Student Number

St. Catherine's School
Waverley

2011

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 3 – 15%
MAY 2011

Mathematics

General Instructions

- There is no reading time.
- Working time – 1 hour
- Write using blue or black pen.
- Board- approved calculators may be used.
- All necessary working should be shown.
- Start each question in a new booklet.

Total marks - 45

Attempt Questions 1-3

The questions are of equal value.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x>0$

Total marks – 45
Attempt Questions 1-3
All questions are of equal value

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 1 (15 Marks) Use the Question 1 Writing Booklet **Marks**

(a) Given $\log_5 3 = 0.682606$ and $\log_5 2 = 0.430676$, calculate the value of:

(i) $\log_5 6$ 2

(ii) $\log_5 50$ 2

Give your answers to 4 decimal places.

(b) Differentiate with respect to x:

(i) $\tan \pi x$ 1

(ii) $e^{3x} \sin x$ 2

(iii) $\cos^2 x$ 2

(c) Find the following indefinite integrals:

(i) $\int \cos\frac{x}{3}\, dx$ 1

(ii) $\int \frac{6x-5}{3x^2-5x+2}\, dx$ 1

(iii) $\int 9e^{3x+5}\, dx$ 1

(d) (i) Find the equation of the ~~tangent~~ to the curve $y = e^x - 2x$ at the point $(1, e-2)$. Leave your answer in terms of e. 2

(ii) Show that this tangent passes through the origin. 1

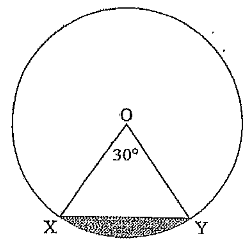
Question 2 (15 Marks) Use the Question 2 Writing Booklet **Marks**

(a) Solve these equations for x:

(i) $e^{\log 3x} = 6$ 1

(ii) $2\log_7 3 = \log_7 x - \log_7 6$ 2

(b) The diagram shows an arc, XY of length $15\pi\ cm$ subtending an angle of $30°$ at O, the centre of the circle.



(i) Show that the radius of the circle is $90\ cm$. 2

(ii) Calculate the shaded area shown in the diagram.
Answer to one decimal place. 2

(iii) Calculate the length of the chord XY. Answer to one decimal place. 2

(c) Evaluate the following definite integrals, leaving your answers in exact form.

(i) $\int_2^3 \frac{dx}{3x-4}$ 3

(ii) $\int_1^e (2t + \frac{2}{t})\, dt$ 3

Question 3 (15 Marks) Use the Question 3 Writing Booklet **Marks**

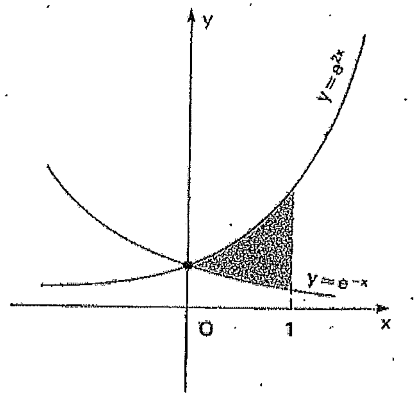
(a) Find:

(i) $\dfrac{d}{dx}\ln\left(\dfrac{x+4}{x-2}\right)$. Answer as a single fraction. **2**

(ii) $\int \tan x \, dx$ **1**

(b) The region under the curve $y=\sqrt{\dfrac{2x}{x^2+1}}$ and bounded by the lines $x=0$ and $x=1$, is rotated about the x-axis. Find the volume of the solid of revolution. **3**

(c) The diagram shows the curves $y=e^{2x}$ and $y=e^{-x}$.

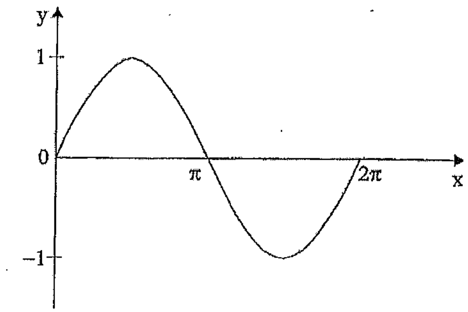


Show that the shaded area between the curves from $x=0$ to $x=1$ is: **3**

$\frac{1}{2}(e^2+\frac{2}{e}-3)\ unit^2$.

Question 3 continued

(d) The graph of $y=\sin x$ for $0 \le x \le 2\pi$, has been drawn below.



NOT TO SCALE

(i) Copy or trace this diagram into your answer booklet.

(ii) On the same axes, draw the graph of $y=2\sin x$. **1**

(iii) On your diagram, shade in the region defined by $\int_0^{\pi}(2\sin x-\sin x)\,dx$ and hence calculate this area. **3**

(iv) Let m be a negative number. By drawing a sketch, or otherwise, show that $\sin x = mx$ has $x=0$ as its only solution satisfying $-\pi \le x \le \pi$. **2**

End of examination

YR 12 MATHEMATICS TASK 3

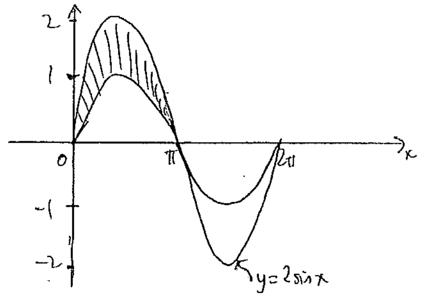
Qn	Solutions	Marks	Comments; Criteria
	Question 1		
	(a) (i) $\log_5 6 = \log_5(2\times 3)$ $= \log_5 2 + \log_5 3$ $= 0.682606 + 0.430676$ $= 1.1133$ (4 d.p)	2	
	(ii) $\log_5 50 = \log_5(25\times 2)$ $= \log_5 25 + \log_5 2$ $= \log_5 5^2 + \log_5 2$ $= 2\log_5 5 + \log_5 2$ $= 2 + 0.430676$ $= 2.4307$ (4 d.p)	2	
	(b) (i) $\frac{d}{dx}\tan \pi x = \pi \sec^2 \pi x$	1	
	(ii) $\frac{d}{dx} e^{3x}\sin x = e^{3x}.\cos x + 3e^{3x}\sin x$ $= e^{3x}(\cos x + 3\sin x)$	2	
	(iii) $\frac{d}{dx}\cos^2 x = \frac{d}{dx}(\cos x)^2$ $= 2(\cos x)(-\sin x)$ $= -2\cos x \sin x$	2	
	(c) (i) $\int \cos\frac{x}{3}\,dx = \frac{1}{\frac{1}{3}}.\sin\frac{x}{3} + C$ $= 3\sin\frac{x}{3} + C$	1	
	(ii) $\int \frac{6x-5}{3x^2-5x+2}\,dx = \ln(3x^2-5x+2) + C$	1	
	(iii) $\int 9e^{3x+5}\,dx = 9.\frac{1}{3}e^{3x+5}$ $= 3e^{3x+5} + C$	1	
	P.T.O.		

Qn	Solutions	Marks	Comments; Criteria
1	continued....		
	(d) (i) $y = e^x - 2x$ $y' = e^x - 2$ At $x = 1$, gradient of tangent $= e^1 - 2$ $\therefore$ Equation of tangent is: $y - (e-2) = (e-2)(x-1)$ $y \cancel{-e} \cancel{+2} = ex \cancel{-e} - 2x \cancel{+2}$ $\therefore y = ex - 2x$ $y = x(e-2)$	2	
	(ii) If tangent passes through (0,0), then (0,0) satisfies equation. i.e. $0 = 0(e-2)$ True. $\therefore$ Tangent does pass through the origin.	1	

Qn	Solutions	Marks	Comments: Criteria
	Question 2		
	(a) (i) $e^{\log 3x} = 6$ $\therefore 3x = 6$ $x = 2$	1	
	(ii) $2\log_7 3 = \log_7 x - \log_7 6$ $\log_7 3^2 = \log_7\left(\frac{x}{6}\right)$ $\log_7 9 = \log_7 \frac{x}{6}$ $\therefore 9 = \frac{x}{6}$ $\therefore x = 54$	2	
	(b) (i) $30^\circ = \frac{\pi}{6}$ $l = r\theta$ $\therefore 15\pi = r.\frac{\pi}{6}$ $r = 15\cancel{\pi} \times \frac{6}{\cancel{\pi}}$ $\therefore r = 90\text{cm}$, as required.	2	
	(ii) Area of minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$ $= \frac{1}{2}.90^2\left(\frac{\pi}{6} - \sin\frac{\pi}{6}\right)$ $= 4050\left(\frac{\pi}{6} - \frac{1}{2}\right)$ $= 95.6\text{ cm}^2$ (1 d.p)	2	
	(iii) By using Cosine Rule: $XY^2 = 90^2 + 90^2 - 2(90)(90)\cos 30$ $= 2170.388\ldots$ $\therefore XY = \sqrt{2170.388\ldots}$ $XY = 46.6\text{cm}$ (1 d.p)	2	

Qn	Solutions	Marks	Comments: Criteria
	Q2 continued...		
	(c) (i) $\int_2^3 \frac{dx}{3x-4} = \frac{1}{3}\int_2^3 \frac{3}{3x-4}\,dx$ $= \left[\frac{1}{3}\ln(3x-4)\right]_2^3$ $= \frac{1}{3}(\ln 5 - \ln 2)$ $= \frac{1}{3}\ln\left(\frac{5}{2}\right)$	3	
	(ii) $\int_1^e \left(2t + \frac{2}{t}\right)dt = \left[t^2 + 2\ln t\right]_1^e$ $= (e^2 + 2\ln e) - (1 + \underbrace{2\ln 1}_{0})$ $= e^2 + 2 - 1$ $= e^2 + 1$	3	

Qn	Solutions	Marks	Comments: Criteria
	Question 3		
	(a) (i) $\frac{d}{dx}\ln\left(\frac{x+4}{x-2}\right)$		
	$= \frac{d}{dx}\left[\ln(x+4) - \ln(x-2)\right]$		
	$= \frac{1}{x+4} - \frac{1}{x-2}$		
	$= \frac{x-2-(x+4)}{(x+4)(x-2)}$		
	$= \frac{\cancel{x}-2\cancel{-x}-4}{(x+4)(x-2)}$		
	$= \frac{-6}{(x+4)(x-2)}$	2	
	(ii) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$		
	$= -1\int \frac{-\sin x}{\cos x} dx$	1	
	$= -\ln \cos x + C$		
	(b) $y = \sqrt{\frac{2x}{x^2+1}}$		
	$\therefore y^2 = \frac{2x}{x^2+1}$		
	$\therefore V = \pi \int_0^1 \frac{2x}{x^2+1} dx$	3	
	$= \pi \left[\ln(x^2+1)\right]_0^1$		
	$= \pi\left[\ln(2) - \cancel{\ln 1}_{0}\right]$		
	$= \pi \ln 2 \ u^3$		

Qn	Solutions	Marks	Comments: Criteria
	Q3 continued ...		
	(c) $A = \int_0^1 (e^{2x} - e^{-x}) dx$		
	$= \left[\frac{1}{2}e^{2x} + e^{-x}\right]_0^1$	$1\frac{1}{2}$	
	$= \left(\frac{1}{2}e^2 + e^{-1}\right) - \left(\frac{1}{2}e^0 + e^0\right)$	$\frac{1}{2}$	
	$= \frac{1}{2}e^2 + \frac{1}{e} - \frac{1}{2} - 1$		
	$= \frac{1}{2}e^2 + \frac{1}{e} - \frac{3}{2}$	3	
	$= \frac{1}{2}\left(e^2 + \frac{2}{e} - 3\right) u^3$, as required.	1	
	(d) (i) (ii) 	1	
	(iii) Shading on diagram	1	
	$A = \int_0^{\pi} (2\sin x - \sin x) dx$		
	$= \left[-2\cos x + \cos x\right]_0^{\pi}$	2	
	$= (-2\cos\pi + \cos\pi) - (-2\cos 0 + \cos 0)$		
	$= -2(-1) + (-1) - (-2+1)$		
	$= 2 - 1 + 2 - 1$		
	$= 2u^2$		
	or more simply: $A = \int_0^{\pi} (2\sin x - \sin x) dx$		
	$= \int_0^{\pi} \sin x \, dx = \left[-\cos x\right]_0^{\pi}$		
	$= -\cos\pi + \cos 0$		
	$= -1 + 1 = 2u^2$		

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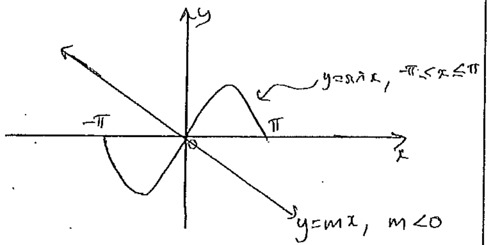
<tr><th>Qn</th><th>Solutions</th><th>Marks</th><th>Comments: Criteria</th></tr>
<tr><td></td><td>Q3 continued ....

(d)



The graph of $y=mx$ passes through the origin.
As $m<0$, gradient of line negative.
$\therefore$ From graph, $x=0$ is only solution for $\sin x = mx$, where $-\pi \leq x \leq \pi$.

— End of examination —</td><td>2</td><td></td></tr>
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