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HSC Trial Examination 2004

Mathematics Extension 2

This paper must be kept under strict security and may only be used on or after the morning of Monday 9 August, 2004, as specified in the NEAP Examination Timetable.

General Instructions

Reading time 5 minutes

Working time 3 hours

Write using blue or black pen.

Board-approved calculators may be used.

A table of standard integrals is provided on page 12.

All necessary working should be shown in every question.

Total marks – 120

Attempt questions 1–8.

All questions are of equal value.

Total marks 120
Attempt Questions 1–8.
All questions are of equal value.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find:

(i) $\displaystyle\int \frac{1}{x\ln x}\,dx$ **2**

(ii) $\displaystyle\int \frac{x\,dx}{x^2+2x+5}$ **3**

(b) Evaluate, using integration by parts, $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx$. **2**

(c) Evaluate, using partial fractions, $\displaystyle\int_1^3 \frac{dx}{x^2+2x}$. **3**

(d) The integral I_n is defined by $I_n = \displaystyle\int_0^1 x^n e^{-x}\,dx$.

(i) Show that $I_n = nI_{n-1} - e^{-1}$. **2**

(ii) Hence show that $I_3 = 6 - 16e^{-1}$. **3**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $u = 4 + 2i$ and $v = 3 - i$.

(i) Find $\bar{u}v$. **1**

(ii) Find v^2. **1**

(b) The complex number z satisfies both equations $|z - 1| = \frac{1}{2}|z|$ and $\arg(z - 1) - \arg z = \frac{\pi}{3}$.

(i) Show that $\frac{z-1}{z} = \frac{1}{2}\text{cis}\left(\frac{\pi}{3}\right)$. **2**

(ii) Hence show that $z = \frac{3 + i\sqrt{3}}{3}$. **2**

(c) (i) In the **same** Argand diagram, sketch the locus of $|z - 3| = 1$ and the locus of $|z| = |z - 2i|$. **2**

(ii) Write down the complex number represented by the point of intersection of the two loci. **1**

(d) (i) Find all solutions of the equation $z^3 = 1$. Leave your solutions in the form $z = \text{cis}\,\theta$. **1**

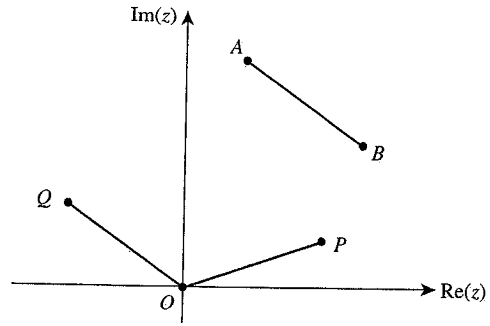
(ii) Let $\omega = \text{cis}\frac{2\pi}{3}$. Show that $\omega^2 + \omega + 1 = 0$. **1**

(iii) By expanding $(z + 2)(z^2 + z + 1)$, or otherwise, solve the equation $z^3 + 3z^2 + 3z + 2 = 0$. **2**

Question 2 continued on page 4

QUESTION 2. (Continued) **Marks**

(e) **2**

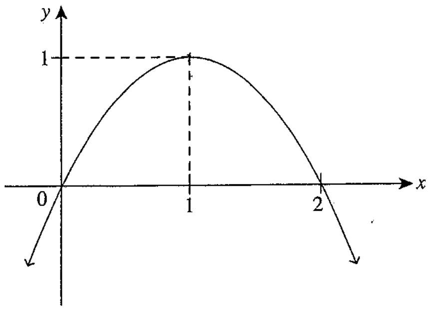


In the Argand diagram above, the intervals AB, OP and OQ are equal in length. Also, $QO \parallel AB$ and $\angle POQ = \frac{2\pi}{3}$, where P has been rotated $\frac{2\pi}{3}$ anticlockwise about O to the point Q.

Let A and B represent the complex numbers u and v respectively. Find, in terms of u and v, the complex number represented by P.

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The graph above shows the curve $y = f(x)$, where $f(x) = x(2 - x)$.

Without the use of calculus, sketch the following curves. Show any intercepts, asymptotes, end points and turning points.

(i) $y = f(2x)$ — 1

(ii) $y = \dfrac{1}{f(x)}$ — 2

(iii) $|y| = f(x)$ — 2

(iv) $y = \ln f(x)$ — 2

(v) $y = f(e^x)$ — 2

(b) Consider the hyperbola $\mathscr{H}$ defined by $9x^2 - 16y^2 = 144$

(i) Find the coordinates of the foci. — 1

(ii) Find the equations of the directrices. — 1

(iii) Find the equations of the asymptotes. — 1

(iv) Sketch $\mathscr{H}$ showing all the above information and all intercepts with the axes. — 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Fifteen new students at a school are distributed evenly among three classes. Given that there are three children with blonde hair among the fifteen and that the students are distributed randomly, find the probability that:

(i) all the children with blonde hair end up in the same class. — 2

(ii) each class gets one child with blonde hair. — 2

(b) Let $P(x_1, y_1)$ be a point on the ellipse $16x^2 + 25y^2 = 400$.

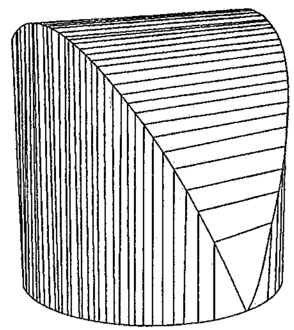
(i) Draw the ellipse, showing all intercepts. — 1

(ii) Write down the eccentricity. — 1

(iii) Show that the equation of the normal at P is $25y_1x - 16x_1y - 9x_1y_1 = 0$. — 2

(iv) The normal at P meets the major axis at N. By using the focus-directrix definition of an ellipse, or otherwise, prove that $\dfrac{NS}{NS'} = \dfrac{PS}{PS'}$, where S and S' are the two foci. — 3

(c) Trieu took a wooden cylinder and carved it into the shape shown in the diagram below.
The base of her shape is a circle with radius 8 cm.
Each vertical cross-section shown in the diagram is a square.



(i) Show that the area A of the cross-section distance x cm from the centre of the base is $A = 4(64 - x^2)$. — 2

(ii) Hence show that the volume V of the art project is given by — 2

$$V = 8\int_0^8 (64 - x^2)\,dx$$

and evaluate the integral.

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that the quartic polynomial $p(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$ has a zero of multiplicity three, factorise the polynomial completely and find all its zeroes. **3**

(b) Let $Q(x) = x^3 + px + q$, where p and q are real and non-zero. Two of the zeroes of $Q(x)$ are $a + ib$ and k, where a, b and k are real and non-zero and $k < 0$.

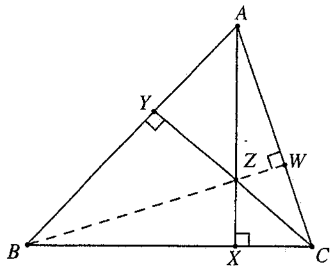
It is known that the graph of $y = Q(x)$ has two turning points.

(i) By a consideration of $Q'(x)$, show that $p < 0$. **1**

(ii) Deduce that $a > 0$. **2**

(iii) Show that $q = 8a^3 + 2ap$. **3**

(c)



In the diagram above, ΔABC is an acute-angled triangle. Altitudes are drawn from A and C to meet BC and AB at X and Y respectively. The altitudes AX and CY intersect at Z. The interval BZ is produced to meet AC at W.

Copy the diagram in your answer booklet.

(i) Show that $BYZX$ is a cyclic quadrilateral. **1**

(ii) Prove that $\angle BZX = \angle BYX$. **1**

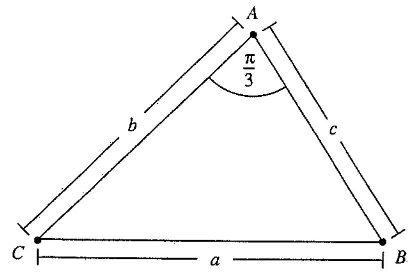
(iii) Prove that $\angle BZX = \angle ACX$. **2**

(iv) Prove that BW is perpendicular to AC. **1**

(v) What general result relating to triangles has been proven in parts (i)–(iv) above? **1**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the diagram below, a triangle ABC has sides of length a, b and c, and $\angle A = \frac{\pi}{3}$.



(i) Show that $a^2 \geq bc$. **2**

(ii) Hence show that the area of the triangle $ABC \leq \frac{a^2\sqrt{3}}{4}$. **1**

(b) Let $I_n = \int_0^{\frac{1}{2}} \frac{(\tan^{-1}2x)^n}{1+4x^2}\,dx$, where n is a positive integer.

(i) By using an appropriate substitution, show that $I_n = \left(\frac{\pi}{4}\right)^{n+1} \times \frac{1}{2(1+n)}$. **2**

(ii) Hence, or otherwise, show that $I_0 \times I_1 \times I_2 \times \ldots \times I_{2n-1} = \left(\frac{\pi}{4}\right)^{n(2n+1)} \times \frac{1}{2^{2n}(2n)!}$. **2**

Question 6 continues on page 9

QUESTION 6. (Continued) **Marks**

(c) A food package of mass m kg has a parachute device attached. It is released from rest from the top of a cliff 100 metres high. During its fall, the only forces acting are gravity, and owing to the parachute, a resistive force of magnitude $\frac{1}{10}mv^2$, where v metres per second is the speed of the package.

After $\frac{1}{2}\ln 99$ seconds, the parachute disintegrates, and then the only force acting on the particle is due to gravity.

The acceleration due to gravity is taken as 10 m s^{-2}. At time t seconds after being dropped, the package has fallen a distance of x metres from the plane, and its speed is v m s^{-1}.

(i) Show that while the parachute is operating, $\ddot{x} = 10 - \frac{v^2}{10}$. Hence show that **4**

$$v = 10\left(\frac{e^{2t} - 1}{e^{2t} + 1}\right) \text{ and } x = 5\ln\left(\frac{100}{100 - v^2}\right).$$

(ii) Find the exact speed of the package and the exact vertical distance fallen just before the parachute disintegrates. **2**

(iii) Find the speed of the package just before it reaches the ground. Give your answer correct to two significant figures. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Prove that $\frac{1}{2}(\cos(m-n)x - \cos(m+n)x) = \sin mx \sin nx$. **2**

(ii) Hence find the general solution of $\sin 3x \sin x = 2\cos 2x + 1$. **3**

(b) The function $f(x)$ is defined by $f(x) = x - \ln(1 + x^2)$.

(i) Show that $f'(x) > 0$, for all $x \neq 1$. **2**

(ii) Deduce that $e^x > 1 + x^2$, for all $x > 0$. **3**

(c) A sequence $t_1, t_2, t_3, \dots, t_n, \dots$ is defined by

$t_1 = 1$,

$3t_{n+1} = 2t_n - 1$, for $n \geq 2$.

(i) Prove by mathematical induction that $t_n = 3\left(\frac{2}{3}\right)^n - 1$ for all integers $n \geq 1$. **3**

(ii) Hence find $\sum_{r=1}^{n} t_r$. **2**

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that, for all positive integers n, **2**

$$1+(1+x)+(1+x)^2+\ldots+(1+x)^n=\frac{(1+x)^{n+1}-1}{x}.$$

(ii) Hence, or otherwise, show that for all integers $n \geq 2$, **2**

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n}{2}=\binom{n+1}{3}.$$

(iii) The polynomial $1-x+x^2-x^3+\ldots+x^{16}-x^{17}+x^{18}$ may be written in the form **2**

$$b_0+b_1w+b_2w^2+\ldots+b_{17}w^{17}+b_{18}w^{18}$$

where $w=x+1$ and $b_0, b_1, \ldots, b_{18}$ are real numbers.
Using the results of parts (i) and (ii) above, or otherwise, find the value of b_2.

(b) The series $\frac{1}{2}+\frac{8}{4}+\frac{27}{8}+\ldots=\sum_{n=1}^{\infty}\frac{n^3}{2^n}$ is not geometric, and as such it is not a routine matter to decide whether or not it converges to a finite sum. Let $y_n=\frac{n^3}{2^n}$.

(i) Show that $\frac{y_n}{y_{n-1}}=\frac{1}{2}\times\left(\frac{n}{n-1}\right)^3$, and hence show that this ratio is greater than 1 when $2 \leq n \leq 4$, but less than 1 when $n \geq 5$. **3**

(ii) Show that $\frac{y_n}{y_{n-1}} \leq 0.98$, for $n \geq 5$. **2**

(iii) Given that $y_4=4$, deduce that $y_n \leq 4\times(0.98)^{n-4}$, for $n \geq 4$, and write down the value of $\lim_{n\to\infty} y_n$. **2**

(iv) By considering $\sum_{n=4}^{\infty} 4\times(0.98)^{n-4}$, deduce that $\sum_{n=1}^{\infty}\frac{n^3}{2^n}<200+\frac{47}{8}$. **2**

End of paper

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Mathematics Extension 2

Solutions and marking guidelines

Question 1

Sample answer | **Syllabus outcomes and marking guide**

(a) (i) $\int \frac{1}{x\ln x}dx$

Let $u = \ln x$.

Then $du = \left(\frac{1}{x}\right)dx$.

Hence $\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln u + c = \ln(\ln x) + c$

E8
- Correct solution.......................2
- Appropriate substitution done correctly OR
- Correct modified primitive or equivalent merit but fails to get the correct solution. . 1

(ii) $\int \frac{x dx}{x^2+2x+5} = \int \frac{\left[\frac{1}{2}(2x+2)-1\right]dx}{x^2+2x+5}$

$= \frac{1}{2}\int \frac{(2x+2)dx}{x^2+2x+5} - \int \frac{dx}{(x+1)^2+4}$

$= \frac{1}{2}\ln(x^2+2x+5) - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$

E8
- Correct solution.......................3
- Decomposes correctly into two known integrals, but fails to obtain correct answer.............................2
- Partially decomposes into one or two known integrals and fails to obtain correct answer.............................1

(b) $\int_0^{\frac{\pi}{2}} x\cos x \, dx$

Let $u = x$ and $v' = \cos x$. Then $u' = 1$ and $v = \sin x$.

Hence $\int_0^{\frac{\pi}{2}} x\cos x dx = [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx$

$= \frac{\pi}{2} - [\sin x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - 1$

E8
- Correct solution.......................2
- Reasonable attempt to use the method of integration by parts...................1

(c) $\frac{1}{x(x+2)} = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)$

$\int_1^3 \frac{dx}{x(x+2)} = \frac{1}{2}\left(\int_1^3 \left(\frac{1}{x} - \frac{1}{x+2}\right)dx\right)$

$= \frac{1}{2}[\ln x - \ln(x+2)]_1^3$

$= \frac{1}{2}\left(\ln\left(\frac{3}{5}\right) - \ln\left(\frac{1}{3}\right)\right)$

$= \frac{1}{2}\ln\frac{9}{5}$

E8
- Correct solution.......................3
- Applies method of partial fractions correctly but fails to get the correct answer.............................2
- Reasonable attempt to use the method of partial fractions.......................1

Question 1 **(Continued)**

Sample answer | **Syllabus outcomes and marking guide**

(d) (i) $I_n = \int x^n e^{-x} dx$

Let $u' = e^{-x}$ and $v = x^n$

Then $u = -e^{-x}$, $v' = nx^{n-1}$

Hence $I_n = [-e^{-x}x^n]_0^1 - \int_0^1 -e^{-x}nx^{n-1}dx$

$= -e^{-1} + nI_{n-1}$

$= nI_{n-1} - e^{-1}$

QED

E8
- Correct solution.......................2
- Reasonable attempt to use the method of integration by parts...................1

(ii) Using the recursion formula

$I_3 = 3I_2 - e^{-1}$

$I_2 = 2I_1 - e^{-1}$

$I_1 = I_0 - e^{-1}$

Also $I_0 = \int_0^1 e^{-x}dx = [-e^{-x}]_0^1$

$= (-e^{-1} + 1)$

Hence $I_1 = -e^{-1} + 1 - e^{-1} = -2e^{-1} +$

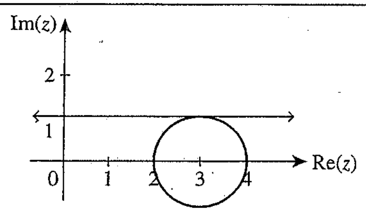
$I_2 = 2(-2e^{-1}+1) - e^{-1} = 2 - 5e^{-1}$

$I_3 = 3(2 - 5e^{-1}) - e^{-1} = 6 - 16e^{-1}$

QED

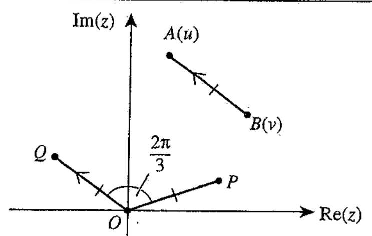
E8
- Correct solution.......................3
- Applies recurrence relation properly, but fails to get correct answer.............2
- Makes reasonable attempt to use the recurrence relation...................1

Question 2

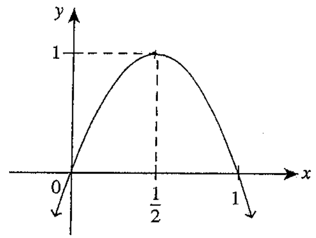
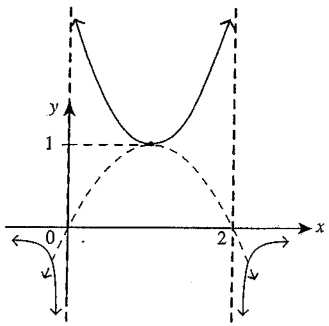
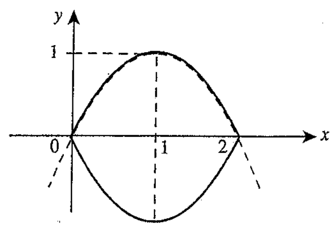
Sample answer	Syllabus outcomes and marking guide
(a) (i) $\bar{u}v = (4-2i)(3-i)$ $= 12 - 10i + 2i^2$ $= 10 - 10i$	E3 • Correct answer 1
(ii) $v^2 = (3-i)^2$ $= 9 - 6i + i^2$ $= 8 - 6i$	E3 • Correct answer 1
(b) (i) $\dfrac{\lvert z-1\rvert}{\lvert z\rvert} = \dfrac{1}{2}$ so $\left\lvert \dfrac{z-1}{z} \right\rvert = \dfrac{1}{2}$ Also $\arg(z-1) - \arg(z) = \dfrac{\pi}{3}$ so $\arg\left(\dfrac{z-1}{z}\right) = \dfrac{\pi}{3}$ Hence $\dfrac{z-1}{z} = \dfrac{1}{2}\text{cis}\dfrac{\pi}{3}$	E3 • Correct solution. . . . . 2 • Reasonable attempt but fails to correctly show the result. 1
(ii) $z - 1 = z\left(\dfrac{1}{2}\text{cis}\dfrac{\pi}{3}\right)$ $z\left(1 - \dfrac{1}{2}\text{cis}\dfrac{\pi}{3}\right) = 1$ $z\left(\dfrac{3}{4} - \dfrac{i\sqrt{3}}{4}\right) = 1$ $z = \dfrac{4}{3 - i\sqrt{3}}$ $= \dfrac{4(3 + i\sqrt{3})}{12}$ $= \dfrac{3 + i\sqrt{3}}{3}$	E3 • Correct solution. . . . . 2 • Substantially correct 1
(c)  The point of intersection represents the complex number $z = 3 + i$.	E8 • Correct diagram with correct point of intersection . . . . 2 • Correct diagram but no attempt at finding the point of intersection OR • An incorrect diagram (with one mistake) and an attempt at finding the point of intersection 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

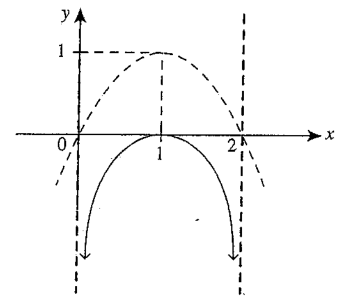
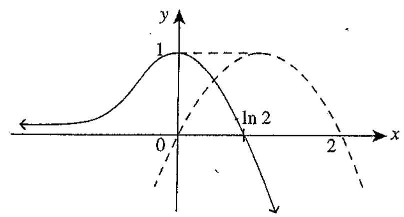
Question 2 (Continued)

Sample answer	Syllabus outcomes and marking guide
(d) (i) Let $z = \text{cis}(\theta)$ (clearly $\lvert z\rvert = 1$) $(\text{cis}\,\theta)^3 = 1$ so $\text{cis}3\theta = 1$, by de Moivre's theorem Hence $3\theta = 0, 2\pi, 4\pi, \ldots$ $\theta = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \ldots$ Hence the roots are $1, \text{cis}\dfrac{2\pi}{3}, \text{cis}\dfrac{4\pi}{3}$	E2, E3, E9 • Correct solution. 1
(ii) The roots are $\text{cis}0 = 1$, $\text{cis}\dfrac{2\pi}{3} = \omega$, and $\text{cis}\dfrac{4\pi}{3} = \omega^2$ From the equation $z^3 - 1 = 0$ sum of roots $= \dfrac{-\text{coefficient } z^2}{\text{coefficient } z^3} = 0$ Hence $1 + \omega + \omega^2 = 0$	E2, E3, E9 • Correct solution. 1
(iii) Now $(z+2)(z^2+z+1)$ $= z^3 + z^2 + z + 2z^2 + 2z + 2$ $= z^3 + 3z^2 + 3z + 2$ the equation $z^3 + 3z^2 + 3z + 2 = 0$ has the same roots as $(z+2)(z^2+z+1) = 0$ so the roots are $z = -2, \omega, \omega^2$	E2, E3, E9 • Correct solution. . . . . 2 • Substantially correct 1
(e)  The vectors BA and OQ are equal. Hence Q represents the complex number $z - u$. OP represents a clockwise rotation of OQ through $\dfrac{2\pi}{3}$. Hence $OP = (u - v) \times \text{cis}\left(-\dfrac{2\pi}{3}\right)$ $= (u - v)\left(\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)\right)$ $= (u - v)\left(-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)$	E3, E9 • Correct solution. . . . . 2 • Either $z - w$ or $-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$ 1

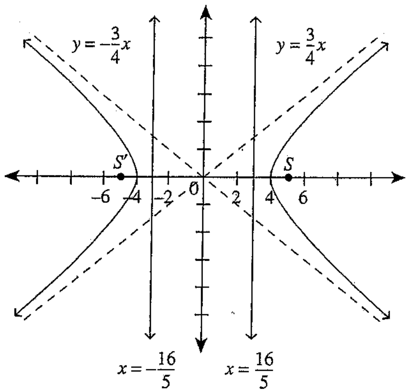
Question 3

Sample answer	Syllabus outcomes and marking guide
(a) (i) $y = f(2x)$ The new graph is a horizontal contraction by a factor of 2. 	E6 • Correct graph. 1
(ii) $y = \frac{1}{f(x)}$ There are vertical asymptotes at $x = 0$ and $x = 2$. There is a horizontal asymptote at $y = 0$. 	E6 • Correct graph. . . . . 2 • Substantially correct, but missing one of intercepts, asymptotes, correct behaviour at end points or turning points 1
(iii) $\|y\| = f(x)$ There are cusps at (0, 0) and (2, 0). 	E6 • Correct graph. . . . . 2 • Substantially correct, but missing one of intercepts, asymptotes, correct behaviour at end points or turning points 1

Question 3 (Continued)

Sample answer	Syllabus outcomes and marking guide
(iv) $y = \ln(f(x))$ y is undefined where $f(x) \le 0$. There are vertical asymptotes at $x = 0$ and $x = 2$. $\ln 1 = 0$ Since $0 < f(x) < 1, y < 0$ As $x \to 0^+, y \to -\infty$ and as $x \to 2^-, y \to -\infty$. 	E6 • Correct graph. . . . . 2 • Substantially correct, but missing one of intercepts, asymptotes, correct behaviour at end points or turning points 1
(v) $y = f(e^x)$ When $x = 0, y = f(1) = 1$ When $x > 0, e^x > 1$, so $f(e^x) < 1$ When $x < 0, e^x < 1$, so $f(e^x) < 1$ As $x \to -\infty, e^x \to 0^+$, so $y \to 0^+$ As $x \to \infty, e^x \to \infty$, so $y \to -\infty$ When $y = 0$, $e^x(e^x - 2) = 0$, that is, $x = \ln 2 \doteqdot 0.7$ 	E6 • Correct graph. . . . . 2 • Substantially correct, but missing one of intercepts, asymptotes, correct behaviour at end points or turning points 1
(b) (i) $9x^2 - 16y^2 = 144$ $\frac{x^2}{16} - \frac{y^2}{9} = 1$ Hence $a = 4$ and $b = 3$. Hence $e^2 = 1 + \left(\frac{b}{a}\right)^2 = \frac{25}{16}$ so $e = \frac{5}{4}$ Hence the foci are at $(ae, 0)$ and $(-ae, 0)$, that is, $(5, 0)$ and $(-5, 0)$	E3 • Correct answer 1
(ii) Directrices are at $x = \pm\frac{a}{e} = \pm\frac{16}{5}$	E3 • Correct answer 1

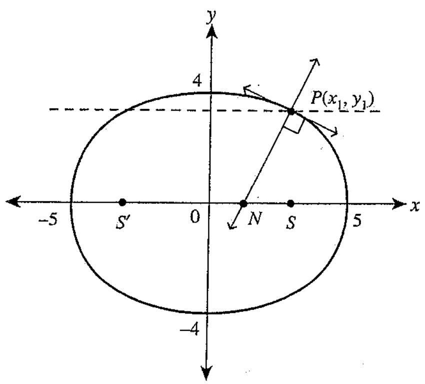
Question 3 (Continued)

Sample answer	Syllabus outcomes and marking guide
(iii) Asymptotes are $y = \pm\frac{b}{a}x = \pm\frac{3}{4}x$	E3 • Correct answers 1
(iv) 	E3 • Correct diagram showing foci, directrices and asymptotes .... 3 • Reasonable diagram showing two of the above .... 2 • Reasonable diagram showing one of the above 1

Question 4

Sample answer	Syllabus outcomes and marking guide
(a) (i) Without restriction There are $\binom{15}{5}$ ways of putting 5 boys into a class, then $\binom{10}{5}$ ways of putting the next 5 boys into a class, leaving $\binom{5}{5}$ ways of putting the remaining boys in class. i.e. $\binom{15}{5}\times\binom{10}{5}\times\binom{5}{5}$ All three blond students in one class: That leaves $\binom{12}{2}$ ways to fill up the rest of that class. Then $\binom{10}{5}$ to fill the next class and $\binom{5}{5}$ to fill the third class. There are $\binom{3}{1} = 3$ ways to choose the class with the three blonds. i.e. $\dfrac{3\times\binom{12}{2}\times\binom{10}{5}\times\binom{5}{5}}{\binom{15}{5}\times\binom{10}{5}\times\binom{5}{5}} = \dfrac{6}{91}$ **Alternative solution:** If we name the blonds A, B and C we can place A first, then B, then C and then the other 12. P(A, B and C end up in the same class) $= 1\times\frac{4}{14}\times\frac{3}{13}$ (the first student, A, can go into any one of the three classes while B and C must go into that class) $= \frac{6}{91}$	PE3 • Correct answer .... 2 • Correct number of outcomes OR One basic error 1

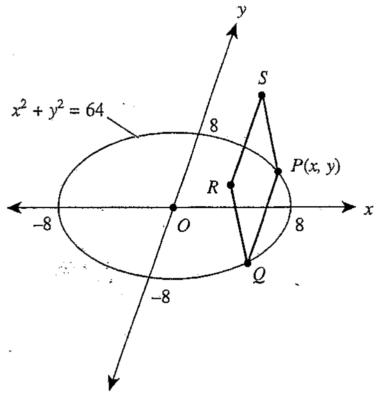
Question 4 (Continued)

Sample answer	Syllabus outcomes and marking guide
(ii) Again, naming the blonds A, B and C then if A goes into class 1 then class 1 can be filled up in $\binom{12}{4}$ ways. With B in class 2 that class can be filled up in $\binom{8}{4}$ ways. With C in the remaining class that can be filled in $\binom{4}{4}$ ways. Now there are 3! ways of distributing A, B and C around the classes. Hence $\dfrac{3!\times\binom{12}{4}\times\binom{8}{4}\times\binom{4}{4}}{\binom{15}{5}\times\binom{10}{5}\times\binom{5}{5}}=\dfrac{25}{91}$ **Alternative solution:** P(A, B and C end up in 3 different classes) $=1\times\dfrac{4}{14}\times\dfrac{3}{13}$ (the second student, B, must go into one of the other two classes and the third student, C, must go into the third class) $=\dfrac{25}{91}$	PE3 • Correct answer .... 2 • One basic error 1
(b) (i) 	E3 • Correct diagram 1

Question 4 (Continued)

Sample answer	Syllabus outcomes and marking guide
(ii) For $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$ $a^2=b^2(1-e^2)$ $e^2=1-\dfrac{b^2}{a^2}$ $=1-\dfrac{16}{25}$ $=\dfrac{9}{25}$ $e=\dfrac{3}{5}$	E3 • Correct solution 1
(iii) $16x^2+25y^2=400$ $32x+50y\dfrac{dy}{dx}=0$ $\dfrac{dy}{dx}=-\dfrac{32x}{50y}$ so the slope of the normal at $P(x_1,y_1)=\dfrac{50y_1}{32x_1}$ The equation of the normal is $y-y_1=\dfrac{25y_1}{16x_1}(x-x_1)$ $16x_1y-16x_1y_1=25y_1x-25y_1x_1$ $25y_1x-16x_1y+9x_1y_1=0$	E8 • Correct solution .... 2 • One basic error in working 1
(iv) The point N is given by $\left(0,\dfrac{9y_1}{25}\right)$. Hence $\dfrac{NS}{NS'}=\dfrac{3-\dfrac{9y_1}{25}}{\dfrac{9y_1}{25}+3}=\dfrac{25-3y_1}{25+3y_1}$ **(A)** Now $SP=ePM$ and $S'P=ePM'$ so $\dfrac{PS}{PS'}=\dfrac{PM}{PM'}$ $=\dfrac{\dfrac{25}{3}-y_1}{y_1+\dfrac{25}{3}}$ $=\dfrac{25-3y_1}{25+3y_1}$ **(B)** From A and B, $\dfrac{NS}{NS'}=\dfrac{PS}{PS'}$	E3 • Correct solution .... 3 • No more than one mistake .... 2 • Attempt to use the foci-directrix definition 1

Question 4 (Continued)

Sample answer	Syllabus outcomes and marking guide
(c) (i)  The area A of the square $PQRS$ is $A=2y^2$ $=4y^2$ $=4(64-x^2)$	E7 • Correct solution . . . . 2 • Reasonable attempt at showing that $A=(2y)^2$ or something similar 1
(ii) Hence the value V is $V=\int_{-8}^{8}4(64-x^2)dx$ $=8\int_0^8(64-x^2)dx$, since the integrand is even $=8\left[64x-\frac{1}{3}x^3\right]_0^8$ $=8\left(512-\frac{512}{3}\right)$ $=2730\frac{2}{3}$ cm^3	E7 • Shows $8\int_0^8(64-x^2)dx$ and correctly evaluated . . . . 2 • One of the above correct 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

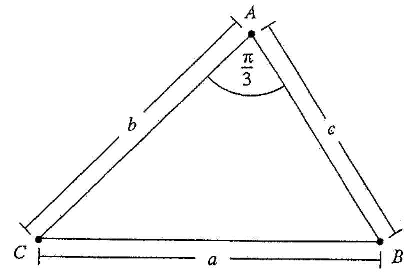
Question 5

Sample answer	Syllabus outcomes and marking guide
(a) $p(x)=x^4-5x^3-9x^2+81x-108$ Let the triple root be α and the other root β. So $p(\alpha)=p'(\alpha)=p''(\alpha)=0$ $p'(x)=4x^3-15x^2-18;$ $p''(x)=12x^2-30x-18=6(2x^2-5x-3)$ $=6(2x+1)(x-3)$ Hence $6(2\alpha+1)(\alpha-3)=0$ so $\alpha=3$ or $-\frac{1}{2}$ But $p\left(-\frac{1}{2}\right)\neq 0$, so $\alpha=3$. Also $3\alpha+\beta=5$ (sum of roots) so $\beta=-4$. Hence $p(x)=(x+4)(x-3)^3$ and its zeros are $x=3, 3, 3, -4$.	E4 • Factorises and finds all the zeros . . . . 3 • Solves the second derivative and finds the multiple root, but fails to finish or obtain correct third root . . . . 2 • Recognises that the second derivative is needed, but fails to find the correct multiple root 1
(b) (i) $Q(x)=x^3+px+q$ and $Q'(x)=3x^2+p$ Since $Q(x)$ has *two turning points*, then $Q'(x)=3x^3+p=0$ has two distinct real solutions. Solving $3x^2+p=0$ $x^2=-\frac{p}{3}$ so p must be negative.	E4 • Correct proof 1
(ii) Given $Q(a+ib)=0$ then since p, q are real, by the conjugate root theorem $Q(a-ib)=0$, $(a+ib)+(a-ib)+k=0$ (sum of the roots) $2a+k=0,\ a=-\frac{k}{2}$ Since $k<0$, a must be positive.	E8 • Correct proof . . . . 2 • Recognises that the conjugate of $a+ib$ is needed 1
(iii) $k(a+ib)(a-ib)=-q$ (product of the roots) $k(a^2-b^2)=-q$ (A) Also $k(a+ib)+k(a-ib)+(a+ib)(a-ib)=p$ (roots two at a time) $2ka+a^2+b^2=p$ $b^2=p-2ka-a^2$ (B) Combining **A** and **B**, $-2a(a^2+p-2(-2a)a-a^2)=-q$ $2a(p+4a^2)=q$ $q=8a^3+2ap$ **QED**	E8 • Correct proof . . . . 3 • Applies the theory connecting coefficients and the sums and products of roots correctly, but fails to obtain correct answer . . . . 2 • Obtains one of the following: $k(a^2+b^2)=-q$, $2a+k=0$, $2ka+a^2+b^2=p$ or their equivalent 1
(c) (i) By the converse of the angle-in-a-semicircle theorem, the circle with diameter BZ passes through X and Y.	E2, PE3 • Correct reason 1
(ii) $\angle BZX=\angle BYX$ (angles on the same arc BX)	E2, PE3 • Correct reason 1

Question 5 (Continued)

Sample answer	Syllabus outcomes and marking guide
(iii) Again by the converse of the angle-in-a-semicircle theorem, the circle with diameter AC passes through X and Y.	E2, PE3 • Correct proof 2
$\angle ACX = \angle BYX$ (exterior angle of cyclic quadrilateral $ACXY$) $= \angle BZX$ (from part (ii))	• Recognises that $SXQP$ is a cyclic quad, but fails to complete the proof satisfactorily 1
(iv) From part (iii), the triangles ΔXZB and ΔWCB are equiangular. Hence $\angle BWC = 90°$	E2, PE3 • Correct reason 1
(v) The altitudes of a triangle are concurrent.	E2, PE3 • Correct statement 1

Question 6

Sample answer	Syllabus outcomes and marking guide
(a) (i)  By the cosine rule, $a^2 = b^2 + c^2 - 2bc \times \cos\frac{\pi}{3}$ $= b^2 + c^2 - bc$ $a^2 = (b-c)^2 + bc$ Hence $a^2 \geq bc$, since $(b-c)^2 \geq 0$	H5, PE3, E2 • Correct solution . . . . 2 • Shows that $a^2 = b^2 + c^2 - bc$ 1
(ii) Area of $\Delta ABC = \frac{1}{2}bc\sin\frac{\pi}{3}$ $= \frac{\sqrt{3}bc}{4}$ $\leq \frac{\sqrt{3}a^2}{4}$, since $a^2 \geq bc$	H5, PE3, E2 • Correct solution 1
(b) (i) $I_n = \int_0^{\frac{1}{2}} \frac{(\tan^{-1}2x)^n}{1+4x^2}du$ Let $u = \tan^{-1}2x$, $du = 2 \times \frac{1}{1+4x^2}dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{4}} u^n du$ $= \frac{1}{2}\left[\frac{u^{n+1}}{n+1}\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{2} \times \frac{\left(\frac{\pi}{4}\right)^{n+1}}{n+1}$ $= \frac{1}{2(n+1)} \times \left(\frac{\pi}{4}\right)^{n+1}$	E8 • Correct solution . . . . 2 • Correctly substitutes but unable to get the required result 1

Question 6 (Continued)

Sample answer

(ii) $I_0 \times I_1 \times I_2 \times I_3 \times I_4 \ldots I_{2n-1}$

$$= \left(\frac{\pi}{4}\right) \times \frac{1}{2\times 1} \times \left(\frac{\pi}{4}\right)^2 \times \frac{1}{2\times 2} \times \left(\frac{\pi}{4}\right)^3 \times \frac{1}{2\times 3} \times \ldots \times \left(\frac{\pi}{4}\right)^{2n} \times \frac{1}{2\times 2n}$$

$$= \left(\frac{\pi}{4}\right) \times \left(\frac{\pi}{4}\right)^2 \times \left(\frac{\pi}{4}\right)^3 \times \left(\frac{\pi}{4}\right)^4 \times \ldots \times \left(\frac{\pi}{4}\right)^{2n} \times \frac{1}{2^{2n}} \times \frac{1}{(2n)!}$$

$$= \left(\frac{\pi}{4}\right)^{1+2+3+4+\ldots+2n} \times \left(\frac{1}{2}\right)^{2n} \times \frac{1}{2^{2n}} \times \frac{1}{(2n)!}$$

$$= \left(\frac{\pi}{4}\right)^{\frac{2n(2n+1)}{2}} \times \left(\frac{1}{2}\right)^{2n} \times \frac{1}{(2n)!}$$

$$= \left(\frac{\pi}{4}\right)^{n(2n+1)} \times \frac{1}{2^{2n}(2n)!}$$

Syllabus outcomes and marking guide

E8

- Correct solution 2
- Correctly obtaining the product, but unable to simplify 1

Question 6 (Continued)

Sample answer

(c) (i) Downwards gravitational force $= mg$

Upwards resistance $= \frac{1}{10}mv^2$

Hence downwards force $= mg - \frac{1}{10}mv^2$

Using $F = m\ddot{x}$

$$\ddot{x} = g - \frac{1}{10}v^2$$

and with $g = 10$

$$\ddot{x} = 10 - \frac{1}{10}v^2$$

$$10\ddot{x} = 100 - v^2$$

Let $\ddot{x} = \frac{dv}{dt}$.

Then $10\frac{dv}{dt} = 100 - v^2 = (10 - v)(10 + v)$

$$10\frac{dv}{dt} = (10 - v)(10 + v)$$

$$\frac{dt}{dv} = \frac{10}{(10 - v)(10 + v)}$$

$$= \frac{1}{2}\left(\frac{20}{(10 - v)(10 + v)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{10 - v} + \frac{1}{10 + v}\right)$$

Integrating, $t = \frac{1}{2}(-\ln|10 - v| + \ln|10 + v|) + c$

$$2t = \ln\left|\frac{10 + v}{10 - v}\right| + c$$

When $t = 0$, $v = 0$

$$0 = \ln 1 + c$$

$$c = 0$$

Hence $2t = \ln\left|\frac{10 + v}{10 - v}\right|$

Now $\frac{10 + v}{10 - v}$ is initially 1,

it can never be zero and varies continuously.

So $2t = \ln\left(\frac{10 + v}{10 - v}\right)$

$$\frac{10 + v}{10 - v} = e^{2t}$$

$$v = 10\left(\frac{e^{2t} - 1}{e^{2t} + 1}\right)$$

Syllabus outcomes and marking guide

E5

- Both answers correct 4
- One answer correct and the other substantially correct 3
- One correct answer with the other substantially NOT correct
  OR
  Both substantially correct 2
- Only one part substantially correct 1

Question 6 **(Continued)**

Sample answer | **Syllabus outcomes and marking guide**

Let $\ddot{x} = v\dfrac{dv}{dx}$

Then $10v\dfrac{dv}{dx} = 100 - v^2$

$\dfrac{dx}{dv} = \dfrac{10v}{100 - v^2}$

$\dfrac{dx}{dv} = -5\left(\dfrac{-2v}{100 - v^2}\right)$

Integrating, $x = -5\ln(100 - v^2) + c_1$

When $x = 0$, $v = 0$.

$0 = -5\ln 100 + c_1$

$c_1 = 5\ln 100$

Hence $x = 5\ln 100 - 5\ln(100 - v^2)$

$= 5\ln\left(\dfrac{100}{100 - v^2}\right)$

(ii)

When $t = \frac{1}{2}\ln 99$

$v = 10\dfrac{(e^{\ln 99} - 1)}{e^{\ln 99} + 1} = \dfrac{10(99 - 1)}{99 + 1}$

$= 10 \times \dfrac{98}{100} = 9.8 \text{ m s}^{-1}$

$x = 5\ln\dfrac{100}{100 - (9.8)^2} = 5\ln\left(\dfrac{2500}{99}\right)$ metres

E5
- Both parts correct 2
- One part correct or both substantially correct 1

(iii) After the parachute disintegrates, only gravity is acting,

so $\ddot{x} = 10$

$v\dfrac{dv}{dx} = 10$

$\dfrac{dx}{dv} = \dfrac{v}{10}$

Integrating, $x = \dfrac{1}{20}v^2 + c_2$

When $v = 9.8$, $x = \ln\dfrac{2500}{99}$

$5\ln\dfrac{2500}{99} = \dfrac{9.8^2}{20} + c_2$

$c_2 = 5\ln\dfrac{2500}{99} - 4.802$

Hence $x = \dfrac{1}{20}v^2 + 5\ln\dfrac{2500}{99} - 4.802$

When $x = 100$,

$100 = \dfrac{1}{20}v^2 + 5\ln\dfrac{2500}{99} - 4.802$

$v \doteqdot 42 \text{ m s}^{-1}$

E5
- Correct solution 2
- Substantially correct 1

Question 7

Sample answer | **Syllabus outcomes and marking guide**

(a) (i) LHS $= \frac{1}{2}(\cos(m - n)x - \cos(m + n)x)$

$= \dfrac{1}{2}\begin{pmatrix}\cos mx\cos nx + \sin mx\sin nx\ - \\ (\cos mx\cos nx - \sin mx\sin nx)\end{pmatrix}$

$= \dfrac{1}{2}\begin{pmatrix}\cos mx\cos nx + \sin mx\sin nx\ - \\ \cos mx\cos nx + \sin mx\sin nx\end{pmatrix}$

$= \frac{1}{2}(2\sin mx\sin nx)$

$= \sin mx\sin nx$

$=$ RHS

H5, HE7
- Correct proof 2
- Partially correct proof 1

(ii) $\sin 3x\sin x = 2\cos 2x + 1$

$\frac{1}{2}(\cos(3 - 1)x - \cos(3 + 1)x) = 2\cos 2x + 1$

$\cos 2x - \cos 4x = 4\cos 2x + 2$

$\cos 4x + 3\cos 2x + 2 = 0$

$\cos 4x = 2\cos^2 2x - 1$

$\cos 4x + 3\cos 2x + 2 = 0$

$2\cos^2 2x + 3\cos 2x + 1 = 0$

$(2\cos 2x + 1)(\cos 2x + 1) = 0$

$\cos 2x = -\frac{1}{2}$ or -1

$2x = 360°n + 120°$ or $360°n - 120°$ or $360°n + 180°$

$x = 180°n + 60°$ or $180°n - 60°$ or $180°n + 90°$

H5, HE7
- Correct answers 3
- Applies (i) correctly but fails to get the correct answers
  OR
  Recognises that $\cos 4x = 2\cos^2 x - 1$ but fails to get the correct answer 2
- Transforms the equation using (i) but goes no further 1

(b) (i) $f(x) = x - \ln(1 + x^2)$

$f'(x) = 1 - \dfrac{2x}{1 + x^2}$

$= \dfrac{1 + x^2 - 2x}{1 + x^2}$

$= \dfrac{(1 - x)^2}{1 + x^2}$

Since the square of a non-zero number is positive,

$f'(x) > 0$, for all $x \neq 1$

H3, PE3, E2
- Correct proof 2
- Differentiates correctly, but fails to finish the proof correctly 1

(ii) Hence $f(x)$ is strictly increasing for all x except $x = 1$.

But $f(0) = 0 - \ln 1$

$= 0$

Hence $f(x) > 0$, for all $x > 0$, because the curve slopes upwards from (0, 0).

Thus $x - \ln(1 + x^2) > 0$, for all $x > 0$

$x > \ln(1 + x^2)$, for all $x > 0$

Taking powers base e of both sides,

$e^x > 1 + x^2$, for all $x > 0$.

H3, PE3, E2
- Correct proof 3
- Shows $f(0) = 0, f'(x) \geq 0$ leads to $f(x) \geq 0$ but fails to complete the proof 2
- Shows that any one of the results immediately above 1

Question 7 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(c) (i) It follows from (a) and (b) by mathematical induction that the result is true for all positive integers n.	HE2 • Correct solution including verification of the initial case . . . . 3 • Establishes the induction step using induction as outlined in section 8.2 of the syllabus but does not verify $n = 1$, or equivalent . . . . 2 • Verifies at least $n = 1$ and makes an appropriate induction hypothesis 1

Theorem: for all integers n, $t_n = 3\left(\frac{2}{3}\right)^n - 1$.

Proof:

1. When $n = 1$, LHS = 1 (given)

$$\text{RHS} = 3 \times \frac{2}{3} - 1 = 1 = \text{LHS}$$

So the result is true for $n = 1$.

2. Suppose the result is true for some positive integer k.

That is, $t_k = 3\left(\frac{2}{3}\right)^k - 1$.*

We now prove the result for $n = k + 1$.

That is, we prove that $t_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$.

$$
\begin{aligned}
\text{LHS} &= t_{k+1} \\
&= \frac{1}{3}(2t_k - 1), \text{ by the formula for } t_{n+1} \\
&= \frac{1}{3}\left(2 \times \left[3\left(\frac{2}{3}\right)^k - 1\right] - 1\right), \text{ by induction*} \\
&= 3 \times \frac{2}{3} \times \left(\frac{2}{3}\right)^k - 1 \\
&= 3\left(\frac{2}{3}\right)^{k+1} - 1 \\
&= \text{RHS}
\end{aligned}
$$

So if the statement is true for $n = k$, then it is also true for $n = k + 1$.

Question 7 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(ii)	H5, E2 • Correct answer . . . . 2 • Applies (i) correctly but fails to complete the solution OR Demonstrates use of geometric series correctly. 1

$$
\begin{aligned}
\sum_{r=1}^{n} t_r &= \sum_{r=1}^{n}\left(3\left(\frac{2}{3}\right)^r - 1\right) \\
&= 3\sum_{r=1}^{n}\left(\frac{2}{3}\right)^r - \sum_{r=1}^{n} 1 \\
&= 3\left(\frac{\frac{2}{3}\left(1 - \left(\frac{2}{3}\right)^n\right)}{1 - \frac{2}{3}}\right) - n, \text{ using the formula for sum of a GP} \\
&= 6\left(1 - \left(\frac{2}{3}\right)^n\right) - n \\
&= 6 - n - 6\left(\frac{2}{3}\right)^n
\end{aligned}
$$

Question 8

Sample answer	Syllabus outcomes and marking guide
(a) (i) $1+(1+x)+(1+x)^2+\ldots+(1+x)^n$ is a geometric series with ratio $(1+x)$ and containing $(n+1)$ terms. Hence LHS $=\dfrac{1((1+x)^{n+1}-1)}{1+x-1}$ $=\dfrac{(1+x)^{n+1}-1}{x}=$ RHS	H5 • Correct solution .... 2 • Attempt to use a geometric series approach 1
(ii) The co-efficient of the term in x^2 on the LHS in (i) is given by $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\ldots+\binom{n}{2}$ **(A)** Consider the expansion of the RHS $\dfrac{(1+x)^{n+1}-1}{x}$ $=\dfrac{\binom{n+1}{0}+\binom{n+1}{1}x+\binom{n+1}{2}x^2+\binom{n+1}{3}x^3+\ldots}{x}$ The co-efficient of x^2 is $\binom{n+1}{3}$ **(B)** From **A** and **B**: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\ldots+\binom{n}{2}=\binom{n+1}{3}$	PE3, E2 • Correct solution .... 2 • Recognising that the coefficient of x^2 needs to be equated and a substantial attempt 1
(iii) $b_0+b_1w+b_2w^2+\ldots+b_{17}w^{17}+b_{18}w^{18}$ $=1-(w-1)+(w-1)^2-(w-1)^3+\ldots+(w-1)^{18}$ $=1+(w-1)+(w-1)^2+(w-1)^3+\ldots+(w-1)^{18}$ Coefficient of w^2 on RHS $=0+0+\binom{2}{2}+\binom{3}{2}+\ldots+\binom{18}{2}$ $=\binom{19}{3}$, by part (ii) Hence, $b_2=\binom{19}{3}$ $=969$	E2, E4 • Correct solution .... 2 • Substantial attempt 1

Question 8 (Continued)

Sample answer	Syllabus outcomes and marking guide
(b) (i) $\dfrac{y_n}{y_{n-1}}=\dfrac{n^3}{2^n}\times\dfrac{2^{n-1}}{(n-1)^3}$ $=\dfrac{1}{2}\left(\dfrac{n}{n-1}\right)^3$ Let $f(n)=\dfrac{1}{2}\left(\dfrac{n}{n-1}\right)^3$ Then $f'(n)=\dfrac{3}{2}\left(\dfrac{n}{n-1}\right)^2\times\dfrac{(n-1)\times1-n\times1}{(n-1)^2}$ $=-\dfrac{3}{2}\dfrac{n^2}{(n-1)^4}$ Clearly $f'(n)<0$ for $n\geq5$ so it is decreasing for $n\geq5$ **(A)** ∴ a decreasing function. Now $f(2)=4$, $f(3)=\dfrac{27}{16}$ and $f(4)=\dfrac{32}{27}$ so $f(n)>1$ for $n=2,3$ and 4 Also $f(5)=\dfrac{125}{128}<1$ So by A, $f(n)<1$ for $n\geq5$.	H5, PE3, E2 • Correct solution .... 3 • Substantially correct with one error .... 2 • Simplification of ratio and some attempt to use it 1
(ii) We know that $f(5)=\dfrac{125}{128}$ <0.98 Since $f(n)$ is decreasing for $n\geq5$, it follows that $f(n)<0.98$, for $n\geq5$.	H5, PE3, E2 • Correct solution .... 2 • Reasonable attempt at solving 1
(iii) $\dfrac{y_n}{4}=\dfrac{y_n}{y_{n-1}}\times\dfrac{y_{n-1}}{y_{n-2}}\times\ldots\times\dfrac{y_6}{y_5}\times\dfrac{y_5}{4}$ $<(0.98)^{n-4}$ $y_n<4\times(0.98)^{n-4}$ Also $(0.98)^{n-4}\to0$ as $n\to\infty$ so $y_n=4(0.98)^{n-4}\to0$ as $n\to\infty$ **Alternative solution:** The sequence $y_4, y_5, y_6, \ldots$ decreases more quickly than a GP with ratio 0.98. So $y_n<y_4\times(0.98)^{n-4}=4\times(0.98)^{n-4}$ Since $\lim\limits_{n\to\infty}4\times(0.98)^{n-4}=0$, it follows that $\lim\limits_{n\to\infty}y_n=0$	H5, PE3, E2 • Both parts correct .... 2 • One part correct 1

Question 8 (Continued)

Sample answer	Syllabus outcomes and marking guide
(iv) $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{2^n} \le \sum_{n=1}^{\infty} 4(0.98)^{n-4}$ $\le \frac{1}{2}+\frac{8}{4}+\frac{27}{8}+4\left(\frac{1}{1-0.98}\right)$ $< \frac{4+16+27}{8}+4\times\frac{1}{\frac{1}{50}}$ $\le \frac{47}{8}+200$	H5, PE3, E2 • Correct solution .... 2 • Establishing the 200 as part of the answer or that $\frac{1}{2}+\frac{8}{4}+\frac{27}{8}=\frac{47}{8}$ 1