SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2011
YEAR 12 Mathematics Extension 2
HSC Task #1

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Answer must be given in simplest exact form.

Total Marks – 70

- Attempt questions 1-6
- Start each new section of a separate answer booklet

Examiner: *D.McQuillan*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Section A

Start each new section on a separate answer booklet.

Question 1

(a) State the exact value of $\cos^{-1}\left(-\frac{1}{2}\right)$ 1

(b) Find $\sqrt{8+15i}$. 2

(c) For the function $f(x) = \sin^{-1}(x^2)$ 6

(i) Find the stationary point and its nature.

(ii) Find the domain and range.

(iii) Hence sketch $y = f(x)$.

(d) Find real numbers A, B and C such that 5

(i) $\frac{2x^2+19x-36}{(x+3)(x-2)^2} \equiv \frac{A}{x+3} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$

(ii) Hence evaluate

$$\int_3^5 \frac{2x^2+19x-36}{(x+3)(x-2)^2}\,dx$$

Question 2

(a) Solve $z^5 + 1 = 0$. 2

(b) Find $g'(a)$ where $g(x)$ is the inverse of $f(x) = x^3 + x + 1$ and $a = 1$. 3

(c) Write down a polynomial equation of the lowest possible degree that has roots $\sqrt{3}+1$ and $2-i$ if the polynomial has 3

(i) Complex coefficients

(ii) Rational coefficients

(d) Let $z = 2(cos\theta + isin\theta)$. 5

(i) Find $\overline{1-z}$

(ii) Show that the real part of $\frac{1}{1-z}$ is $\frac{1-2cos\theta}{5-4cos\theta}$

(iii) Express the imaginary part of $\frac{1}{1-z}$ in terms of θ.

End of Section A

Section B

Start each new section on a separate answer booklet.

Question 3

(a) Evaluate 6

(i) $\int_{-\pi}^{\pi} \sin^2(x)\ dx$

(ii) $\int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos\theta - \sin\theta + 1}$

(b) Find the equation whose roots are each one more than the roots of $x^3 - 7x + 6 = 0$. 2

(c) Factorise $x^6 + 1$ into its 3

(i) real factors.

(ii) linear factors.

Question 4

(a) Let α, β and λ be the zeros of the polynomial 4

$$P(x) = 3x^3 + 7x^2 + 11x + 51.$$

(i) Find $\alpha^2 + \beta^2 + \gamma^2$.

(ii) Using part (i), or otherwise, determine how many of the zeros of $P(x)$ are real. Justify your answer.

(b) In a History examination, Rahib is asked to put five historical events; A, B, C, D and E into chronological order. 4

(i) If he knows that A occurred sometime before B and otherwise guesses his answer, what is his probability of being correct?

(ii) What is the probability of being correct if he knows that A occurred sometime before D and that D occurred sometime before B?

(c) Prove the identity $2\sin^{-1}x = \cos^{-1}(1 - 2x^2)$ where $x \geq 0$. 3

End of Section B

Section C

Start each new section on a separate answer booklet.

Question 5

(a) Let

$$C(x) = \sum_{k=0}^{5} \frac{x^k}{k!}$$

Prove that $C(x) = 0$ has no double roots. 3

(b) Sketch the locus of z such that 6

(i) $|z-1| = 2$

(ii) $|z-1| = Re(z)$

(iii) $\arg(z-1) = \arg(z+2)$

Question 6

(a) Find

(i) $\int e^x \cos x \, dx$

(ii) $\int \frac{1}{x^2\sqrt{x^2+4}} dx$

(iii) $\int \frac{\sqrt{1+x}}{\sqrt{1-x}} dx$

(b) A parliamentary committee of 8 is to be formed by selecting from a pool of 2 Greens, 6 Labor and 9 Liberal party members. How many committees can be formed if 5

(i) The committee must have 1 Green, 3 Labor and 4 Liberal members.

(ii) The committee must have at least 1 Green, 1 Labor and 4 Liberal members.

(iii) The committee must have 1 Green, 3 Labor and 4 Liberal members and if the Labor transport spokesman is selected on the committee then the Liberal Minister for Transport will make sure he is definitely selected on the committee.

End of Exam

Sydney Boys' High School

121153

Student No.:

Paper: Maths Ext 2

Section: A

Sheet No.: 1 of 2 for this Section.

good...!!!

Q.No	Tick	Mark
1		14
2		13
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Q1.

(a) $\frac{2\pi}{3}$

(b) $x^2 - y^2 = 8$ $\quad 2xy = 15$

$x^2 + y^2 = 17$

$2x^2 = 25$

$x^2 = \frac{25}{2}$

$x = \pm \frac{5}{\sqrt{2}}$

$= \pm \frac{5\sqrt{2}}{2}$

$y = \pm \frac{30}{10\sqrt{2}}$

$= \frac{\pm 3}{\sqrt{2}}$

$= \frac{\pm 3\sqrt{2}}{2}$

Ans: $\pm\left(\frac{5\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i\right)$

(c) $f'(x) = \frac{2x}{\sqrt{1-x^4}}$

$x = 0$ when stationary

$f(0) = 0$

$f(1) = \frac{\pi}{2}$

$f(-1) = \frac{\pi}{2}$

OK

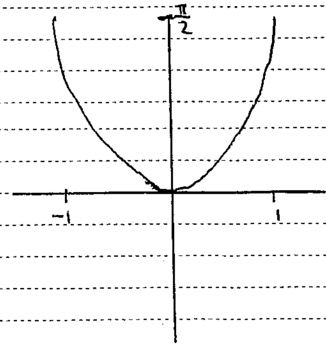
$\therefore$ relative minimum

(ii) $-1 \le x^2 \le 1$

$\therefore -1 \le x \le 1$

$0 \le y \le \frac{\pi}{2}$

(iii)



(d) $A(x-2)^2 + B(x+3) + C(x+3)(x-2)$
$= 2x^2 + 19x - 36$

$A(x^2 - 4x + 4) + Bx + 3B + C(x^2 + x - 6)$
$= Ax^2 - 4Ax + 4A + Bx + 3B + Cx^2 + Cx - 6C$

$A + C = 2 \quad A = 2 - C$ ①
$-4A + B + C = 19$ ②
$4A + 3B - 6C = -36$ ③

①+③
~~4B~~
$4B - 5C = -17$
$4(2-C) + 3B - 6C = -36$
$8 - 4C + 3B - 6C = -36$
$3B - 10C = -44$
$8B - 10C = -34$

$5B = 10$
$B = 2$

$8 - 5C = -17$
$-5C = -25$
$C = 5$
$A = -3$

$$\frac{2x^2+19x-36}{(x+3)(x-2)^2} = \frac{-3}{(x+3)} + \frac{2}{(x-2)^2} + \frac{5}{(x-2)}$$

(ii) $\int_3^5 \frac{-3}{x+3} + \frac{2}{(x-2)^2} + \frac{5}{(x-2)}$

$= \left[-3\ln(x+3) + -\frac{2}{(x-2)} + 5\ln(x-2)\right]_3^5$

$= (-3\ln 8 - \frac{2}{3} + 5\ln 3) - (-3\ln 6 - 2 + 5\ln 1]$

$= -3\ln 8 - \frac{2}{3} + 5\ln 3 + 3\ln 6 + 2$

$= 3\ln\frac{3}{4} + \frac{4}{3} + 5\ln 3$

2. (a) $z^5 = -1$
$z^5 = \text{cis}(\pi + 2k\pi)$
$z = \text{cis}\left(\frac{\pi + 2k\pi}{5}\right)$
where $k = 0, 1, 2, 3, 4$

$\therefore z = \text{cis}\frac{\pi}{5}, \text{cis}\frac{3\pi}{5}, \text{cis}\,\pi, \text{cis}(-\frac{\pi}{5}), \text{cis}(-\frac{3}{5}$

~~[illegible]~~ ~~$x = y^3 + y$ [illegible]~~

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121152

Student No.:

Paper: Maths Ext 2

Section: A

Sheet No.: 2 of 2 for this Section.

Q.No	Tick	Mark
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2. b) ~~Let $y = x^3+x+1$~~

$\frac{dy}{dx}$

$x = y^3 + y + 1$ $\qquad x = a = 1$

~~$x = y(y^2+1)+1$~~ $\qquad y^3 + y = 0$

$\frac{dx}{dy} = 3y^2 + 1$ $\qquad y(y^2+1) = 0$

$\therefore y = 0$

$\frac{dy}{dx} = \frac{1}{3y^2+1}$

$= \cancel{\frac{1}{3(y^2+3)-2}}$

$= \frac{1}{3(0)+1}$

$= 1$

(c) $(x-(\sqrt{3}+i))(x-(2-i))$

(i)

$= x^2 - (\sqrt{3}x + x) - (2-i)x + 2\sqrt{3} + 2 - \sqrt{3}i - i$

$= x^2 - (\sqrt{3}+1-2+i)x + 2\sqrt{3} + 2 - \sqrt{3}i - i$

$= x^2 - (\sqrt{3}-1+i)x + (2\sqrt{3}+2-\sqrt{3}i-i)$

(ii) $(x-(\sqrt{3}+i))(x+(\sqrt{3}-i))(x-(2-i))(x-(2+i))$

$= (x^2-2x-2)(x^2-4x+5)$

$= x^4 - 2x^3 - 2x^2 - 4x^3 + 8x^2 + 8x \cancel{\quad} + 5x^2 - \ldots$

$= x^4 - 6x^3 + 11x^2 - 2x - 10$

(d) (i) $\overline{1-z} = \overline{1-2\cos\theta-2i\sin\theta} = 1 - 2\cos\theta + 2i\sin\theta$

~~$1-(2\cos\theta - 2i\ldots)$~~

(ii) $\dfrac{1}{1-2(\cos\theta+i\sin\theta)} \times \dfrac{1-2(\cos\theta-i\sin\theta)}{1-2(\cos\theta-i\sin\theta)}$

$= \dfrac{1-(2\cos\theta - i\sin\theta)}{1-4\cos\theta+4(\cos^2\theta+\sin^2\theta)}$ ✓ ✓

$= \dfrac{1-2\cos\theta+i\sin\theta}{5-4\cos\theta}$

$\therefore$ real part is $\dfrac{1-2\cos\theta}{5-4\cos\theta}$ ✓

(iii) imaginary part is

$\dfrac{i\sin\theta}{5-4\cos\theta}$ ✓

4 ✓

3

10

Sydney Boys' High School

121154

Student No.:

Paper: Maths Ext 2

Section: B

Sheet No.: 1 of 2 for this Section.

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(a) (i) $2\int_0^{\pi} \sin^2 x \, dx$

$= 2\int_0^{\pi} \frac{1}{2}(1-\cos 2x)\, dx$ ✓

$= \int_0^{\pi} (1-\cos 2x)\, dx$

$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$

$= \pi - 0$

$= \pi$ ✓

(ii) $\int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos\theta - \sin\theta + 1}$ $\qquad t = \tan\frac{\theta}{2}$

$d\theta = \frac{2}{1+t^2}\, dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} + \frac{1+t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$ ✓

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t^2}{2-2t} \cdot \frac{2}{1+t^2}\, dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{2-2t}\, dt$ ✓

$= \left[-\ln(2-2t)\right]_0^{\frac{1}{\sqrt{3}}}$ ✓

$= -\ln\left(2 - \frac{2}{\sqrt{3}}\right) + \ln 2$

$= \ln 2 - \ln\left(2 - \frac{2\sqrt{3}}{3}\right)$

$= \ln 2 - \ln\left(\frac{6-2\sqrt{3}}{3}\right)$ ½

simplify

(b) $X = x+1$

$x = X-1$

$(x-1)^3 - 7(x-1) + 6 = 0$ ✓

$x^3 - 3x^2 + 3x - 1 - 7x + 7 + 6 = 0$

$x^3 - 3x^2 - 4x + 12 = 0$ ✓

(c) (i)

$$\begin{array}{r} x^4 - x^2 + 1 \\ x^2+1 \overline{)\, x^6 + 0x^4 + 0x^2 + 1} \\ x^6 + x^4 \\ \hline -x^4 + 0x^2 \\ -x^4 - x^2 \\ \hline x^2 + 1 \end{array}$$

(i) $(x^2+1)(x^4 - x^2 + 1)$ ½

~~(ii) $(x+i)(x-i)\times(x^4-x^2+1)$~~

~~$x^4 - x^2 + 1$ $\qquad r = -x^2$, $a = 1$~~

~~$x^6 = -1$~~

~~$x = cis\left(\frac{2k\pi + \pi}{6}\right)$~~

~~$k = 0, 1, 2, 3, 4, 5$~~

~~$x = cis\frac{\pi}{6}, cis\frac{-3\pi}{6}, cis\frac{3\pi}{6}, cis\frac{-5\pi}{6}$~~

~~$(-x^2)^3 - 1$~~

(ii) roots of $(x^2+1)(x^4-x^2+1)$

are amongst roots of $x^6=-1$

$x = \text{cis}\,\frac{2k+1}{6}\pi$

$k=0,1,2,3,4,5$

$x = \text{cis}\,\frac{\pi}{6}, \frac{3\pi}{6}, \text{cis}\,\frac{5\pi}{6}, \text{cis}\,\frac{-\pi}{6}, \text{cis}\,\frac{-3\pi}{6}, \text{cis}\,\frac{-5\pi}{6}$

$\therefore (x^2+1)(x^4-x^2+1)$

~~$= (x+i)(x-i)(x-\text{cis}\frac{\pi}{6})(x-\text{cis}\frac{3\pi}{6})$~~

$x^6+1 \quad = (x+i)(x-i)(x-\text{cis}\frac{\pi}{6})(x-\text{cis}(\frac{-\pi}{6}))(x-\text{cis}(\frac{5\pi}{6}))(x-\text{cis}(\frac{5\pi}{6}$

Sydney Boys' High School

121646

Student No.: 22051776

Paper:

Section: B

Sheet No.: 2 of 2 for this Section.

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4. (a) $3(\sqrt{x})^3 + 7(\sqrt{x})^2 + 11\sqrt{x} + 51$

$= 3x\sqrt{x} + 7x + 11\sqrt{x} + 51$

$3x\sqrt{x} + 11\sqrt{x} = -7x - 51$

$\sqrt{x} = \frac{-7x-51}{3x+11}$

$x = \frac{49x^2+714x+2601}{9x^2+66x+121}$

$49x^2+714x+2601 = 9x^3+66x^2+121x$

$9x^3+17x^2-593x-2601=0$

$\therefore \alpha^2+\beta^2+\gamma^2 = -\frac{17}{9}$ (sum of roots)

(ii) a polynomial with real coefficients have roots that come in conjugate pairs.

As $\alpha^2+\beta^2+\gamma^2 = \frac{-17}{9}$ ✓

there ~~is one~~ must be one pair of imaginary roots for this to be true. ✓

$\therefore$ one real root

(b) (i) $\frac{\frac{5!}{2!}}{\cancel{5!}} = 0.5$ ✓ Probability $= \frac{1}{\frac{5!}{2!}} = \frac{1}{60}$

50% probability

(ii) $\frac{\frac{5!}{3!}}{\cancel{5!}} = \cancel{\frac{4}{6}}$ ✓ $\frac{1}{\frac{5!}{3!}} = \frac{3!}{5!} = \frac{1}{20}$.

chance

(c) 0

(c) To prove $2\sin^{-1}x = \cos^{-1}(1-2x^2)$

Let $\alpha = \sin^{-1}x \Rightarrow \sin\alpha = x$

$\beta = \cos^{-1}(1-2x^2) \Rightarrow \cos\beta = 1-2x^2$

To prove that $2\alpha = \beta$

$\Rightarrow \cos 2\alpha = \cos\beta = 1-2x^2$

Proof:

$LHS = \cos 2\alpha = 1-2\sin^2\alpha$

$= 1-2x^2 = RHS$.

Sydney Boys' High School

123498

Student No.:

Paper:

Section: C

Sheet No.: 1 of 1 for this Section.

Q.No	Tick	Mark
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(a) $c(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$

Let α be a double root

$c(\alpha) = 1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}+\frac{\alpha^5}{5!} = 0$

$\therefore \alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}+\frac{\alpha^5}{5!} = -1$

$c'(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!} = -\frac{x^5}{5!}$

when $c'(x)=0$ $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!} = 0$ (1)

if there is a double root subbing (1) back into $c(x)$ then $0+\frac{x^5}{5!} = 0 \Rightarrow x=0$

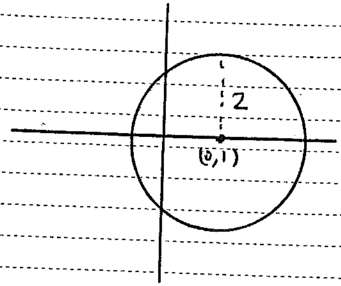
however this is impossible as $x \neq 0$ Why?

But $c(0) = 1$

$\therefore x=0$ is not a root.

2½

(b) (i)



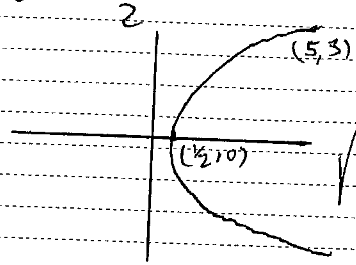
circle centre $(0,1)$ radius 2

(ii) $(x-1)^2 + y^2 = x^2$

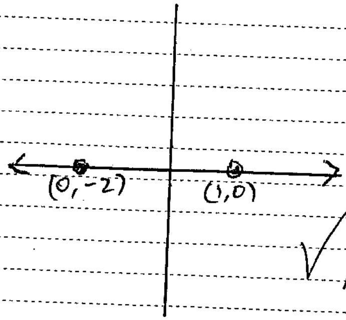
$x^2 - 2x + 1 + y^2 = x^2$

$y^2 = 2x - 1$

$\frac{y^2+1}{2} = x$



(iii)



6. (a)

$I = \int e^x \cos x \, dx \qquad u = \cos x \quad v' = e^x$

$u' = -\sin x \quad v = e^x$

$= e^x \cos x + \int e^x \sin x \, dx$

$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \qquad u = \sin x \quad v' = e^x$

$u' = \cos x \quad v = e^x$

$\therefore I = e^x \cos x + e^x \sin x - I$

$2I = e^x \cos x + e^x \sin x + c$

$\therefore I = \frac{1}{2}(e^x \cos x + e^x \sin x) + c$

~~(ii) $\int \frac{1}{x^2\sqrt{x^2+4}} dx \qquad x = 2\sec\theta \quad \frac{dx}{d\theta} = 2\sec\theta\tan\theta$~~

~~$= \int \frac{1}{\sec^2\theta\sqrt{\sec^2\theta+4}} \sec\theta\tan\theta \, d\theta \qquad dx = \sec\theta\tan\theta \, d\theta$~~

~~$= \int \frac{1}{2\tan\theta\sec^2\theta} \sec\theta\tan\theta \, d\theta$~~

~~$= \int \frac{1}{2\sec\theta} d\theta$~~

~~$= \frac{1}{2}\sin\theta + c$~~

~~$= \frac{1}{2}\sin(\cos^{-1}(\frac{2}{x})) + c$~~

(b) (i) ${}^2C_1 \times {}^6C_3 \times {}^9C_4 = 5040$

Sorry I accidentally skipped question

1(a) (ii) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ $x = 2\tan\theta$ $\theta = \tan^{-1}\frac{\theta}{2}$

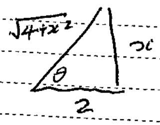
$dx = 2\sec^2\theta\, d\theta$

$= \int \frac{1}{4\tan^2\theta\sqrt{4+4\tan^2\theta}}\, 2\sec^2\theta\, d\theta$

$= \int \frac{2\sec^2\theta}{4\tan^2\theta\, 2\sec\theta}\, d\theta$ $\operatorname{cosec}\theta = \frac{\sqrt{4+x^2}}{x}$

$= \frac{1}{4}\int \cot\theta \operatorname{cosec}\theta\, d\theta$

$= -\frac{\sqrt{4+x^2}}{4x} + c$



(iii) $\int \frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx$ $u = \sqrt{1-x}$?

(b) (ii) Cases

2	6	9
Green	LAB	LIB
1	1	6
1	2	5
1	3	4
2	1	5
2	2	5

$^2C_1 \times {}^6C_1 \times {}^9C_6 + {}^2C_1 \times {}^6C_2 \times {}^9C_5 + {}^2C_1 \times {}^6C_3 \times {}^9C_4 + {}^2C_2 \times {}^6C_1 + {}^9C_5$

$+ {}^2C_2 \times {}^6C_2 \times {}^9C_4 = 12474$

(iii) LAB not selected $^2C_1 \times {}^5C_3 \times {}^9C_4 = 2520$

LAB selected $^2C_1 \times {}^5C_2 \times {}^8C_3 = 1120$

$\therefore$ Total: 3640