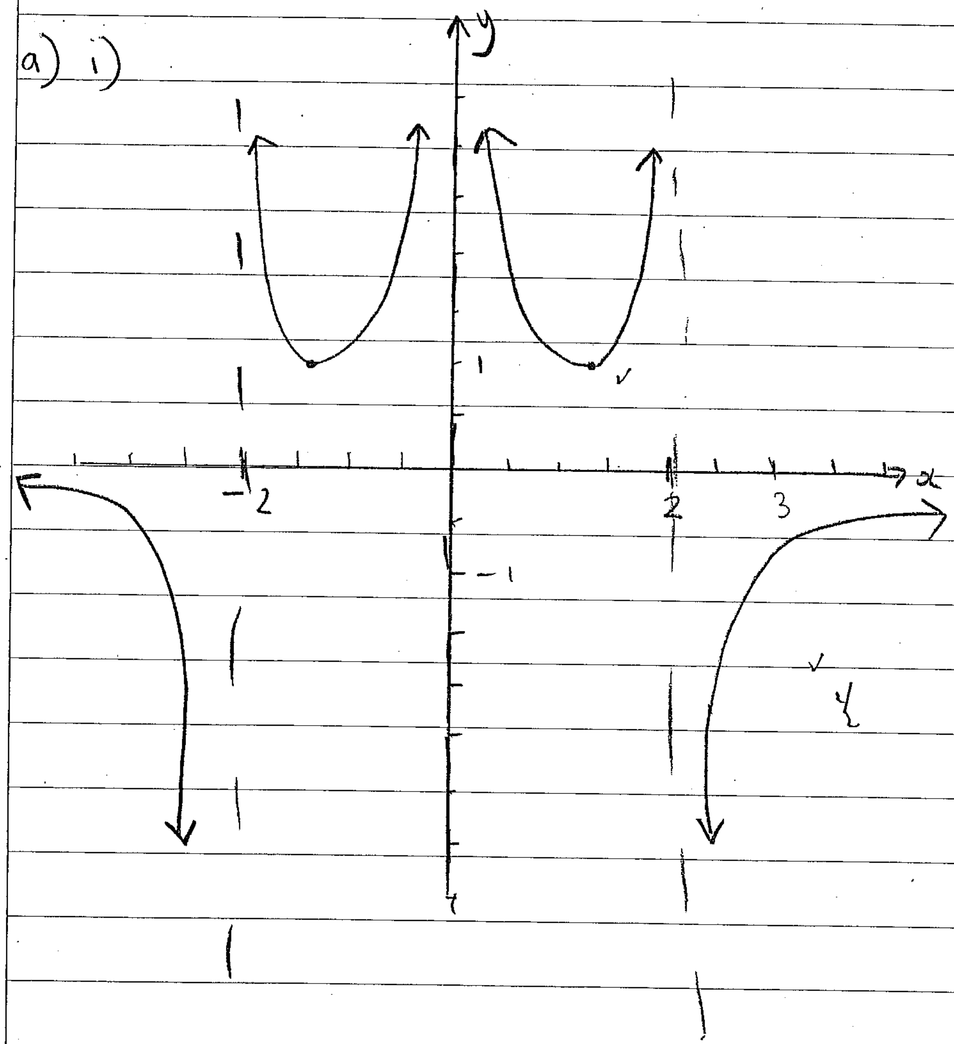
$\therefore z_4 = (i+1)z_2 + \underbrace{z_1(1-i) - iz_1(1-i)}_{\cancel{z_1} - z_1 i - z_1 i \cancel{z_1}}$

$(1+2i)z_2$

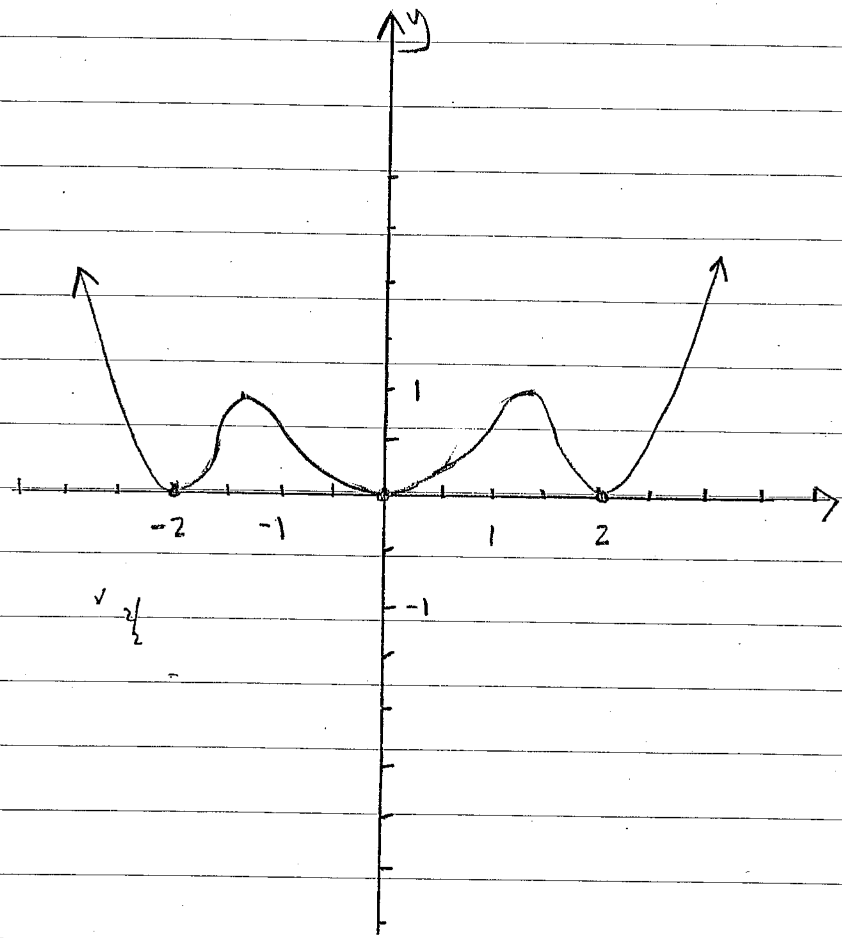
$z_4 = (i+1)z_2 - 2z_1 i$ 1/2

Question 2

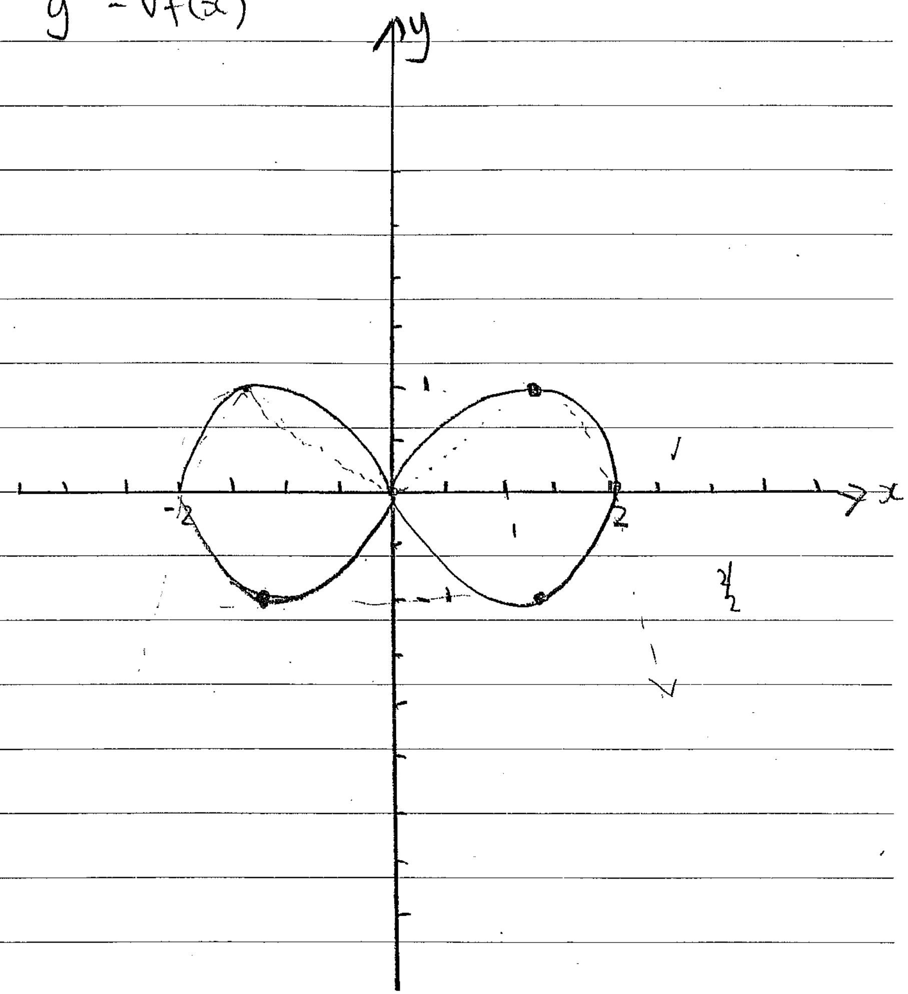
a) i)



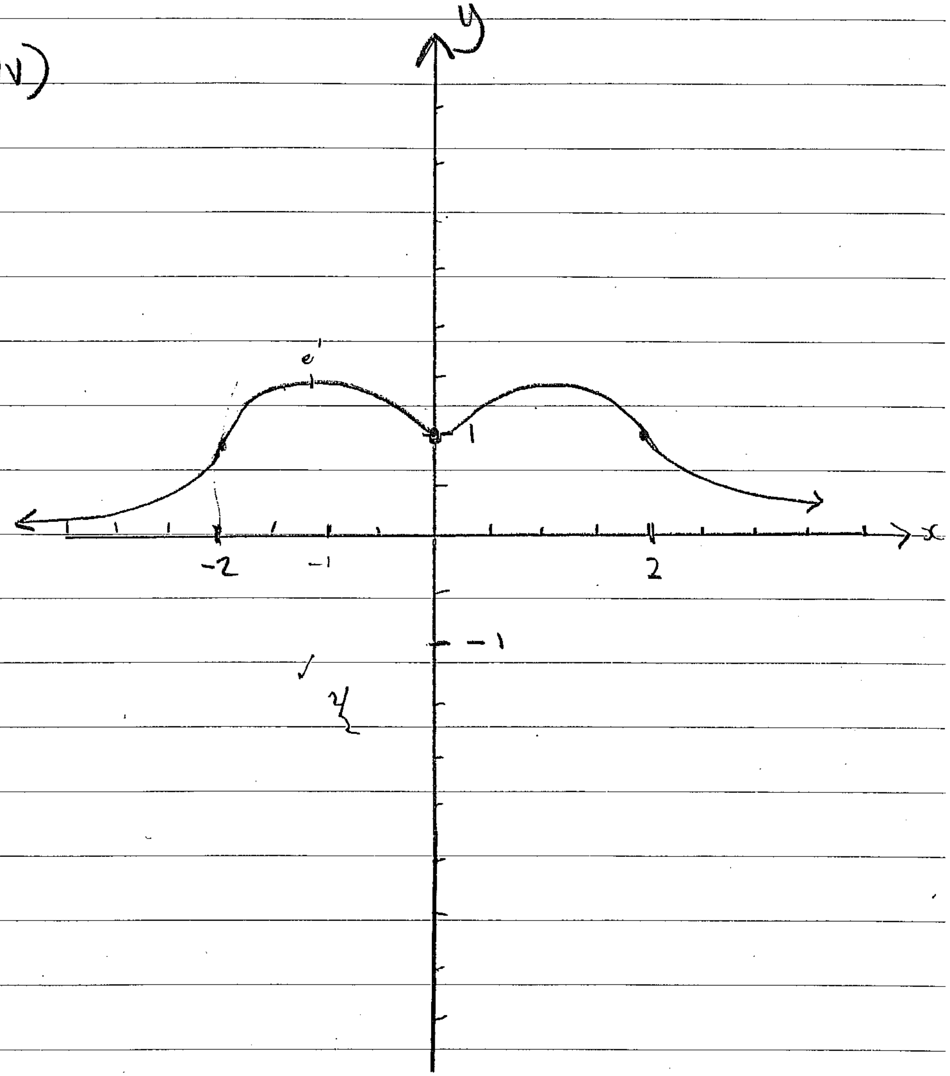
ii)



iii) $y = \sqrt{f(x)}$



iv)



b) i) $2x^{2} + 8y\frac{dy}{dx} = 0$

$$\cancel{8y}\frac{dy}{dx} = \frac{-2x^2}{8y}$$

$$\frac{dy}{dx} = \frac{-x^{2}}{4y}$$

½

ii) $\frac{dy}{dx} = 0$

$$\frac{-x^2}{4y} = 0 \qquad \frac{-(1)^2}{4\left(\frac{\sqrt{3}}{2}\right)}$$

$$\cancel{x^2 = 0} \qquad = \frac{-1}{2\sqrt{3}} = m$$

$$\cancel{x = 0}$$

$$\cancel{0 + 4y^2 = 4}$$

$$\cancel{y^2 = 1}$$

$$\cancel{y = \pm 1}$$

$$(y - y_1) = m(x - x_1)$$

$$y - \frac{\sqrt{3}}{2} = \frac{-1}{2\sqrt{3}}(x - 1)$$

✓

$$2\sqrt{3}y - 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = -1(x-1)$$

$$2\sqrt{3}y - \frac{6}{2} = -x + 1$$

$$x + 2\sqrt{3}y - 3 - 1 = 0$$

$$x + 2\sqrt{3}y - 4 = 0$$

✓ ½

iii) vertical gradients

$$y' = \frac{-x}{4y}$$

$$4y = 0$$

$$y = 0$$

$$x^2 + 0 = 4$$

$$x^2 = 4$$

$$x = \pm 2$$

$(2, 0)$ $(-2, 0)$ ✓ points with ~~g~~ vertical tangent

$$y' = \frac{-x}{4y}$$

$y' = 0$ (for horizontal tangents)

$-x = 0$

$x = 0$

$4y^2 = 4$

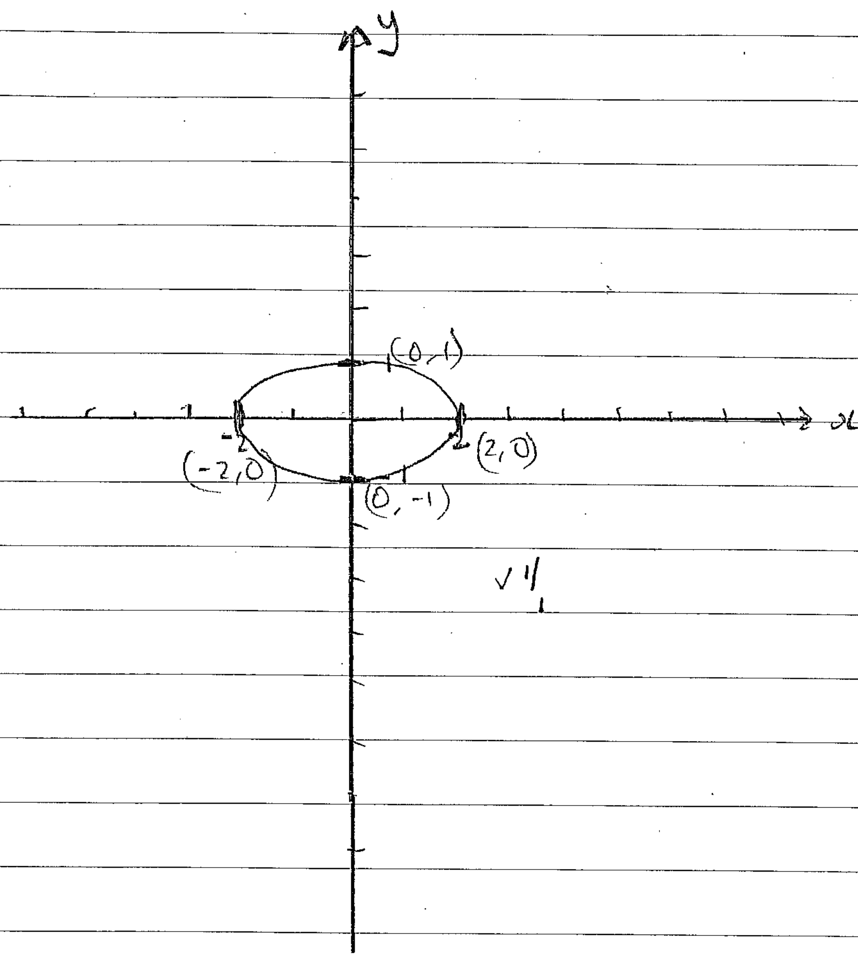
$y^2 = 1$

$y = \pm 1$

✓

$(0, 1)$, $(0, -1)$ points on curve where tangents are horizontal

$2/2$



✓ 1/1

Question 3

a) $(2+3i)$ is a zero

as coefficients of z are real

$(2-3i)$ is also a zero ((conjugate ✓

$\left(z-(2+3i)\right)\left(z-(2-3i)\right)$

$= z^2 - (2-3i)z - (2+3i)z + (2+3i)(2-3i$

$= z^2 - 2z \cancel{+3iz} - 2z \cancel{-3iz} + 4 \cancel{-6i} \cancel{+6i} + 9$

$= z^2 - 4z + 13$

$$
\begin{array}{r}
z \quad -3 \quad ✓ \\
z^2-4z+13 \,\overline{)\, z^3 - 7z^2 + 25z - 39} \\
z^3 - 4z^2 + 13z \\
\hline
-3z^2 + 12z - 39 \\
-3z^2 + 12z - 39 \\
\hline
0
\end{array}
$$

$P(x) = (z^2 - 4z + 13)(z-3)$

$\therefore$ zeros are $2+3i$

$2-3i$

3 ✓ $\quad 3/3$

b) i) must be ~~one degree~~ less than $(x+2)^2$ ✓ (quadratic) degree 2

1/1

ii) $P(x) = (x+2)^2 Q(x) + R(x)$

$P'(x) = (x+2)^2 Q'(x) + 2Q(x)(x+2)$

$u = (x+2)^2 \qquad + R'(x)$ ✓

$u' = 2(x+2) \qquad$ at most remainder is linear $\therefore$

$v = Q(x) \qquad \boxed{R(x) = ax+b}$

$v' = Q'(x)$

$P(-2) = 6 \qquad P'(-2) = 6$

$P(-2) = (-2+2)^2 Q(-2) + R(-2)$

$R(-2) = 6$ ✓ $\quad$ i.e. $-2a + b = 6$

$P'(-2) = (-2-2)^2 Q'(-2) + 2Q(-2)(-2+$

$+ R'(-2)$

$R'(-2) = 6 \quad$ i.e. $a = 6$

$R(x) = R'(x)$

~~$+ R(x)$~~

$R(x) = R'(x) = 0 \qquad 1/3$

$\therefore \boxed{R(x) = 0}$ X

therefore → $6(-2)$ ... $= 6 \qquad -12 + b = 6$

$b = 18$

c) $x = \alpha, \beta, \gamma$

i) let $X = \alpha^2$

$X = x^2$

$\sqrt{X} = x$

$p(\sqrt{X}) = 0$

$p(\sqrt{X}) = (\sqrt{X})^3 + 3p(\sqrt{X}) - 1$

$X\sqrt{X} + 3p\sqrt{X} - 1 = 0$ ✓

$(\sqrt{X}(X+3p))^2 = (1)^2$

$X(X+3p)^2 = 1$ ✓

$X(X^2 + 6pX + 9p^2) = 1$

$X^3 + 6pX^2 + 9Xp^2 - 1 = 0$

$\therefore\ x^3 + 6px^2 + 9p^2x - 1 = 0$ ✓

2/2

ii) let $X = \frac{\beta\gamma}{\alpha}$

$\alpha\beta\gamma = -\frac{d}{a}$

$= 1$

$\alpha X \times \alpha = \beta\gamma \times \alpha$

$X\alpha^2 = 1$

$X = \frac{1}{\alpha^2}$ ✓

$X = \frac{1}{x^2}$

$x^2 = \frac{1}{X}$

$x = \frac{1}{\sqrt{X}}$

$p\left(\frac{1}{\sqrt{X}}\right) = 0$

$\left(\frac{1}{\sqrt{X}}\right)^3 + 3p\left(\frac{1}{\sqrt{X}}\right) - 1 = 0$ ✓

~~$(\sqrt{X})^{-3}$~~ $\frac{1}{X\sqrt{X}} + \frac{3p}{\sqrt{X}} = 1$

$1 + 3p\cancel{X} - X\sqrt{X} = 0$

$(+X\sqrt{X})^2 = (+1 + 3pX)^2$

$X^3 = 1 + 6pX + 9p^2X^2$

$X^3 - 9p^2X^2 - 6pX - 1 = 0$ ✓ 3/3

$\therefore$ ✓ $x^3 - 9p^2x^2 - 6px - 1 = 0$

write in terms of → just means polynomial must have p in it

~~$x(x^2 - 9p^2x(-6p)) = 1$~~

~~$x^2 - 9p^2x - 6p = \frac{1}{x}$~~

~~$-9p^2x - 6p = \frac{1}{x} - x^2$~~

~~$-9p^2x - 6p + (-3)^2 = \frac{1}{x} - x^2 + 9$~~

~~$(3p - 3)^2$~~

d) $\dfrac{a(x+2) + b(x-3)(x+2) + c(x-3)}{(x-3)^2(x+2)}$

$\therefore a(x+2) + b(x-3)(x+2) + c(x-3)^2$

$= 16x - 43$

LHS $= ax + 2a + b(x^2 - x - 6) + c(x^2 - 6x + 9)$

$= ax + 2a + \cancel{bx^2} - bx - 6b + \cancel{x^2c} - \cancel{6xc} + 9c$

$= (b+c)x^2 + (a - b - 6c)x + 2a - 6b + 9c$ ✓

$\therefore b + c = 0 \qquad a - b - 6c = 16 \qquad 2a - 6b + 9c$

$b = -c \qquad a - (-c) - 6c = 16 \qquad -43$

$a - 5c = 16$

✓ $a = 16 + 5c$

$2(16 + 5c) - 6(-c) + 9c = -43$

$32 + 10c + 6c + 9c = -43$

$25c = -75$

$b = -(-3) \qquad c = -3$ ✓

$b = 3$ ✓ $\qquad a = 16 + 5(-3)$

$a = 1$ ✓ $\quad \frac{3}{3}$

$= \boxed{\dfrac{1}{(x-3)^2} + \dfrac{3}{x-3} - \dfrac{3}{x+2}}$

Question 4

a) $\omega^3 = 1$ (complex cube root unity)

$\omega^3 - 1 = 0$

$(\omega - 1)(\omega^2 + \omega + 1) = 0$ ✓

$\therefore 1 + \omega + \omega^2 = 0$ ✓ 1

b) i) $\angle HMR = 90°$ (tangent to circle and radius of circle touch at $90°$) ✓

$\angle RMK = 90°$ (tangent to circle and radius of circle touch at $90°$) ✓

$\angle RQK = 90°$ (" ")

$\angle HPR = 90°$ (" ") ✓

$\therefore$ HPRM and MRQK are cyclic as have pair of opposite angles which are supplementary ✓

$\frac{2}{2}$

①

c) step ① Prove true for $n = 1, 2$

$U_2 = 2 + 3^2$ $\quad$ $U_1 = 2 + 3^1$

$= 11$ ✓ true $\quad$ ✓ $= 5$ ✓ true

✓

Step ② Assume true for $n = k$

✓ $n = k + 1$

$U_k = 2 + 3^k$ $\quad$ $U_{k+1} = 2 + 3^{k+1}$

✓ ✓

Step ③ Required to prove true for $n = k + 2$

$U_{k+2} = 2 + 3^{k+2}$

✓

$U_{k+2} = 4(U_{k+1}) - 3U_k$ ✓

$= 4(2 + 3^{k+1}) - 3(2 + 3^k)$

$= 8 + 4.3.3^k$ ✓ $- 6 - 3.3^k$

$= 2 + (12 - 3)3^k$

$= 2 + 9.3^k$

$= 2 + 3^2.3^k$

$= 2 + 3^{k+2}$ ✓

∴ proven true! ✓

①

Step ④ ∴ by the principal of mathematic induction proven true for all Integers n ✓ 4/4

d) $z^5 = -1$

$z^5 = cis(\pi + 2k\pi)$

$z = cis\left(\frac{\pi + 2k\pi}{5}\right)$ ✓

$k = 0$ $\quad$ $k = 1$ $\quad$ $k = 2$

$z_0 = cis\left(\frac{\pi}{5}\right)$ ✓ $\quad$ $z_1 = cis\left(\frac{3\pi}{5}\right)$ ✓ $\quad$ $z_2 = cis\left(\frac{\cancel{5}\pi}{\cancel{5}}\right)$

$z_2 = -1$

$k = 3$ $\quad$ $k = 4$

$\bar{z}_1 = cis\left(\frac{-3\pi}{5}\right)$ ✓ $\quad$ $\bar{z}_0 = cis\left(\frac{-\pi}{5}\right)$ ✓ $\quad$ 2/2

~~i) $(z+1)(z^4 - z^3 + z^2 - z + 1)$~~

~~$(z+1)($~~

~~$(x+1)(x - z_1)(x - \bar{z}_1)(x - \bar{z}_0)(x - z_0)$~~