Waverley College
Year 12 Extension 1Mathematics
Mid Semester Examination
Term 2 2011

TIME ALLOWED: 120 MINUTES

NAME: ______________

TEACHER: __________

INSTRUCTIONS:

Attempt all questions
Start each question on a new booklet
Calculators may be used
Write in blue or black pen only
Show all necessary working
Marks may be deducted for careless or badly arranged work

Question 1	/12
Question 2	/12
Question 3	/12
Question 4	/12
Question 5	/12
Question 6	/12
Question 7	/12
Total	/84 5
%	

Total marks – 84
Attempt Questions 1-7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Express $6\times 3^n + 3^{n+1}$ is simplest form. 1

(b) Let A be the point $(-3,8)$ and let B be the point $(-5,-6)$. Find the coordinates of the P that divides the interval AB internally in the ratio $1:3$ 2

(c) Solve $\dfrac{2}{x+5} \le 1$. 3

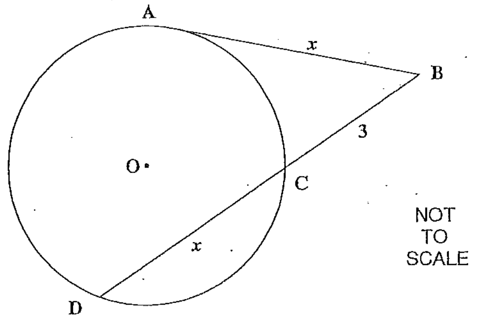
(d) Evaluate $\displaystyle\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx$, using the substitution $u = 1-x$. 4

(e) Evaluate $\displaystyle\lim_{x\to 0}\left\{\frac{\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{x}\right\}$. 2

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $\cos^2 x + 2\cos 2x = \frac{1}{2}$ for $0 \le x \le \pi$. 3

(b) 3



Copy this diagram in your answer booklet

In the diagram, AB is a tangent to a circle ACD, while BCD is a secant intersecting the circle at C and D. Given that $AB = CD = x$ and $BC = 3$, find the simplified exact value of x.

(c) Find the acute angle between the lines $3x - 4y + 3 = 0$ and $y = 2x - 5$ to the nearest minute. 2

(d) A particle moves in simple harmonic motion about a fixed point O. The amplitude of the motion is 2 m and the period is $\frac{2\pi}{3}$ seconds. Initially the particle moves from O with a positive velocity.

(i) Explain why the displacement x, in metres, of the particle at time t seconds, is given by 1

$$x = 2\sin 3t$$

(ii) Find the speed of the particle when $\sqrt{3}$ m from O. 2

(iii) What is the maximum speed reached by the particle? 1

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove by Mathematical Induction that $3^n + 5$ is divisible by 8 for all **ODD** positive integers n. 3

(b) Consider the function $f(x) = \frac{4}{x^2 - 1}$

(i) Find the vertical asymptotes. 1

(ii) Show that $y = 0$ is the horizontal asymptote. 1

(iii) Prove that $f(x) = \frac{4}{x^2 - 1}$ is an even function. 1

(iv) Prove that $(0, -4)$ is the only turning point. 2

(v) Given that there are no points of inflexion, sketch the curve of $f(x) = \frac{4}{x^2 - 1}$. 2

(c) Evaluate $\int_{-2}^{0} \frac{dx}{4 + x^2}$, leaving your answer in exact form. 2

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $\sin 2\theta = \sqrt{2}\cos\theta$ for $0 \le \theta \le 2\pi$. **4**

(b)

(i) Show that $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{\pi}{4}$. **3**

(ii) Hence or otherwise, find the volume obtained when the graph of $y = \sin x + 1$, $0 \le x \le \frac{\pi}{2}$ is rotated about the x *axis*. **3**

(c) By the use of the "*t formulae*" where $t = \tan\frac{x}{2}$ prove that **2**

$$\sec^2 x = 1 + \tan^2 x.$$

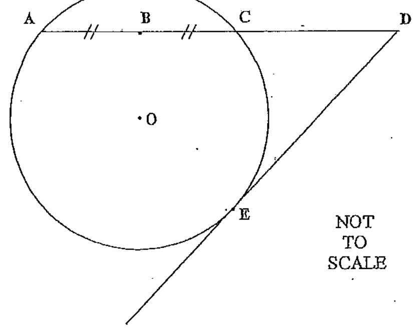
Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\sin^{-1}\left(\cos\frac{3\pi}{4}\right) + \cos^{-1}\left(\cos\frac{3\pi}{4}\right)$. **2**

(b) (i) Find the domain and range of the function $y = 3\sin^{-1}\left(\frac{x}{2}\right)$. **2**

(ii) Draw a neat sketch of $y = 3\sin^{-1}\left(\frac{x}{2}\right)$. **2**

(c)



Copy this diagram in your answer booklet

In the diagram O is the centre of the circle, DE is a tangent and ACD is a secant to the circle where $AB = BC$.

(i) Prove that $OEDB$ is cyclic. **2**

(ii) Hence prove $\angle EBD = \angle DOE$. **1**

(d) (i) Express $\sqrt{3}\sin x - \cos x$ in the form $A\sin(x-\alpha)$ and find A and α (radians). **2**

(ii) Hence find the value of a so that $\sqrt{3}\sin x - \cos x - a = 0$ has no solutions. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Differentiate $x\cos^{-1}x-\sqrt{1-x^2}$. **2**

(ii) Hence $\int_0^1 \cos^{-1}x+1\,dx$. **2**

(b) A parabola has parametric equations $x=6t$ and $y \pm 3t^2$. P is the point on the parabola where $t=p$.

(i) Show that the tangent to the parabola at P has the equation $y=px-3p^2$. **2**

(ii) If Q is the point on the parabola where $t=1-p$ and P and Q are Distinct, find the equation of the tangent at Q and show that the tangents at P and Q meet at the point $T(3,3p-3p^2)$. **4**

(iii) Write down the locus of T in Cartesian form. **2**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In a particular equatorial African swamp a colony of tsetse flies increases its population according to the differential equation $\frac{dp}{dt}=k(P-10000)$, where k is the growth rate of the colony. Initially there were 15000 tsetse flies and after 6 months there were 25000 tsetse flies.

(i) Show that $P=10000+P_oe^{kt}$ is a solution of this differential equation. **2**

(ii) Determine the growth rate k and P_0 in exact form. **2**

(iii) Determine the number of tsetse flies after 1year. **2**

(b) A missile is launched upwards from a submarine 40m below sea level at an angle θ to the horizontal with speed of 30ms^{-1}.

After reaching its maximum height after $\frac{3\sqrt{3}}{2}s$ the missile strikes a frigate located 3km away in a horizontal direction with respect to the sea level axis. Assuming that the acceleration due to gravity, g is 10ms^{-2} and neglecting any air or water resistance:

(i) Show that the parametric equations of motion are given by: $x=30t\cos\theta$ and $y=30t\sin\theta-5t^2-40$. **2**

(ii) Find the angle of projection θ. **2**

(iii) Find the time taken to strike the frigate. **2**

END OF PAPER

12 Term 2 MID-YEAR HSC EXAM 2011 Solutions

Q1

(a) $6 \times 3^n + 3^{n+1} = 2 \times 3 \times 3^n + 3^{n+1}$

$= 2 \times 3^{n+1} + 3^{n+1}$

$= 3^{n+1}(2+1)$

$= 3 \times 3^{n+1}$

$= 3^{n+2}$ ✓

(b)

$P = \left(\dfrac{lx_1 + kx_2}{k+l}, \dfrac{ly_1 + ky_2}{k+l} \right)$

where $k=1$, $l=3$, $x_1=-3$, $y_1=8$

$x_2 = 5$, $y_2 = -6$

So

$P = \left(\dfrac{3 \times -3 + 1 \times 5}{4}, \dfrac{3 \times 8 + 1 \times -6}{4} \right)$

$-\frac{14}{4}$ ✓ $\frac{9}{2}$ ✓

$\therefore P = \left(-3\frac{1}{2}, 4\frac{1}{2} \right)$

(c) Critical points method

$\dfrac{2}{x+5} \leq 1 \quad \therefore x \neq -5$ ✓

Consider $\dfrac{2}{x+5} = 1$

$2 = x+5$

$x = -3$ ✓

Look at number line

$\longleftarrow \circ_{-5} \quad \bullet_{-3} \longrightarrow$

Test For $x=-6$: $\dfrac{2}{-6+5} \leq 1$, $\dfrac{2}{-1} \leq 1$ True

$\therefore$ Extremities.

So $\dfrac{2}{x+5} \leq 1$ for $x \leq -5$ or $x \geq -3$ but $x \neq -5$

Solution $x < -5$ or $x \geq -3$ ✓

(d) $u = 1-x$

$\dfrac{du}{dx} = -1$

$-du = dx$

$x=0 \rightarrow u=1$

$x=-3 \rightarrow u=4$ ✓

Let $\displaystyle\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx = I$

$I = -\displaystyle\int_4^1 \frac{1-u}{u^{\frac{1}{2}}}\,du$ ✓

$= \displaystyle\int_1^4 u^{-\frac{1}{2}} - u^{\frac{1}{2}}\,du$

$= \left[2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right]_1^4$ ✓

$= \left[\left(4 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right) \right] = 2 - \frac{14}{3}$

$= -2\frac{2}{3}$ ✓

(e)

$\displaystyle\lim_{x \to 0} \left\{ \frac{\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{x} \right\} = \lim_{x \to 0} \left\{ \frac{\sin x}{2x} \right\}$ ✓

$= \dfrac{1}{2} \displaystyle\lim_{x \to 0} \left\{ \frac{\sin x}{x} \right\}$

$= \dfrac{1}{2}$ ✓

Alternate Solution to Question 1 (c)

Multiplying both sides by $(x+5)^2$

$\dfrac{2}{x+5} \leq 1, \quad x \neq -5$

$2(x+5) \leq (x+5)^2$

$2x + 10 \leq x^2 + 10x + 25$

$0 \leq x^2 + 8x + 15$

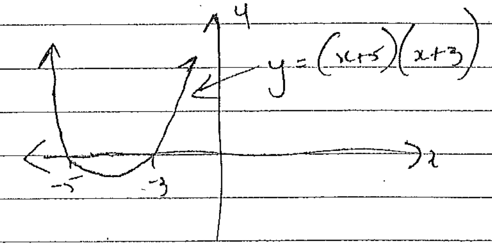
$0 \leq (x+5)(x+3)$

From the graph of $y=(x+5)(x+3)$ we see that:

$y \geq 0$ for $x \leq -5$ or $x \geq 3$

but $x \neq -5$ from above so

$x < -5$ or $x \geq 3$



Q2

(a) $\cos^2 x + 2\cos 2x = \frac{1}{2}$

$\cos^2 x + 2\left[2\cos^2 x - 1\right] = \frac{1}{2}$ ✓

$\cos^2 x + 4\cos^2 x - 2 = \frac{1}{2}$

$5\cos^2 x - 2 = \frac{1}{2}$

$5\cos^2 x = 2\frac{1}{2}$

$5\cos^2 x = \frac{5}{2}$

$\cos^2 x = \frac{1}{2}$ ✓

$\cos x = \pm \frac{1}{\sqrt{2}}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}$ ✓ for $0 \le x \le \pi$

(b)

Noting that $AB^2 = BC \times BD$ (ΔADC is similiar ΔABD)

$x^2 = 3(x+3)$ ✓

$x^2 = 3x + 9$

$x^2 - 3x - 9 = 0$

$x = \frac{3 \pm \sqrt{9+36}}{2}$ ✓

$= \frac{3 \pm \sqrt{45}}{2} = \frac{3+3\sqrt{5}}{2}, \; x > 0$

(c) For $3x - 4y + 3 = 0$ For $y = 2x - 5$

$M_1 = \frac{3}{4}$ ✓ $M_2 = 2$

Now for acute angle θ between lines

$\tan\theta = \left|\frac{M_1 - M_2}{1 + M_1 M_2}\right| = \left|\frac{\frac{3}{4} - 2}{1 + \frac{3}{4} \times 2}\right|$

$\therefore \tan\theta = \frac{1}{2}$

$\therefore \theta = 26^\circ 34'$ ✓ (to nearest minute)

(d)(i) The particle starts from the centre of motion and moves with positive velocity so the general form of the displacement-time function is

$x = a\sin nt$ ✓

Now $a = 2$ and period $= \frac{2\pi}{n}$

So $n = 3$

So the required equation is

$x = 2\sin 3t$

(ii) $x = 2\sin 3t$

when $x = \sqrt{3}$, $\begin{array}{c|c} S & A \\ \hline T & C \end{array}$

$\sin 3t = \frac{\sqrt{3}}{2}$

$3t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$

$t = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}$ ✓

Now $x = 2\sin 3t$

$\frac{dx}{dt} = 6\cos 3t$

When $t = \frac{\pi}{9}$ $\frac{dx}{dt} = 6\cos\frac{\pi}{3}$ $\begin{array}{c|c} S & A \\ \hline T & C \end{array}$

$= 3$

(check when $t = \frac{2\pi}{9}$,

$\frac{dx}{dt} = 6\cos\frac{2\pi}{3} = 6 \times \left(-\frac{1}{2}\right) = -3$

Speed $= |-3| = 3$ So the Speed is $3\,ms^{-1}$ ✓

(iii) Method 1

For SHM the maximum speed occurs at the centre of motion ie at O.

$x = 2\sin 3t$ $\begin{array}{c|c} S & A \\ \hline T & C \end{array}$

$0 = 2\sin 3t$

$\sin 3t = 0$

$3t = 0, \pi, 2\pi \ldots$

$t = 0, \frac{\pi}{3}, \frac{2\pi}{3} \ldots$

$\frac{dx}{dt} = 6\cos 3t$

when $t = 0$,

$\frac{dx}{dt} = 6\cos 0$

$= 6(1)$

$= 6$

(Check when $t = \frac{\pi}{3}$

$\frac{dx}{dt} = 6\cos\pi$

$= -6$

Speed $= |-6| = 6$)

Maximum speed reached by particle is $6\,ms^{-1}$ ✓

Method 2

Maximum Speed occurs when $\frac{d^2x}{dt^2} = 0$

$\frac{d^2x}{dt^2} = -18\sin 3t$

$0 = -18\sin 3t$

$\sin 3t = 0$

$3t = 0, \pi, 2\pi \ldots$

$t = 0, \frac{\pi}{3}, \frac{2\pi}{3} \ldots$

$\frac{dx}{dt} = 6\cos 3t$

when $t = 0$

$\frac{dx}{dt} = 6\cos 0$

$= 6$

(check when $t = \frac{\pi}{3}$

$\frac{dx}{dt} = 6\cos\pi$

$= -6$

Speed $= |-6|$

$= 6$)

Maximum speed reached by particle is $6\,ms^{-1}$

Maximum Speed reached by particle is $6\,ms^{-1}$ ✓

Q3

(a) 3^n+5

Step 1

Prove the statement is true for $n=1$

$3+5=8$ (True)

Step 2

Assume the statement is true for $n=k$ (where k is an odd integer)

Hence $3^k+5=8M$ (M is an element of Integers)

$$3^k=8M-5$$

Step 3

Prove the statement is true for $n=k+2$

$$\begin{aligned} f(k+2) &= 3^{k+2}+5 \\ &= 9[3^k]+5 \\ &= 9[8M-5]+5 \quad \text{from Step 2} \\ &= 72M-40 \\ &= 8[9M-5] \end{aligned}$$

(hence divisible by 8 as 8 is a factor)

Hence, if the statement is true for $n=k$, it is true for $n=k+2$. But the statement is true for $n=1$ $\therefore$ the statement is true for $n=3$ etc. Therefore, the statement is true for all odd integers n.

(b) (i) $x=\pm 1$

(b) (ii)

Horizontal asymptote $= \lim_{x\to\infty}\left(\frac{4}{x^2-1}\right)$

$$= \lim_{x\to\infty}\left(\frac{\frac{4}{x^2}}{1-\frac{1}{x^2}}\right)$$

$$=0$$

$\therefore y=0$ is the horizontal asymptote.

(iii)

$$f(x)=\frac{4}{x^2-1}$$

$$f(-1)=\frac{4}{(-x)^2-1}=\frac{4}{x^2-1}$$

$$f(x)=f(-x)$$

$\therefore f(x)=\frac{4}{x^2-1}$ is an even function.

(iv)

$$f(x)=\frac{4}{x^2-1}$$

$$f(x)=4(x^2-1)^{-1}$$

$$f'(x)=-4(x^2-1)^{-2}\times 2x = \frac{-8x}{(x^2-1)^2}$$

Stationary points occur when $f'(x)=0$

So $\frac{-8x}{(x^2-1)^2}=0$

$-8x=0$

$x=0$

$f(0)=-4$

Stationary point at $(0,4)$

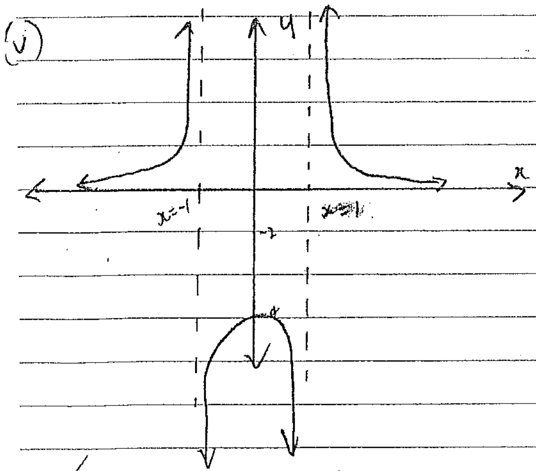
Check concavity

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
$f'(x)$	$+$	0	$-$

max

$\therefore (0,-4)$ is concave down maximum

(v)



✓ Has asymptotes and turning points in the correct positions

✓ Correct answer

(c)

$$\int_{-2}^{0}\frac{dx}{4+x^2}=\frac{1}{2}\int_{-2}^{0}\frac{2}{4+x^2}\,dx$$

$$=\frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{0}$$

$$=\frac{1}{2}\left[\tan^{-1}0+\tan^{-1}1\right]$$

$$=\frac{\pi}{8}$$

Q4

(a) $\sin 2\theta = \sqrt{2}\cos\theta \qquad 0 \le \theta \le 2\pi$

$\sqrt{2}\sin\theta\cos\theta = \sqrt{2}\cos\theta$

$2\sin\theta\cos\theta - \sqrt{2}\cos\theta = 0 \quad (\div\sqrt{2})$

$\frac{2}{\sqrt{2}}\sin\theta\cos\theta - \frac{\sqrt{2}}{\sqrt{2}}\cos\theta = 0$

$\sqrt{2}\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(\sqrt{2}\sin\theta - 1) = 0$

$\cos\theta = 0 \quad$ or $\quad \sqrt{2}\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

(b)(i) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos 2x\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right]$

$= \frac{\pi}{4}$

(ii) $V = \pi\int_0^{\frac{\pi}{2}} (\sin x + 1)^2\,dx = \pi\int_0^{\frac{\pi}{2}} (\sin^2 x + 2\sin x + 1)\,dx$

$= \pi\left(\frac{\pi}{4}\right) + \pi\int_0^{\frac{\pi}{2}} (2\sin x + 1)\,dx$

$= \frac{\pi^2}{4} + \pi\left[-2\cos x + x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi^2}{4} + \pi\left[\left(0 + \frac{\pi}{2}\right) - (-2 + 0)\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi^2}{4} + \frac{\pi^2}{2} + 2\pi$

$= \left(\frac{3\pi^2}{4} + 2\pi\right)$ unit^3

(iii) $\sec x = \frac{1+t^2}{1-t^2}$ and $\tan x = \frac{2t}{1-t^2}$

Prove: $\sec^2 x = 1 + \tan^2 x$

$\left(\frac{1+t^2}{1-t^2}\right)^2 = 1 + \left(\frac{2t}{1-t^2}\right)^2$

RHS $=$

$1 + \frac{4t^2}{(1-t^2)^2}$

$= \frac{(1-t^2)^2 + 4t^2}{(1-t^2)^2}$

$= \frac{1 - 2t^2 + t^4 + 4t^2}{(1-t^2)^2}$

$= \frac{1 - 2t^2 + t^4 + 4t^2}{(1-t^2)^2}$

$= \frac{t^4 + 2t^2 + 1}{(1-t^2)^2}$

$= \frac{(1+t^2)^2}{(1-t^2)^2}$

$= \left(\frac{1+t^2}{1-t^2}\right)^2$

$= LHS$

Q5(a)

$\sin^{-1}\left(\cos\frac{3\pi}{4}\right)+\cos^{-1}\left(\cos\frac{3\pi}{4}\right)$

$=\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ ✓

$=-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\left[\pi-\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$

$=-\frac{\pi}{4}+\left[\pi-\frac{\pi}{4}\right]$

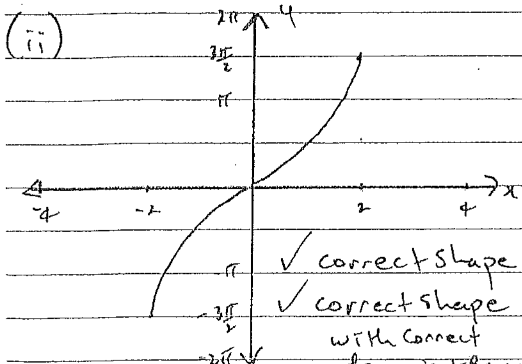
$=\frac{\pi}{2}$ ✓

(b)

(i) Domain $=\{x: -2\le x\le 2\}$ ✓

Range $=\left\{y: -\frac{3\pi}{2}\le y\le \frac{3\pi}{2}\right\}$ ✓

(ii)



✓ correct shape

✓ correct shape with correct domain & range

2 marks

(c)

(i)

OB is perpendicular to AC

(a radius meets midpoint of a chord at right angles) ✓

OE is perpendicular to ED

(a tangent meets a radius at right angles)

Since $\angle DBO+\angle OED=180°$

$\therefore$ OEDC is cyclic ✓

(ii)

$\angle EBD=\angle DOE$ ✓

(angles at the circumference standing on the same arc (ED) are equal)

(d)

$\sqrt{3}\sin x-\cos x=2\left[\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x\right]$

$=2\sin(x-\alpha)$

$=2\sin\left(x-\frac{\pi}{6}\right)$ ✓

$\tan\alpha=\frac{1}{\sqrt{3}}$ ✓

$\therefore A=2$ and $\alpha=\frac{\pi}{6}$

$\sqrt{3}\sin x-\cos x-a=0$ ✓

$\sqrt{3}\sin x-\cos x=a$

$2\sin\left(x-\frac{\pi}{6}\right)=a$

As $-2\le 2\sin\left(x-\frac{\pi}{6}\right)\le 2$

$\therefore$ for no solutions

$a<-2$ or $a>2$ ✓

Q6(a)

(i)

$$\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$$

$$= \left(\cos^{-1}x\right)\times 1 + x \times \left(\frac{-1}{\sqrt{1-x^2}}\right) - \left[\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \times -2x\right]$$

$$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$$

$$= \cos^{-1}x$$

(ii)

$$\int_0^1 \cos^{-1}x + 1\, dx = \left[x\cos^{-1}x - \sqrt{1-x^2} + x\right]_0^1$$

$$= \left[(0-0+1)-(0-1+0)\right]$$

$$= 2$$

(b)

(i) $x = 6t$ $\qquad y = 3t^2$

$\frac{dx}{dt} = 6$ $\qquad \frac{dy}{dt} = 6t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 6t \times \frac{1}{6}$$

$$= t$$

Hence $M = p$ $\quad (6p, 3p^2)$

$$y - 3p^2 = p(x - 6p)$$

$$y - 3p^2 = px - 6p^2$$

$$y = px - 3p^2$$

(ii) For the tangent at Q

Substitute $t = 1-p$.

Hence the equation is:

$$y = (1-p)x - 3(1-p)^2$$

For T Solve Simultaneously:

$y = px - 3p^2$ and $y = (1-p)x - 3(1-p)^2$

$$px - 3p^2 = (1-p)x - 3(1-p)^2$$

$$px - (1-p)x = 3p^2 - 3(1-p)^2$$

$$px - x + px = 3\left[p^2 - (1-p)^2\right]$$

$$(2p-1)x = 3\left[p^2 - (1-p)^2\right]$$

$$(2p-1)x = 3\left[(p-(1-p))(p+(1-p))\right]$$

$$(2p-1)x = 3(2p-1)$$

$x = 3$ $\qquad p \neq \frac{1}{2}$

(Since P & Q are distinct)

Therefore $T = (3, 3p - 3p^2)$

(iii)

$x = 6p = 3$ $\quad \therefore p = \frac{1}{2}$

$$y = 3p - 3p^2$$

$$= 3\left(\frac{1}{2}\right) - 3\left(\frac{1}{2}\right)^2$$

$$= \frac{3}{2} - \frac{3}{4}$$

$$= \frac{3}{4}$$

The intersection of the tangents must lie outside the parabola

Therefore the locus of T is given by

$x = 3$ $\quad y < \frac{3}{4}$

27

(a) $\frac{dP}{dt} = k(P - 10000)$ —— ①

(i) $P = 10000 + P_0 e^{kt}$ —— ②

Sub ② into ①:

LHS of ① $= \frac{dP}{dt}$

$= \frac{d}{dt}\left(10000 + P_0 e^{kt}\right)$

$= kP_0 e^{kt}$

$= k(P - 10000)$ (from ②)

$=$ RHS of ①

$\Rightarrow P = 10000 + P_0 e^{kt}$ is a solution of the differential equation.

(ii) when $t=0$ $P = 15000$

$\therefore 15000 = 10000 + P_0 e^0$

$\therefore P_0 = 5000$

$\therefore P = 10000 + 5000e^{kt}$

when $t=6$, $P = 25000$

$\therefore 25000 = 10000 + 5000e^{6k}$

$\therefore k = \frac{1}{6}\ln 3$

(iii)

Now $P = 10000 + 5000e^{(\frac{1}{6}\ln 3)t}$

when $t = 12$, $P = ?$

$\therefore P = 10000 + 5000e^{(\frac{1}{6}\ln 3).12}$

$= 10000 + 5000e^{\ln 9}$

$= 1000 + 45\,000$

$= 55000$

$\therefore$ After 1 year there are 55000 tsetse flies.

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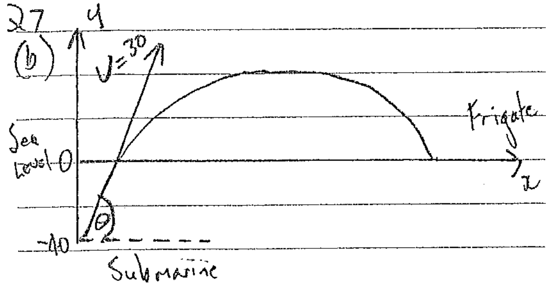
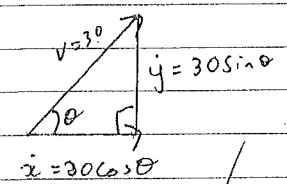
(b)





(i) Initially $\ddot{x} = 0$, $\ddot{y} = -10$

$\therefore \dot{x} = C_1$, $\dot{y} = -10t + C_2$

when $t=0$ $\dot{x} = 30\cos\theta$, $\dot{y} = 30\sin\theta$

$\therefore 30\cos\theta = C_1$, $30\sin\theta = C_2$

$\therefore \dot{x} = 30\cos\theta$, $\dot{y} = -10t + 30\sin\theta$

$\therefore x = 30t\cos\theta + C_3$, $y = -5t^2 + 30t\sin\theta + C_4$

when $t=0$, $x=0$, $y=-40$

$\therefore C_3 = 0$, $C_4 = -40$

$\therefore x = 30t\cos\theta$, $y = 30t\sin\theta - 5t^2 - 40$ are the parametric equations of motion

(ii) Maximum height occurs when $\dot{y} = 0$ after $\frac{3\sqrt{3}}{2}$ s.

Now as $\dot{y} = -10t + 30\sin\theta$

$\therefore 0 = -\frac{30\sqrt{3}}{2} + 30\sin\theta$

$\therefore \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \theta = \frac{\pi}{3}$

$\therefore$ angle of projection, $\theta = \frac{\pi}{3}$ or 60°

(iii) The missile strikes the frigate when $x = 3000$

Now as $x = 30t\cos\theta$

$\therefore 3000 = 30 \times t \times \cos\frac{\pi}{3}$

$\therefore t = \frac{3000}{30 \times \frac{1}{2}}$

$\therefore t = 200$

$\therefore$ Missile strikes the frigate after ~~2000~~ 200 seconds