SYDNEY GRAMMAR SCHOOL

2011 Assessment Examination

FORM VI
MATHEMATICS EXTENSION 2

Thursday 12th May 2011

General Instructions
- Writing time — 1 hour 30 minutes
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 72
- All six questions may be attempted.
- All six questions are of equal value.

Collection
- Write your candidate number clearly on each booklet.
- Hand in the six questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist
- SGS booklets — 6 per boy
- Candidature — 85 boys

Examiner
TCW

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Let $z = 3 + i$ and $w = 2 - 2i$.

(i) Express $(\overline{z})^2$ in the form $a + ib$, where a and b are real. $\boxed{2}$

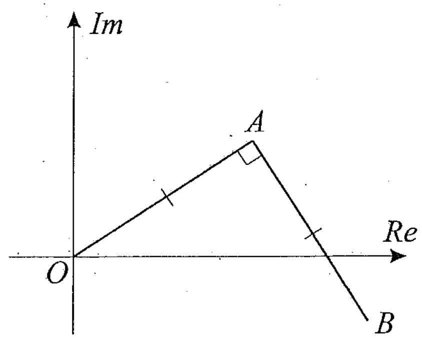
(ii) Find $\mathrm{Re}\left(\dfrac{w}{z}\right)$. $\boxed{2}$

(b) Let $z_1 = -\sqrt{3} + i$ and $z_2 = 3 - 3i$.

(i) Write z_1 and z_2 in modulus–argument form. $\boxed{2}$

(ii) Hence, or otherwise, express $\dfrac{z_1}{z_2}$ in modulus–argument form. $\boxed{2}$

(c)



In the Argand diagram above, $OA = AB$ and $OA \perp AB$. The point A represents the number ω. Find the complex number represented by B in terms of ω. $\boxed{2}$

(d) The point C represents $4 + 8i$ and the point D represents $-4 - 8i$. The locus of z is defined by $|z - 4 - 8i| = |z + 4 + 8i|$. Describe the locus of z and find its Cartesian equation. $\boxed{2}$

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. Marks

(a) Find $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$. 2

(b) Evaluate $\displaystyle\int_0^2 \frac{4+x}{4-x}\,dx$. 3

(c) Use the substitution $t=\tan\frac{\theta}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta}$. 4

(d) Use integration by parts to find $\displaystyle\int \sin^{-1}x\,dx$. 3

QUESTION THREE (12 marks) Use a separate writing booklet. Marks

(a) Consider the ellipse $\mathcal{E}$ with equation $16x^2+25y^2=400$.

(i) Show that the eccentricity of $\mathcal{E}$ is $\frac{3}{5}$. 1

(ii) Find the coordinates of the foci S and S'. 1

(iii) Find the equations of the directrices. 1

(iv) Sketch the ellipse $\mathcal{E}$, showing its x and y-intercepts, foci and directrices. 2

(v) The point $P(x,y)$ lies on $\mathcal{E}$. Use the eccentricity of $\mathcal{E}$ to prove that $PS+PS'$ is independent of the position of P. 2

(b) Write down the equations of the asymptotes of the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{1}=1$ and find the acute angle between them. 2

(c) Find the values of k for which the equation $\dfrac{x^2}{6-k}+\dfrac{y^2}{k-3}=1$ represents

(i) a circle, 1

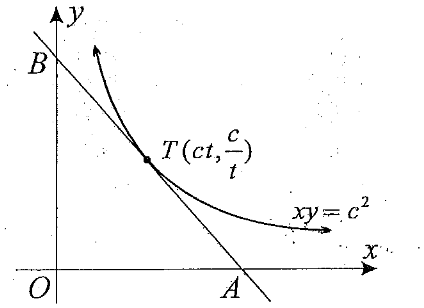
(ii) an ellipse. 2

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. Marks

(a) The point P represents the complex number $z=-2i$. The points Q and R represent the complex numbers w_1 and w_2, which are the square roots of z. Plot the points P, Q and R on the Argand diagram, given that $\mathrm{Im}(w_1)>0$. 2

(b)



The diagram above shows the hyperbola $xy=c^2$. The tangent at the point $T(ct,\frac{c}{t})$ cuts the asymptotes of the hyperbola at the points A and B. The point O is the origin.

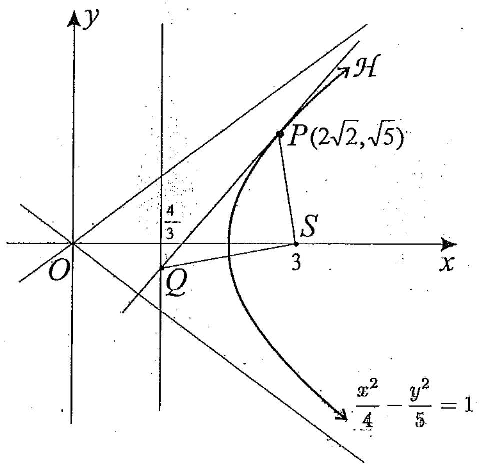
(i) Show that the tangent at T has equation $x+t^2y=2ct$. 2

(ii) Prove that the area of $\triangle AOB$ is constant. 2

Exam continues next page ...

QUESTION FOUR (Continued)

(c)



The diagram above shows only one branch of the hyperbola $\mathcal{H}$ with equation $\frac{x^2}{4}-\frac{y^2}{5}=1$. It has a focus at $S(3,0)$ and a directrix with equation $x=\frac{4}{3}$.

(i) Show that the equation of the tangent to $\mathcal{H}$ at the point $P(2\sqrt{2},\sqrt{5})$ is $\sqrt{10}\,x-2y=2\sqrt{5}$. $\boxed{3}$

(ii) The tangent at P cuts the directrix at Q. Show that $\angle PSQ$ is a right angle. $\boxed{3}$

Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. Marks

(a) Sketch the region in the complex plane which satisfies $\frac{\pi}{4}\le\arg(z-i)\le\frac{3\pi}{4}$ and $|z-i|\le 2$ simultaneously. $\boxed{3}$

(b) Let $I_n=\int_0^{\frac{\pi}{3}}\tan^n x\,dx$, for integers $n\ge 2$.

(i) Show that $I_n=\frac{1}{n-1}\left(\sqrt{3}\right)^{n-1}-I_{n-2}$. $\boxed{2}$

(ii) Hence find the exact value of $\int_0^{\frac{\pi}{3}}\tan^5 x\,dx$. $\boxed{2}$

(c) (i) Given that $u=x+\frac{1}{x}$, show that $\frac{1}{u^2+1}=\frac{x^2}{x^4+3x^2+1}$. $\boxed{1}$

(ii) Use the substitution $u=x+\frac{1}{x}$ to show that $\int_1^2\frac{x^2-1}{x^4+3x^2+1}\,dx=\tan^{-1}\left(\frac{1}{12}\right)$. $\boxed{4}$

QUESTION SIX (12 marks) Use a separate writing booklet. Marks

(a) The point $P(a\cos\theta,b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The point $Q(a\cos\theta,a\sin\theta)$ lies on the auxiliary circle $x^2+y^2=a^2$ and the point O is the origin.

(i) Show that the normal at P has equation $ax\sin\theta-by\cos\theta=(a^2-b^2)\cos\theta\sin\theta$. $\boxed{2}$

(ii) Find the equation of OQ. $\boxed{1}$

(iii) Find and describe the locus of the intersection of the normal at P and the line OQ. $\boxed{3}$

(b) Let $I=\int_0^1\frac{x^n}{x+1}\,dx$, $J=\int_0^1\frac{x^{n+1}}{x+1}\,dx$ and $K=\int_0^1\frac{x^n}{x^2+1}\,dx$

where n is a positive integer.

(i) Arrange I, J and K in ascending order. Justify your answer. $\boxed{2}$

(ii) Show that $\frac{x^6}{x+1}=(x^5-x^4+x^3-x^2+x-1)+\frac{1}{x+1}$. $\boxed{1}$

(iii) Use $n=5$ to prove that $\frac{62}{90}<\ln 2<\frac{63}{90}$. $\boxed{3}$

END OF EXAMINATION

Question 1

(a) (i) $(\bar{z})^2 = (3-i)^2$ ✓

$= 9 - 6i - 1$

$= 8 - 6i$ ✓

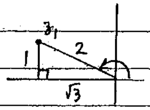
(ii) $\dfrac{w}{z} = \dfrac{2-2i}{3+i} \times \dfrac{3-i}{3-i}$

$= \dfrac{6 - 2i - 6i - 2}{10}$

$= \dfrac{2}{5} - \dfrac{4}{5}i$ ✓

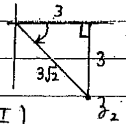
$Re\left(\dfrac{w}{z}\right) = \dfrac{2}{5}$ ✓

(b) (i) $z_1 = -\sqrt{3} + i$



$z_1 = 2 \text{ cis } \dfrac{5\pi}{6}$

$z_2 = 3 - 3i$



$z_2 = 3\sqrt{2} \text{ cis}\left(-\dfrac{\pi}{4}\right)$

✓ moduli ✓ arguments

(ii) $\dfrac{z_1}{z_2} = \dfrac{\sqrt{2}}{3} \text{ cis}\left(\dfrac{5\pi}{6} + \dfrac{\pi}{4}\right)$

$= \dfrac{\sqrt{2}}{3} \text{ cis}\left(\dfrac{13\pi}{12}\right)$ ✓ mod ✓ arg

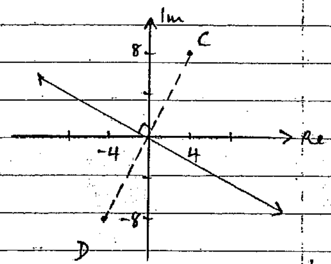
$= \dfrac{\sqrt{2}}{3} \text{ cis}\left(-\dfrac{11\pi}{12}\right)$

(c) $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ ✓

$= w - iw$ ✓

$= w(1-i)$

(d) The locus of z is the perpendicular bisector of CD ✓



$y = -\frac{1}{2}x$ ✓

/12

Question 2

(a) $\displaystyle\int \frac{1}{x^2 - 4x + 13}\,dx$

$= \displaystyle\int \frac{1}{(x-2)^2 + 3^2}\,dx$ ✓

$= \dfrac{1}{3}\tan^{-1}\left(\dfrac{x-2}{3}\right) + c$ ✓

(b) $\displaystyle\int_0^2 \frac{4+x}{4-x}\,dx$

$= \displaystyle\int_0^2 \frac{-(4-x) + 8}{4-x}\,dx$ ✓

$= \displaystyle\int_0^2 -1\,dx + \int_0^2 \frac{8}{4-x}\,dx$

$= -\left[x\right]_0^2 - 8\left[\ln|4-x|\right]_0^2$ ✓

$= -2 - 8(\ln 2 - \ln 4)$

$= 8\ln 2 - 2$ ✓

(c) Let $t = \tan\frac{\theta}{2}$

$\theta = 2\tan^{-1} t$

$d\theta = \dfrac{2}{1+t^2}\,dt$

θ	$\frac{\pi}{2}$	0
t	1	0

✓

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta} = \int_0^1 \frac{\frac{2}{1+t^2}}{1+\frac{2t}{1+t^2}}\,dt$ ✓

$= \displaystyle\int_0^1 \frac{2}{(1+t)^2}\,dt$

$= 2\displaystyle\int_0^1 (1+t)^{-2}\,dt$ ✓

$= 2\left[-\dfrac{1}{1+t}\right]_0^1$

$= 2 \times \dfrac{1}{2}$

$= 1$ ✓

(d) $\displaystyle\int \sin^{-1}x\,dx$

$= \displaystyle\int 1 \times \sin^{-1}x\,dx$

$= x\sin^{-1}x - \displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$ ✓

$= x\sin^{-1}x + \dfrac{1}{2}\displaystyle\int -2x(1-x^2)^{-\frac{1}{2}}\,dx$ ✓

$= x\sin^{-1}x + \cancel{\frac{1}{2}} \times \dfrac{1}{\cancel{\frac{1}{2}}}(1-x^2)^{\frac{1}{2}} + c$

$= x\sin^{-1}x + \sqrt{1-x^2} + c$ ✓

/12

Question 3

(a) $16x^2 + 25y^2 = 400$

$\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

(i) $b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$ ✓

$e^2 = 1 - \dfrac{16}{25}$

$e = \dfrac{3}{5}$ $(e>0)$

(ii) foci : $(\pm ae, 0)$

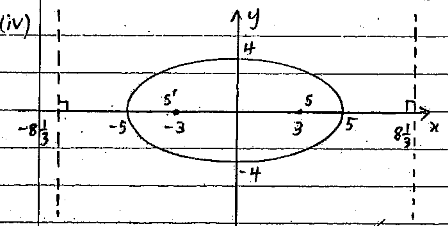
$S = (3,0)$

$S' = (-3,0)$ ✓

(iii) directrices : $x = \pm \dfrac{a}{e}$

$x = \dfrac{25}{3}$, $x = -\dfrac{25}{3}$ ✓

(iv)



$\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ ✓ intercepts

✓ foci, directrices

(v) By definition,

$\dfrac{PS}{PM} = \dfrac{PS'}{PM'} = e$

(where the horizontal through P meets the directrices at M and M')

$PS + PS' = ePM + ePM'$

$= e\,(PM + PM')$ ✓

$= e\,(MM')$

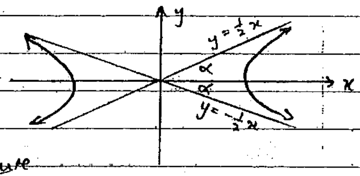
$= \dfrac{3}{5} \times 2 \times \dfrac{25}{3}$

$= 10$ units ✓

which is independent of the position of P.

(b) $\dfrac{x^2}{2^2} - \dfrac{y^2}{1^2} = 1$

asymptotes : $y = \pm \frac{1}{2}x$ ✓



Acute Angle between asymptotes $= 2\tan^{-1}\frac{1}{2}$ ✓

$\doteqdot 53^\circ$

(nearest degree)

[OR $\tan^{-1}\frac{4}{3} \doteqdot 53^\circ$]

(c) $\dfrac{x^2}{6-k} + \dfrac{y^2}{k-3} = 1$

(i) circle : $6-k = k-3$

$k = 4\frac{1}{2}$ ✓

(ii) ellipse :

$6-k>0$ and $k-3>0$ ✓

$k<6$ and $k>3$

$3<k<6$ ✓

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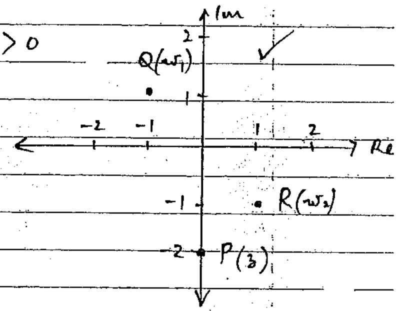
Question 4

(a) Let $(a+ib)^2 = -2i$, for real a, b.

$$\left.\begin{aligned} a^2-b^2 &= 0 \\ 2ab &= -2 \\ ab &= -1 \end{aligned}\right\} \quad \begin{aligned} &a=1,\ b=-1 \\ &\text{OR} \\ &a=-1,\ b=1 \end{aligned}$$

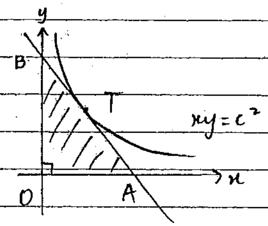
$w_1 = -1+i$, since $\operatorname{Im}(w_1)>0$

$w_2 = 1-i$ ✓



✓

(b)



(i) $T: x = ct,\ y = \dfrac{c}{t}$ [OR $\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$]

$\dfrac{dy}{dx} = \dfrac{-\frac{c}{t^2}}{c}$

$= -\dfrac{1}{t^2}$ ✓

Tangent at T :

$y - \dfrac{c}{t} = -\dfrac{1}{t^2}(x - ct)$

$t^2y - ct = -x + ct$ ✓

$x + t^2y = 2ct$

(ii) B: $x=0$, $y = \dfrac{2ct}{t^2}$

$= \dfrac{2c}{t}$

A: $y=0$, $x = 2ct$ ✓

Area of $\Delta AOB = \frac{1}{2} \times 2ct \times \dfrac{2c}{t}$

$= 2c^2$ which is constant

✓

(c) (i)

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

$$\frac{2x}{4} - \frac{2y}{5}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{5x}{4y}$$

At $P(2\sqrt{2}, \sqrt{5})$, gradient of tangent $= \frac{10\sqrt{2}}{4\sqrt{5}}$

$$= \frac{\sqrt{10}}{2}$$

Tangent at P:

$$y - \sqrt{5} = \frac{\sqrt{10}}{2}\left(x - 2\sqrt{2}\right)$$

$$2y - 2\sqrt{5} = \sqrt{10}\,x - 2\sqrt{20}$$

$$\sqrt{10}\,x - 2y = 4\sqrt{5} - 2\sqrt{5}$$

$$\sqrt{10}\,x - 2y = 2\sqrt{5}$$

(ii) Given $S = (3,0)$ and directrix $x = \frac{4}{3}$,

when $x = \frac{4}{3}$, $2y = \sqrt{10}\times\frac{4}{3} - 2\sqrt{5}$

$$y = \frac{2}{3}\sqrt{10} - \sqrt{5}$$

$$\therefore Q = \left(\frac{4}{3}, \frac{2}{3}\sqrt{10} - \sqrt{5}\right)$$

$$m_{PS} = \frac{\sqrt{5}}{2\sqrt{2}-3} \qquad m_{QS} = \frac{\sqrt{5} - \frac{2}{3}\sqrt{10}}{3 - \frac{4}{3}}$$

$$= \frac{3}{5}\left(\sqrt{5} - \frac{2}{3}\sqrt{10}\right)$$

$$= \frac{3\sqrt{5} - 2\sqrt{10}}{5}$$

$$m_{PS}\times m_{QS} = \frac{\sqrt{5}}{2\sqrt{2}-3}\times\frac{\sqrt{5}\left(3 - 2\sqrt{2}\right)}{5}$$

$$= \frac{-5\left(2\sqrt{2}-3\right)}{5\left(2\sqrt{2}-3\right)}$$

$$= -1$$

$\therefore$ $PS \perp QS$ and $\angle PSQ = 90°$.

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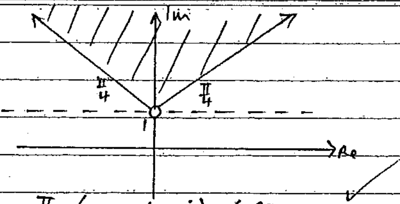
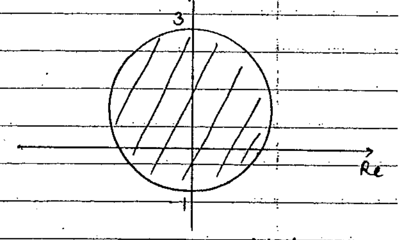
Question 5

(a)

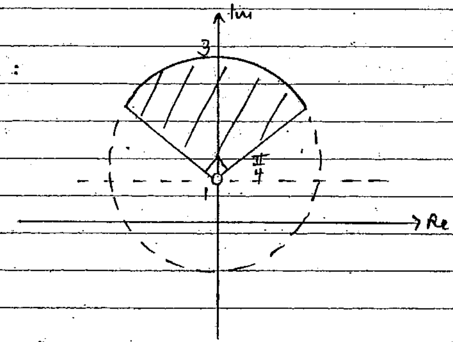


$$\frac{\pi}{4} \le \arg(z-i) \le \frac{3\pi}{4}$$



$$|z-i| \le 2$$

Intersection:



note: $z \ne i$

(b) (i)

$$I_n = \int_0^{\frac{\pi}{3}} \tan^{n-2}x\left(\sec^2 x - 1\right)dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^2 x\tan^{n-2}x\,dx - \int_0^{\frac{\pi}{3}} \tan^{n-2}dx$$

$$= \left[\frac{1}{n-1}\tan^{n-1}x\right]_0^{\frac{\pi}{3}} - I_{n-2}$$

$$= \frac{1}{n-1}\left(\sqrt{3}\right)^{n-1} - I_{n-2}$$

(ii)

$$I_5 = \frac{(\sqrt{3})^4}{4} - I_3$$

$$= \frac{9}{4} - \left(\frac{(\sqrt{3})^2}{2} - I_1\right)$$

$$= \frac{9}{4} - \frac{3}{2} + \int_0^{\frac{\pi}{3}} \tan x\,dx$$

$$= \frac{3}{4} + \left[-\ln|\cos x|\right]_0^{\frac{\pi}{3}}$$

$$= \frac{3}{4} - \ln\frac{1}{2} + \ln 1$$

$$= \frac{3}{4} + \ln 2$$

(c) (i) $u = x + \frac{1}{x}$, LHS $= \frac{1}{u^2+1}$

$= \frac{1}{(x+\frac{1}{x})^2+1}$

$= \frac{1}{x^2+3+\frac{1}{x^2}} \times \frac{x^2}{x^2}$

$= \frac{x^2}{x^4+3x^2+1}$

$=$ RHS

(ii) $\frac{du}{dx} = 1 - \frac{1}{x^2}$

x	2	1
u	$2\frac{1}{2}$	2

$du = \frac{x^2-1}{x^2}\,dx$

$$\int_1^2 \frac{x^2-1}{x^4+3x^2+1}\,dx = \int_1^2 \frac{x^2-1}{x^2} \times \frac{x^2}{x^4+3x^2+1}\,dx$$

$$= \int_2^{2\frac{1}{2}} \frac{1}{1+u^2}\,du$$

$$= \left[\tan^{-1} u\right]_2^{\frac{5}{2}}$$

$$= \tan^{-1}\frac{5}{2} - \tan^{-1} 2$$

Let $\alpha = \tan^{-1}\frac{5}{2}$ and $\beta = \tan^{-1} 2$

$\tan\alpha = \frac{5}{2}$ $\quad$ $\tan\beta = 2$

$\tan(\alpha-\beta) = \frac{\frac{5}{2}-2}{1+\frac{5}{2}\times 2}$

$= \frac{\frac{1}{2}}{1+5}$

$= \frac{1}{12}$

$\therefore \quad \alpha - \beta = \tan^{-1}\left(\frac{1}{12}\right)$

$$\int_1^2 \frac{x^2-1}{x^4+3x^2+1}\,dx = \tan^{-1}\left(\frac{1}{12}\right)$$

12

Question 6

(a) $P(a\cos\theta, b\sin\theta)$ $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$Q(a\cos\theta, a\sin\theta)$

(i) At P: $x = a\cos\theta$, $y = b\sin\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta}$

Normal at P: gradient $= \frac{a\sin\theta}{b\cos\theta}$

Equation: $y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$

$ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$

(ii) Equation of OQ: $y = \frac{a\sin\theta}{a\cos\theta}x$

$y = \frac{\sin\theta}{\cos\theta}x$

(iii) $ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$ —— (1)

$y\cos\theta = x\sin\theta$ —— (2)

sub LHS of (2) in (1):

$ay\cancel{\cos\theta} - by\cancel{\cos\theta} = (a^2-b^2)\sin\theta\cancel{\cos\theta}$, $\cos\theta \neq 0$

$y(\cancel{a-b}) = (\cancel{a-b})(a+b)\sin\theta$

$y = (a+b)\sin\theta$, $\cos\theta \neq 0$

sub RHS of (2) in (1): $x = (a+b)\cos\theta$, $\sin\theta \neq 0$

Now $x^2+y^2 = (a+b)^2(\sin^2\theta + \cos^2\theta)$

$x^2+y^2 = (a+b)^2$ for $\cos\theta \neq 0$, $\sin\theta \neq 0$.

So the locus is the circle centre $(0,0)$ radius $(a+b)$ for $\theta \neq \frac{k\pi}{2}$, where k is an integer.

When $\theta = \frac{k\pi}{2}$, the normal at P and OQ are the same line, so the locus is the x and y axes.

ie The locus of the intersection of the normal at P and the line OQ is the circle $x^2+y^2=(a+b)^2$, the line $x=0$ and the line $y=0$.

(b) (i) For $0 < x < 1$, where n is a positive integer,

$x^{n+1} < x^n$

$\frac{x^{n+1}}{x+1} < \frac{x^n}{x+1}$

$\therefore \quad J < I$

$x^2 < x$

$x^2+1 < x+1$

$\frac{1}{x^2+1} > \frac{1}{x+1}$

$\frac{x^n}{x^2+1} > \frac{x^n}{x+1}$

$\therefore \quad K > I$

Hence $J < I < K$.

(ii) $(x+1)\left(x^5 - x^4 + x^3 - x^2 + x - 1 + \frac{1}{x+1}\right)$

$= (x+1)(x^5 - x^4 + x^3 - x^2 + x - 1) + 1$

$= x^6 - x^5 + x^4 - x^3 + x^2 - x + x^5 - x^4 + x^3 - x^2 + x - 1 + 1$

$= x^6$

$\therefore \quad x^6 \div (x+1) = x^5 - x^4 + x^3 - x^2 + x - 1 + \frac{1}{x+1}$

(iii) when $n=5$, $J = \int_0^1 \frac{x^6}{x+1}\,dx$

$= \int_0^1 \left(x^5 - x^4 + x^3 - x^2 + x - 1 + \frac{1}{x+1}\right) dx$

$= \left[\frac{x^6}{6} - \frac{x^5}{5} + \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + \ln|x+1|\right]_0^1$

$= \frac{1}{6} - \frac{1}{5} + \frac{1}{4} - \frac{1}{3} + \frac{1}{2} - 1 + \ln 2$

$= \ln 2 - \frac{37}{60}$

$\frac{x^5}{x+1} = \frac{1}{x}\left(\frac{x^6}{x+1}\right)$

$= x^4 - x^3 + x^2 - x + 1 - \frac{1}{x} + \frac{1}{x(x+1)}$

$= x^4 - x^3 + x^2 - x + 1 + \frac{1 - x - 1}{x(x+1)}$

$= x^4 - x^3 + x^2 - x + 1 - \frac{1}{x+1}$

when $n=5$, $I = \int_0^1 \frac{x^5}{x+1}\,dx$

$= \int_0^1 x^4 - x^3 + x^2 - x + 1 - \frac{1}{x+1}\,dx$

$= \left[\frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1|\right]_0^1$

$= \frac{1}{5} - \frac{1}{4} + \frac{1}{3} - \frac{1}{2} + 1 - \ln 2$

$= \frac{47}{60} - \ln 2$

$$\begin{array}{r|l} & x^3 - x \\ x^2+1 & x^5 + 0x^4 + 0x^3 + 0x \\ & x^5 \quad\quad + x^3 \\ & \quad -x^3 + 0 \\ & \quad -x^3 - x \\ & \quad\quad\quad x \end{array}$$

when $n=5$, $K = \int_0^1 \frac{x^5}{x^2+1}\,dx$

$= \int_0^1 x^3 - x + \frac{x}{x^2+1}\,dx$

$= \left[\frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ln(x^2+1)\right]_0^1$

$= \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\ln 2$

$= \frac{1}{2}\ln 2 - \frac{1}{4}$

From part (i),

$J < I$

$\ln 2 - \frac{37}{60} < \frac{47}{60} - \ln 2$

$2\ln 2 < \frac{84}{60}$

$\ln 2 < \frac{7}{10}$

$I < K$

$\frac{47}{60} - \ln 2 < \frac{1}{2}\ln 2 - \frac{1}{4}$

$\frac{62}{60} < \frac{3}{2}\ln 2$

$\frac{31}{45} < \ln 2$

$\therefore \quad \frac{31}{45} < \ln 2 < \frac{7}{10}$

$\frac{62}{90} < \ln 2 < \frac{63}{90}$

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