NSW INDEPENDENT SCHOOLS

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2000

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt **ALL** questions.
- **ALL** questions are of equal value.
- Write your Student Name / Number on every page of the question paper and your answer sheets.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board approved calculators may be used.
- The answers to the seven questions are to be handed in separately, clearly marked Question 1, Question 2 etc.
- ***This question paper must not be removed from the examination room.***

STUDENT NUMBER / NAME: ....

Student Name / Number ....

Question 1 (Start a new page) Marks

a. Show that the exact value of $\cos 15°$ is $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ 2

b. For what values of x $(x \neq 0)$ does the geometric series 4

$1 + \dfrac{2x}{x+1} + \left(\dfrac{2x}{x+1}\right)^2 + \ldots$ have a limiting sum?

c. Use the table of standard integrals to find $\displaystyle\int_0^4 \frac{1}{\sqrt{9+x^2}}\,dx$ 2

d. Six men and five women are arranged at random in a row so that each woman is between two men. 4

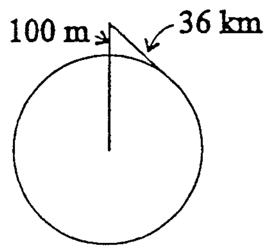
i. How many such arrangements are possible?

ii. What is the probability that two specified men, A and B, sit at the ends of the row?

Question 2 (Start a new page)

a. From a cliff 100 metres high, the straight line distance to the horizon is 36 kilometres. 3

Calculate the radius of the earth.



b. A spherical bubble is expanding so that its volume is increasing at a constant rate of 50 mm^3 per second. 3

What is the rate of increase of its surface area when the radius is 8 mm?

c. Show that $\tan^{-1}\left(\dfrac{1}{4}\right) + \tan^{-1}\left(\dfrac{3}{5}\right) = \dfrac{\pi}{4}$ 2

d. In the expansion of $(\sqrt[5]{x} + \sqrt[3]{x})^9$, find the term(s) where the power of x is an integer. 4

Student Name / Number ....

Question 3 (Start a new page)

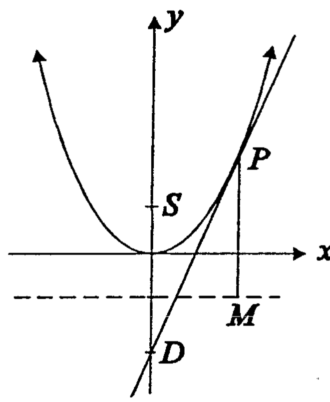
Marks

a. i. Show that $\dfrac{\sec^2 x}{\tan x} = \dfrac{1}{\sin x \cos x}$ **4**

ii. Use the substitution $u = \tan x$ to show that $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sin x \cos x} = \log_e 3$

b. The point P($2ap$, ap^2) lies on the parabola defined by $x^2 = 4ay$. **4**

The line PM is drawn parallel to the axis of the parabola to meet the directrix in M. S is the focus of the parabola.



i. State why SP is equal to PM.

ii. The tangent at P meets the y-axis at D. Find the coordinates of D.

iii. Show that SPMD is a rhombus.

c. Use the Principle of Mathematical Induction to prove that, for all positive integers, n, **4**

$$\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} = \frac{n}{4n+1}$$

Question 4 (Start a new page)

a. The point C(–6, 1) divides the interval AB externally in the ration 3:1. If A has coordinates (0, 4), find the coordinates of B **2**

b. i. Express $4\sin\theta - 3\cos\theta$ in the form $A\sin(\theta - \alpha)$, $A > 0$, $0 < \alpha < 90^\circ$ **4**

ii. Find all solutions of $4\sin\theta - 3\cos\theta = 1$ for $0 \le \theta \le 360^\circ$

Question 4 is continued on the next page.

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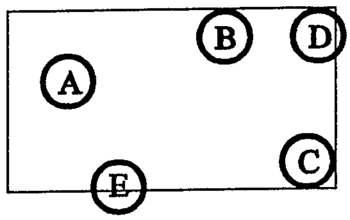
Question 4 (continued) — Marks

c. At the Easter Show, there is a new game in which a small hoop of radius 100 mm is to be thrown onto a rectangular table 3 metres by 2 metres. If the hoop lands so that no part of it extends past the edge of the table, a prize is won. If part of the hoop extends over the edge of the table, no prize is won. (In the diagram, hoops A, B and C would win prizes but hoops D and E would not) **2**

Assuming that the hoop lands on the table, what is the probability of winning a prize with one throw?



d. The quadratic equation $x^2 + 6x + c = 0$ has two real roots. These roots have opposite signs and differ by $2n$, where $n \neq 0$. **4**

i. Show that $n^2 = 9 - c$

ii. Find the set of all possible values of n.

Question 5 (Start a new page)

a. A factory machining car parts finds that 98% are machined correctly. From a sample of 40 car parts, calculate to 3 decimal places the probability that **4**

i. exactly 38 of the parts are correctly machined.

ii. less than three parts are incorrectly machined.

b. i. Show that the equation $\log_e x + x^2 - 4x = 0$ has a root between $x = 3$ and $x = 4$. **4**

ii. Using $x = 3.5$ as a first approximation, find a better approximation using Newton's method once.

c. i. Show that $\cos 4x = 8(\cos^4 x - \cos^2 x) + 1$ **4**

ii. Hence or otherwise solve $\cos^2 x - \cos^4 x = \frac{1}{16}, \quad 0 \le x \le \frac{\pi}{2}$

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Question 6 (Start a new page) **Marks**

a. An F18 jet is climbing at a speed of 504 kilometres per hour at an angle of $30°$ to the horizontal. When the jet is 600 metres above the ocean, it drops a flare from a wing. The only force acting on the flare is gravity. **5**

Take $g = 10$ ms^{-2}.

i. Find the time taken for the flare to hit the ocean.

ii. Calculate the maximum height reached by the flare.

iii. What is the horizontal distance travelled by the flare?

b. The velocity, v ms^{-1}, of a particle moving in Simple Harmonic Motion along the x-axis is given by the expression **7**

$$v^2 = 28 + 24x - 4x^2$$

i. Between what two points is the particle oscillating?

ii. What is the amplitude of the motion?

iii. Find the acceleration of the particle in terms of x.

iv. Find the period of the oscillation.

v. If the particle starts from the point furthest to the right, draw a graph of the displacement of the particle against time over two complete periods.

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Question 7 (Start a new page) **Marks**

a. The arc of the curve $y = \frac{1}{2}(1 + \sin x)$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is rotated about the x-axis. **4**

Find the volume of the solid formed.

b. i. Use the substitution $u = \cos x$ to evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$, leaving your answer as a fraction. **8**

ii. Given $y = \sin^{2n-1} x \cos x$, where n is a positive integer, find an expression for $\frac{dy}{dx}$ in terms of powers of $\sin x$

iii. Hence show that $\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{2n-1}{2n} \int_0^{\frac{\pi}{2}} \sin^{2n-2} x \, dx$, where n is a positive integer.

iv. Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$ in terms of π

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Q1.(a) $\cos(A-B) = \cos A\cos B + \sin A \sin B$

$\cos(45-30) = \cos 45 \cos 30 + \sin 45 \sin 30$

$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

$\cos 15 = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(b) $|r| < 1$

$\left|\frac{2x}{x+1}\right| < 1$

either $\frac{2x}{x+1} < 1$

Critical points at $x=-1$ and

$\frac{2x}{x+1} = 1$

$2x = x+1 \Rightarrow x = 1$

$x<-1 \{ \quad -1<x<1 \{ \quad x>1$

$-1 \quad 0 \quad 1$

Test $x=0$: true $\therefore -1 < x < 1$

or $\frac{2x}{x+1} > -1$

Critical points at $x=-1$ and

$\frac{2x}{x+1} = -1$

$2x = -x-1 \Rightarrow x = -\frac{1}{3}$

$x<-1 \{ \quad \{ x > -\frac{1}{3}$

$-1 \quad -\frac{1}{3} \quad 0$

Test $x=0$: true $\therefore x<-1$ + $x > -\frac{1}{3}$

$\therefore$ solution is: $-\frac{1}{3} < x < 1, \ x \neq 0$

(c) $\int_0^4 \frac{1}{\sqrt{9+x^2}}\,dx$

$= \left[\ln\left(x + \sqrt{9+x^2}\right)\right]_0^4$

$= \ln\left(4+\sqrt{9+4^2}\right) - \ln\left(0+\sqrt{9+0}\right)$

$= \ln 9 - \ln 3$

$= \ln 3$

(d) (i) ${}^6P_6 \times {}^5P_5 = 86400$

(ii) Ignore the women:

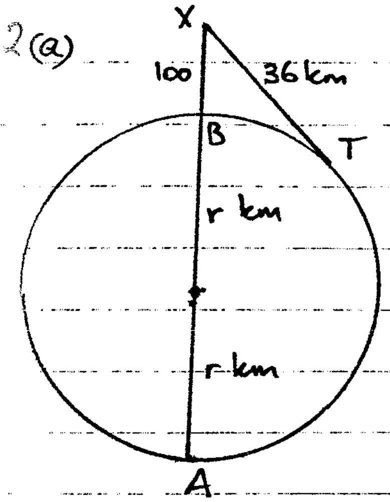
Number of permutations with A + B at the ends is ${}^2P_2 \times {}^4P_4 = 48$

$\therefore P(A + B \text{ are at the ends})$

$= \frac{48}{720}$

$= \frac{1}{15}$

2(a)



Now

$$BX \cdot AX = TX^2$$

$$\therefore \ .1 \times (2r + .1) = 36^2$$

$$2r + .1 = \frac{36^2}{.1}$$

$$r = \frac{1}{2}\left(\frac{36^2}{.1} - .1\right)$$

$$= 6479.95 \text{ km}$$

$$\sim 6480 \text{ km}$$

(b) $\dfrac{dV}{dt} = 50 \quad ; \quad r = 8$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{50}{4\pi r^2}$$

and $S = 4\pi r^2$

$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$$= 8\pi r \cdot \frac{50}{4\pi r^2}$$

$$= \frac{100}{r}$$

$\therefore$ if $r = 8$, $\dfrac{dS}{dt} = 12.5 \text{ mm}^2/\text{s}$

(c) Let $\theta = \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right)$

$$\tan\theta = \tan\left(\tan^{-1}\left(\tfrac{1}{4}\right) + \tan^{-1}\left(\tfrac{3}{5}\right)\right)$$

$$= \frac{\tan\tan^{-1}\left(\frac{1}{4}\right) + \tan\tan^{-1}\left(\frac{3}{5}\right)}{1 - \tan\tan^{-1}\left(\frac{1}{4}\right) \times \tan\tan^{-1}\left(\frac{3}{5}\right)}$$

$$= \frac{1/4 + 3/5}{1 - 1/4 \times 3/5}$$

$$\tan\theta = 1$$

$$\therefore \theta = \pi/4$$

(d) $$\left(x^{1/5} + x^{1/3}\right)^9 = \sum_{r=0}^{9} {}^9C_r \left(x^{1/5}\right)^{9-r} \left(x^{1/3}\right)^r$$

$$= \sum_{r=0}^{9} {}^9C_r \, x^{\frac{9-r}{5}} \cdot x^{\frac{r}{3}}$$

$$= \sum_{r=0}^{9} {}^9C_r \, x^{\frac{27+2r}{15}}$$

Integer powers occur when $\dfrac{27+2r}{15}$ is an integer

This occurs when $r = 9$

$$\Rightarrow \frac{27 + 2\times 9}{15} = 3$$

$\therefore$ the term is ${}^9C_9 \, x^3$

$$= x^3$$

3a(i) $\text{LHS} = \dfrac{\sec^2 x}{\tan x}$

$= \dfrac{1/\cos^2 x}{\sin x/\cos x}$

$= \dfrac{1}{\cos x \sin x} = \text{RHS}$ //

(ii) $I = \displaystyle\int_{\pi/6}^{\pi/3} \frac{\sec^2 x \, dx}{\tan x}$

if $u = \tan x$, $du = \sec^2 x \, dx$

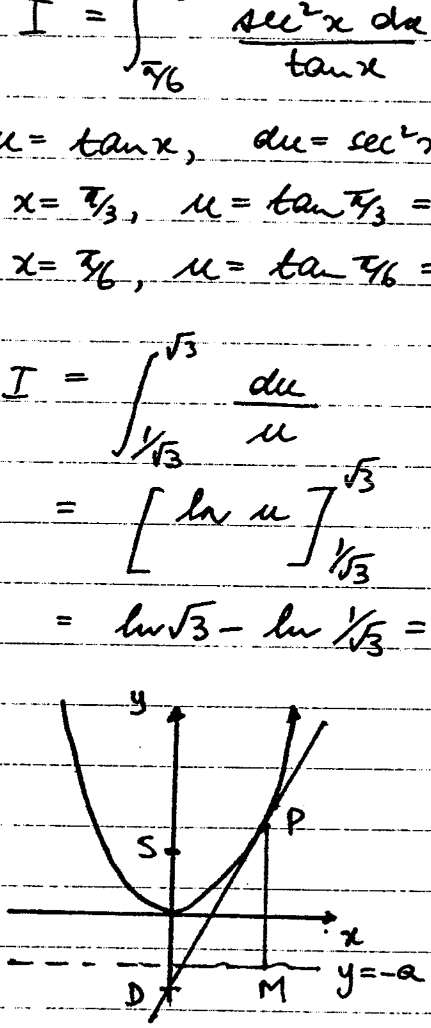
if $x = \pi/3$, $u = \tan \pi/3 = \sqrt{3}$

$x = \pi/6$, $u = \tan \pi/6 = 1/\sqrt{3}$

$\therefore I = \displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u}$

$= \Big[\ln u\Big]_{1/\sqrt{3}}^{\sqrt{3}}$

$= \ln\sqrt{3} - \ln 1/\sqrt{3} = \ln 3$



) Parabola is locus of points equidistant from focus, S, and directrix, $y=-a$

$\therefore PS = PM$

) Tangent at P: $y = px - ap^2$

At $x=0$, $y = -ap^2$

$\therefore D(0, -ap^2)$

$d_{PM} = ap^2 - -a = a(p^2+1)$

$d_{SP} = a - -ap^2 = a(1+p^2)$

. $PM = PS = SD$ and $SD \parallel PM$

so SPMD is a rhombus.

(c) $S(n): \displaystyle\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} = \frac{n}{4n+1}$

$S(1)$: $\text{LHS} = \dfrac{1}{(4-3)(4+1)} = 1/5$

$\text{RHS} = \dfrac{1}{4+1} = 1/5 = \text{LHS}$

$\therefore S(1)$ is true

Assume $n=k$:

i.e. $S(k): \displaystyle\sum_{r=1}^{k} \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$

Prove $n=k+1$

i.e. $S(k+1): \displaystyle\sum_{r=1}^{k+1} \frac{1}{(4r-3)(4r+1)} = \frac{k+1}{4k+5}$

$\text{LHS} = \dfrac{k}{4k+1} + \dfrac{1}{(4k+1)(4k+5)}$

$= \dfrac{k(4k+5)+1}{(4k+1)(4k+5)}$

$= \dfrac{4k^2+5k+1}{(4k+1)(4k+5)}$

$= \dfrac{(4k+1)(k+1)}{(4k+1)(4k+5)}$

$= \dfrac{k+1}{4k+5} = \text{RHS}$

$\therefore$ If $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, so $S(2)$ is true, whence $S(3)$ is true and so on for all positive integer values of n.

4. (a) $A(0,4)$ & $B(x_2,y_2)$ & $k:l=-3:1$

$x=\frac{kx_2+lx_1}{k+l} \Rightarrow -6=\frac{-3x_2+1\times 0}{-3+1}$

$12=-3x_2 \Rightarrow x_2=-4$

$y=\frac{ky_2+ly_1}{k+l} \Rightarrow 1=\frac{-3y_2+1\times 4}{-3+1}$

$-2=-3y_2+4 \Rightarrow y_2=2$

$\therefore B(-4,2)$

(b)(i) $A\sin(\theta-\alpha)=A\sin\theta\cos\alpha-A\cos\theta\sin\alpha$

$\therefore A\cos\alpha=4$

$A\sin\alpha=3$

whence $\alpha=\tan^{-1}(3/4)$ & $A=5$

$\therefore 4\sin\theta-3\cos\theta=5\sin(\theta-\alpha)$

where $\alpha=\tan^{-1}(3/4)$

(ii) $5\sin(\theta-\alpha)=1$

$\sin(\theta-\alpha)=1/5$

$\theta-\alpha=11°32',\ 168°28'$

$\therefore \theta=11°32'+36°52'=48°24'$

and $\theta=168°28'+36°52'=205°20'$

(c) Landing area $=3000\times 2000$

Area where hoop does not protrude

$=2800\times 1800$

$\therefore P(\text{wins prize})=\frac{2800\times 1800}{3000\times 2000}$

$=0.84$

(d) Sum of roots is -6

Product of roots is c

(i) Assume the roots are α, $\alpha+2n$

Then $\alpha+\alpha+2n=-6$

$\Rightarrow \alpha=-n-3$

But $\alpha\times(\alpha+2n)=c$

$(-n-3)\times(-n-3+2n)=c$

$-n^2+9=c$

so $n^2=9-c$

(ii) Since the roots are opposite in sign, the product must be negative

$\therefore c<0$

but $c=9-n^2$ (above)

$\therefore 9-n^2<0$

$n^2>9$

& $n<-3,\ n>3$

15. (a) Let p = probability of correctly machined part $= 0.98$

q = prob. of incorrectly machined part $= 0.02$

X = no. of correctly machined parts.

$P(X=r) = {}^{40}C_r\,(0.98)^r\,(0.02)^{40-r}$

(i) $P(X=38) = {}^{40}C_{38}\,(0.98)^{38}\,(0.02)^2$
$= 0.145$

(ii) $P(X \geq 38) = P(X=38) + P(X=39) + P(X=40)$
$= 0.1448 + {}^{40}C_{39}\,(.98)^{39}\,.02 + {}^{40}C_{40}\,(.98)^{40}$
$= 0.954$

b) let $f(x) = \log_e x + x^2 - 4x$

(i) $f(3) = \ln 3 + 9 - 12 < 0$
$f(4) = \ln 4 + 16 - 16 > 0$
$\therefore$ root exists between $x = 3$, $x = 4$

(ii) $f'(x) = \frac{1}{x} + 2x - 4$

$\therefore x_1 = x - \dfrac{\ln x + x^2 - 4x}{1/x + 2x - 4}$

$= 3.5 - \dfrac{\ln 3.5 + 3.5^2 - 4\times 3.5}{1/3.5 + 2\times 3.5 - 4}$

$= 3.6513$

$\therefore$ Better approximation is $x = 3.65$

c) (i) $\cos 4x = \cos 2 \times 2x$
$= 2\cos^2 2x - 1$
$= 2\,[2\cos^2 x - 1]^2 - 1$
$= 2\,(4\cos^4 x - 4\cos^2 x + 1) - 1$
$= 8\cos^4 x - 8\cos^2 x + 2 - 1$
$= 8(\cos^4 x - \cos^2 x) + 1$

(ii) $\cos^2 x - \cos^4 x = \frac{1}{16}$

$\therefore \cos 4x = 8 \cdot -\frac{1}{16} + 1$
$= \frac{1}{2}$

$4x = \frac{\pi}{3}$ or $4x = \frac{5\pi}{3}$

$\therefore x = \frac{\pi}{12}$ $\quad\therefore x = \frac{5\pi}{12}$

6.(a) $\ddot{x} = 0$ $\qquad \ddot{y} = -10$

$\dot{x} = V\cos\alpha$ $\qquad \dot{y} = -10t + V\sin\alpha$

$x = Vt\cos\alpha$; $y = -5t^2 + Vt\sin\alpha + 600$

Also 504 km/hr $= 140$ m/s

$\therefore \dot{x} = 70\sqrt{3}$ $\qquad \dot{y} = -10t + 70$

$x = 70\sqrt{3}.t$ $\qquad y = -5t^2 + 70t + 600$

(i) $y = 0 \Rightarrow -5t^2 + 70t + 600 = 0$

$-5(t^2 - 14t - 120) = 0$

$-5(t-20)(t+6) = 0$

$\Rightarrow t = 20$ seconds

(ii) $\dot{y} = 0 \Rightarrow -10t + 70 = 0$

$t = 7$

At $t = 7$, $y = -5 \times 7^2 + 70 \times 7 + 600$

$= 845$ metres

(iii) At $t = 20$, $x = 70\sqrt{3} \times 20$

$= 2424 \cdot 87$

$\doteqdot 2 \cdot 425$ kilometres

(b) (i) $v^2 = 28 + 24x - 4x^2$

if $v = 0 \Rightarrow 28 + 24x - 4x^2 = 0$

$4(7 + 6x - x^2) = 0$

$-4(x^2 - 6x - 7) = 0$

$-4(x-7)(x+1) = 0$

$x = -1$ and $x = 7$

$\therefore$ Oscillates between $x = -1$, $x = 7$

(ii) midpoint of motion is $x = 3$

$\therefore$ amplitude is 4m

(iii) $v^2 = 28 + 24x - 4x^2$

$\frac{1}{2}v^2 = 14 + 12x - 2x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 12 - 4x$

$\therefore a = 12 - 4x$

(iv) $a = -4(x-3)$

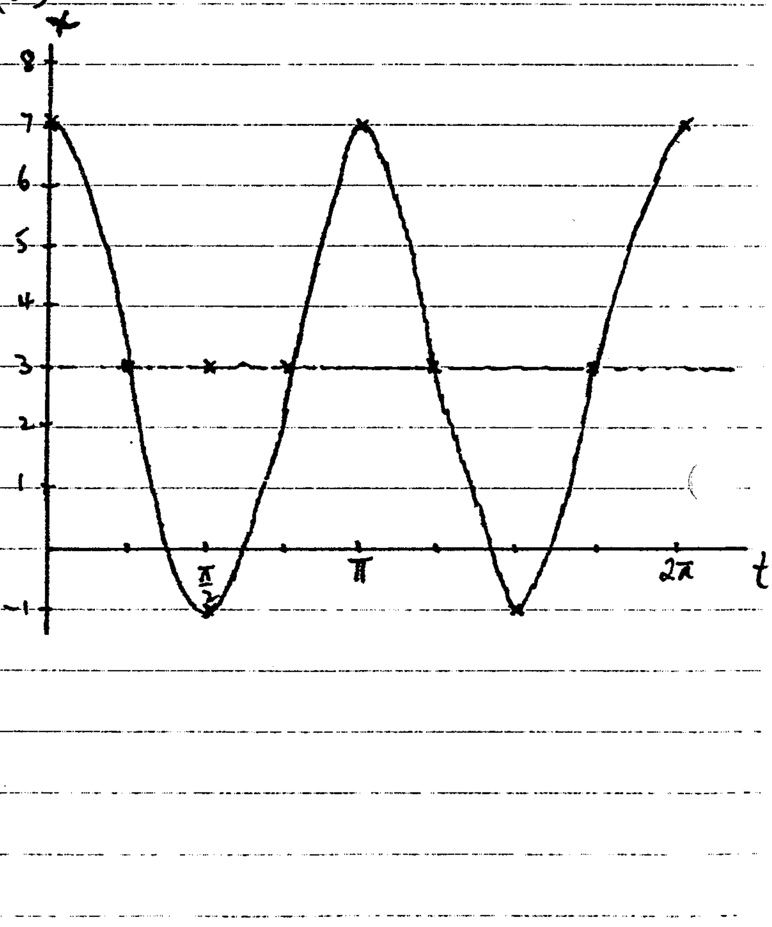
$= -2^2(x-3)$

$\therefore n = 2$

Period, $T = \frac{2\pi}{n}$

$= \pi$ seconds

(v)



7.(a) $V = \pi \int_a^b y^2\,dx$

$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}(1 + 2\sin x + \sin^2 x)\,dx$

$= \frac{\pi}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + 2\sin x + \sin^2 x\,dx$

$= \frac{\pi}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + 2\sin x + \frac{1}{2}(1 - \cos 2x)\,dx$

$= \frac{\pi}{4}\left[x - 2\cos x + \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_{-\pi/2}^{\pi/2}$

$= \pi/4 \left[\left\{\frac{\pi}{2} - 0 + \frac{1}{2}\left(\frac{\pi}{2} - 0\right)\right\}\right.$

$\left. - \left\{-\frac{\pi}{2} - 0 + \frac{1}{2}\left(-\frac{\pi}{2} - 0\right)\right\}\right]$

$= \frac{\pi}{4}\left(\frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4}\right)$

$= \frac{3\pi^2}{8}$

b)(i) $u = \cos x \quad du = -\sin x.dx$

if $x = 0, u = 1; \quad x = \frac{\pi}{2}, u = 0$

$\therefore I = \int_0^{\pi/2} \sin^5 x.dx$

$= \int_1^0 (1 - u^2)^2 . -du$

$= \int_0^1 1 - 2u^2 + u^4 .du$

$= \left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right]_0^1$

$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$

(ii) $\frac{dy}{dx} = (2n-1)\sin^{2n-2}x.\cos x.\cos x$
$+ \sin^{2n-1}. -\sin x$

$= (2n-1)\sin^{2n-2}x(1 - \sin^2 x) + -\sin^{2n}x$

$= (2n-1)\sin^{2n-2}x - (2n-1)\sin^{2n}x - \sin^{2n}x$

$= (2n-1)\sin^{2n-2}x - 2n\sin^{2n}x$

(iii) $\therefore \int (2n-1)\sin^{2n-2}x - 2n\sin^{2n}x.dx$

$= \sin^{2n-1}x\cos x + C$

$\& \int_0^{\frac{\pi}{2}} (2n-1)\sin^{2n-2}x - 2n\sin^{2n}x\,dx$

$= \left[\sin^{2n-1}x\cos x\right]_0^{\pi/2} = 0$

$\therefore \int_0^{\pi/2} 2n\sin^{2n}x\,dx = \int_0^{\pi/2}(2n-1)\sin^{2n-2}x\,dx$

$\int_0^{\pi/2}\sin^{2n}x\,dx = \frac{2n-1}{2n}\int_0^{\pi/2}\sin^{2n-2}x\,dx$

(iv) $\int_0^{\pi/2}\sin^6 x\,dx = \frac{5}{6}\int_0^{\pi/2}\sin^4 x\,dx$

$\int_0^{\pi/2}\sin^4 x\,dx = \frac{3}{4}\int_0^{\pi/2}\sin^2 x\,dx$

$\int_0^{\pi/2}\sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2}dx = \frac{1}{2}\left[x\right]_0^{\pi/2}$

$= \pi/4$

$\therefore \int_0^{\pi/2}\sin^4 x\,dx = 3/4 . \pi/4 = 3\pi/16$

$\& \int_0^{\pi/2}\sin^6 x\,dx = \frac{5}{6} \times \frac{3\pi}{16} = \frac{15\pi}{96}$

$= \frac{5\pi}{32}$