Neap:

HSC Trial Examination 2008

Mathematics Extension 2

This paper must be kept under strict security and may only be used on or after the morning of Monday 11 August, 2008 as specified in the Neap Examination Timetable.

General instructions

Reading time – 5 minutes

Working time – 3 hours

Write using black or blue pen

Board-approved calculators may be used

A table of standard integrals is provided at the back of this paper

All necessary working should be shown in every question

Total marks – 120

Attempt Questions 1–8

All questions are of equal value

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2008 HSC Mathematics Extension 2 Examination.

Total marks 120
Attempt Questions 1–8
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find $\int \frac{x^2}{1+x^2}dx$. **2**

(ii) Use integration by parts to evaluate $\int_0^1 2x\tan^{-1}x\,dx$. **3**

(b) Complete the square to find $\int \frac{dx}{\sqrt{x^2-6x+7}}$. **2**

(c) (i) Find constants a and b such that $\frac{20x^2-4x+30}{(x^2+3)(5x-2)} \equiv \frac{ax}{x^2+3} + \frac{b}{5x-2}$. **2**

(ii) Hence show that $\int_0^1 \frac{20x^2-4x+30}{(x^2+3)(5x-2)}dx = \ln 3$. **2**

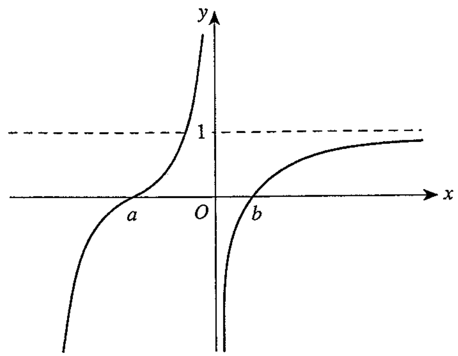
(d) Using the substitution $t = \tan\frac{\theta}{2}$, find the exact value of $\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+4\cos\theta}$. **4**

End of Question 1

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows the graph of $y = f(x)$. On separate axes, showing the x-intercepts a and b and the y-intercept 1 as indicated on the diagram, sketch graphs of the following.

(i) $y = f(|x|)$ — 1

(ii) $y = \dfrac{1}{f(x)}$ — 3

(iii) $y = e^{f(x)}$ — 2

(b) (i) By using polynomial division, or otherwise, express $\dfrac{x^4+1}{x^2+1}$ in the form $x^2 + m + \dfrac{n}{x^2+1}$, where m and n are integers. — 1

(ii) Hence describe the behaviour of $\dfrac{x^4+1}{x^2+1}$ as $x \to \pm\infty$. — 1

(iii) **Without using calculus**, and given that $(0, 1)$ is a **maximum** turning point, sketch the graph of $\dfrac{x^4+1}{x^2+1}$ on a number plane, showing any intercepts and asymptotes. — 2

(c) The roots of the polynomial equation $x^3 - 2x^2 + 3x + 1 = 0$ are α, β, and γ.

(i) Find the monic polynomial equation with roots α^2, β^2, and γ^2. — 2

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. — 3

End of Question 2

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 4 - 3i$ and $w = 2 + i$. Find the following in the form $x + iy$.

(i) $z + 3w$ — 1

(ii) $\dfrac{\bar{z}}{w}$ — 1

(b) Let $\alpha = -\sqrt{3} + i$ and $\beta = 1 - i$.

(i) Express $\bar{\alpha}$ and β in modulus–argument form. — 2

(ii) Find $\bar{\alpha}\beta$ in modulus–argument form. — 2

(iii) Hence, or otherwise, find the exact value of $\tan\dfrac{11\pi}{12}$. Express your answer in its simplest form. — 2

(c) Sketch the region defined by $1 \le |z - 2 + 3i| \le 3$ and $\dfrac{\pi}{4} < \arg(z - 2 + 3i) < \dfrac{2\pi}{3}$. — 3

(d) The complex number z is such that $z \ne i$ and $|z| = 1$, and $w = \dfrac{2+z}{i-z}$.

(i) Find an expression for z in terms of w. — 1

(ii) Explain why $|w + 1| = |w + 2i|$. — 1

(iii) Hence, or otherwise, find the locus of w and describe it geometrically. — 2

End of Question 3

Marks

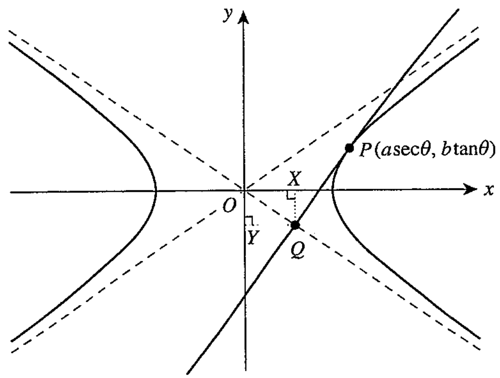
Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$. **1**

(ii) Hence or otherwise evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin 7x \cos 2x \, dx$. **2**

(b) P is a point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

A tangent drawn at P meets an asymptote of the hyperbola at Q. Perpendiculars drawn from Q meet the x- and y-axes at X and Y respectively.



(i) Show that the equation of the tangent to the hyperbola is $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$. **2**

(ii) Prove that Q has coordinates $\dfrac{a}{\sec\theta+\tan\theta}, -\dfrac{b}{\sec\theta+\tan\theta}$. **2**

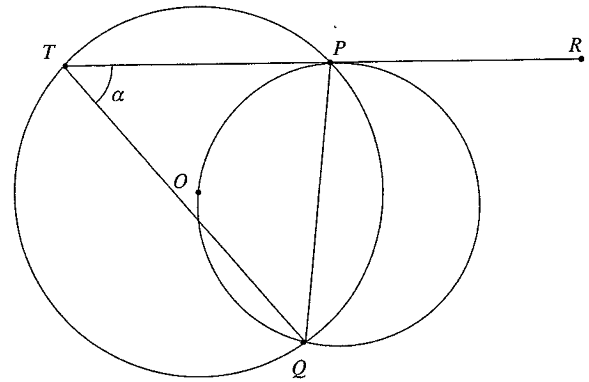
(iii) Show that P lies on the line XY. **3**

Question 4 continues on page 6

Marks

Question 4 (continued)

(c) Two circles intersect at P and Q as shown. The smaller circle passes through the centre, O, of the larger circle.
The tangent to the smaller circle RPT cuts the larger circle at T.
PQ bisects $\angle RQO$.



Let $\angle PTQ=\alpha$.

(i) Show that ΔPQT is isosceles. **3**

(ii) Show that P is the midpoint of RT. **2**

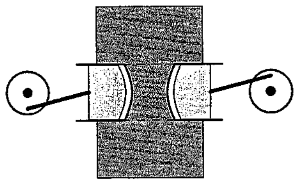
End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

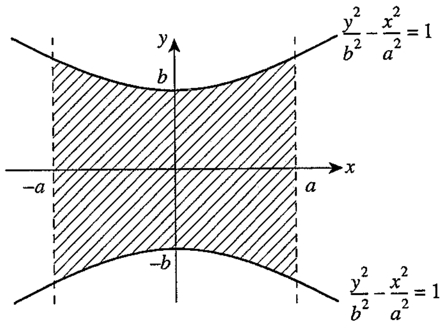
(a) In 1832, the great French mathematician Évariste Galois was shot and killed in a duel. According to some sources, his opponent was a superior marksman. Suppose that, on each shot he fired, the probability of Galois inflicting a mortal wound on his opponent was 0.4, while on each shot his opponent fired, the probability of inflicting a mortal wound on Galois was 0.6. If Galois fired first, calculate his probability of surviving the duel. **3**

(b) An engineer is working on an experimental internal combustion engine, shown in cross-section below.



The casting of the cylinder head is hyperbolic and its cross-section is described by an equation of the form $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

The region between the cross-sections of the cylinders is shaded in the diagram below.



(i) Show that the area of this region can be determined by evaluating the integral **3**

$$\frac{4b}{a}\int_0^a \sqrt{a^2 + x^2}\,dx$$

(ii) By applying the change of variables $x = a\tan\theta$ and explicitly evaluating the integral above, show that this area is equal to $2ab[\sqrt{2} + \ln(\sqrt{2} + 1)]$ square units. **3**

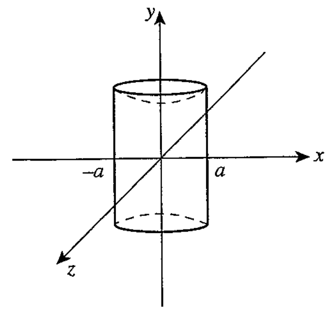
Note: $\int \sec^3\theta\, d\theta = \frac{1}{2}[\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)]$

Question 5 continues on page 8

Marks

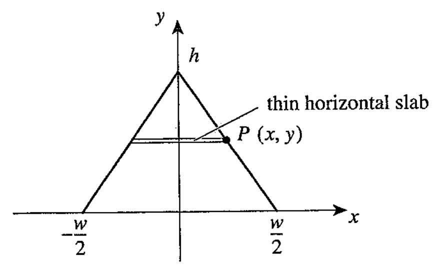
Question 5 (continued)

(iii) The volume of metal in the cylinder head between the two cylinders can be found by rotating the region bounded by graphs of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ and the lines $x = \pm a$ about the y-axis, thus generating a solid of revolution. **3**



Using the method of cylindrical shells, or otherwise, show that the volume of this solid of revolution is equal to $\frac{4\pi ba^2}{3}[2\sqrt{2} - 1]$ cubic units.

(c) The following diagram shows a vertical cross-section of a square-based pyramid. Its height is h and the length of the sides of its base is w. A thin horizontal slab centred about the y ordinate is also shown meeting the arbitrary point $P\ (x, y)$. **3**



By reference to the diagram, or by other means, derive the formula for the volume of a pyramid of arbitrary height h on a square base of side length w.

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the rectangular hyperbola $xy = c^2$, where $c > 0$.

(i) P and Q are points on the parabola with coordinates $\left(cp, \frac{c}{p}\right)$ and $\left(cq, \frac{c}{q}\right)$ respectively. **2**

Prove that the equation of the chord joining P and Q is given by $x + pqy = c(p + q)$.

(ii) The chord PQ cuts the x- and y-axes at M and N respectively. **3**

Prove that $PN = QM$.

(b) The roots of the equation $t^2 - 2t + 2 = 0$ are α and β.

Prove that $\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{\sin n\theta}{\sin^n\theta}$, where $\cot\theta = x + 1$. **4**

(c) The complex numbers u and v are such that $v = -\frac{1}{\sqrt{2}}(1 - i)u$.

(i) Plot the points A, B and C on an Argand diagram, where A and B represent u and v respectively, and C represents $u + v$. Mark in the size of $\angle AOB$ and indicate other key features on your diagram. **3**

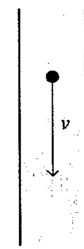
(ii) Show that $\frac{|u - v|^2}{|u + v|^2} = 3 + 2\sqrt{2}$. **3**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

Consider a particle falling through a fluid as shown in the diagram below.



When the particle's velocity is within a certain range, the resistive frictional force on the particle is proportional to its velocity. That is, the resistive force may be written as $F_{fr} = kv$, where k (kg s^{-1}) is a constant and the particle's velocity is v (m s^{-1}).

(a) If the particle falls vertically from rest, show that its terminal velocity, v_t, is given by $v_t = \frac{g}{k}$, where g (m s^{-2}) is the acceleration due to gravity. **2**

(b) If the particle is projected vertically upward into the resistive fluid with speed v_t, show that after t seconds its speed, v (m s^{-1}), and height, x (m), are given by:

(i) $v = v_t(2e^{-kt} - 1)$ **3**

(ii) $x = \frac{v_t}{k}(2 - kt - 2e^{-2kt})$ **3**

(c) Hence show that the greatest height that the particle can reach is $x_{max} = \frac{v_t}{k}(1 - \ln 2)$. **3**

(d) A ball bearing falling from rest through castor oil reaches a terminal velocity of 1.8 m s^{-1}.

(i) Determine the value of k in this situation. Take $g = 10.0$ m s^{-1}. **1**

(ii) What is the maximum height that this ball bearing could reach if it were projected vertically upward into the castor oil at speed $v = v_t$? **2**

(iii) Explain why the ball bearing is not at its maximum height halfway through its time of flight. **1**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $I_n = \int_0^{\frac{\pi}{4}} \sec^n x dx$, where $n = 0, 1, 2, \ldots$

(i) Prove that $I_{n+2} = \frac{2^{\frac{n}{2}}}{n+1} + \frac{n}{n+1} I_n$. **3**

(ii) Evaluate I_6. **2**

(iii) Prove that $\frac{d}{dx}\ln(\sec x + \tan x) = \sec x$. **1**

(iv) Evaluate I_5. **2**

(b) Two sequences of positive integers, $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$, are defined by $x_1 = 2$, $y_1 = 1$ and by equating rational and irrational parts in the equation
$x_{n+1} + \sqrt{3}y_{n+1} = (x_n + \sqrt{3}y_n)^2$ for $n = 1, 2, 3, \ldots$

(i) Prove that an equivalent definition is $x_1 = 2$, $y_1 = 1$ and by equating rational and irrational parts in the equation $x_{n+1} - \sqrt{3}y_{n+1} = (x_n - \sqrt{3}y_n)^2$ for $n = 1, 2, 3, \ldots$ **1**

(ii) Prove by induction that $x_n^2 - 3y_n^2 = 1$, for all positive integers, n. **3**

(iii) Prove that $\frac{x_n}{y_n}$ and $\frac{3y_n}{x_n}$ tend to the same limit from above and below, respectively, and find the value of the limit. **3**

End of paper

Neap:

HSC Trial Examination 2008

Mathematics Extension 2

Solutions and marking guidelines

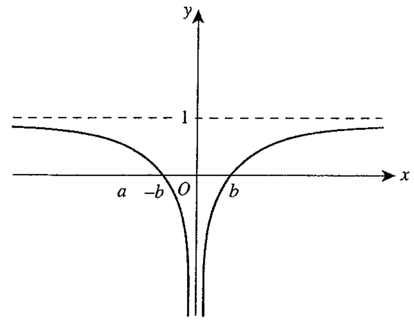
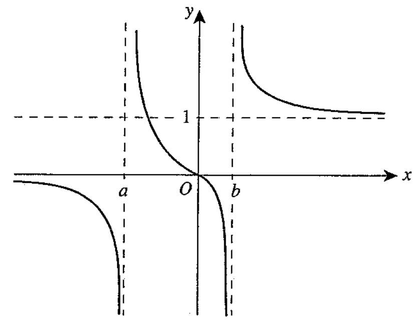
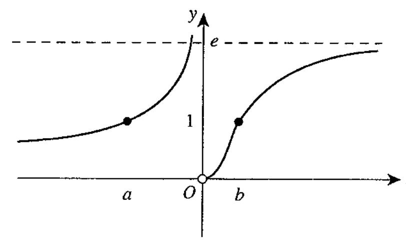
Question 1

	Sample answer	Syllabus outcomes and marking guide
(a) (i)	$\int \frac{x^2}{1+x^2}dx = \int \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}dx$ $= \int\left(1 - \frac{1}{1+x^2}\right)dx$ $= x - \tan^{-1}x + c$	E8 • Finds $\int\left(1 - \frac{1}{1+x}\right)dx$ and reaches the correct answer ... 2 • Finds $\int\left(1 - \frac{1}{1+x}\right)dx$ OR • States the correct answer ... 1
(ii)	$\int_0^1 2x\tan^{-1}x\,dx = [x^2\tan^{-1}x]_0^1 - \int_0^1 \frac{x^2}{1+x^2}dx$ $= \tan^{-1}(1) - [x - \tan^{-1}x]_0^1$ (from (a)) $= \frac{\pi}{4} - \left(1 - \frac{\pi}{4}\right)$ $= \frac{\pi}{2} - 1$	E8 • Correct use of integration by parts technique AND • Correct application of (a) and substitution of limits in first integral AND • Correct solution ... 3 • Any two of the above ... 2 • Any one of the above ... 1
(b)	$\int \frac{dx}{\sqrt{x^2 - 6x + 7}} = \int \frac{dx}{\sqrt{(x-3)^2 - 2}}$ Let $u = x - 3$ $\therefore du = dx$ $\int \frac{dx}{\sqrt{x^2 - 6x + 7}} = \int \frac{du}{\sqrt{u^2 - 2}}$ $= \ln[u + \sqrt{u^2 - 2}] + c$ $= \ln[x - 3 + \sqrt{x^2 - 6x + 7}] + c$	E8 • Correctly completes the square on the denominator AND • Correctly applies the standard integrals to get the correct answer (use of u as a substitute is not necessary) ... 2 • One of the above ... 1
(c) (i)	$20x^2 - 4x + 30 \equiv ax(5x-2) + b(x^2+3)$ $= (5a+b)x^2 - 2ax + 3b$ $\therefore a = 2$ and $b = 10$	E8 • Finds $a = 2$ and $b = 10$ AND • Correctly writes RHS in general form, or correctly uses two substitutions of x ... 2 • One of the above ... 1
(ii)	$\int_0^1 \frac{20x^2 - 4x + 30}{(x^2+3)(5x-2)}dx = \int_0^1 \frac{2x}{x^2+3}dx + \int_0^1 \frac{10}{5x-2}dx$ $= [\ln(x^2+3)]_0^1 + [2\ln\|5x-2\|]_0^1$ $= \ln 4 - \ln 3 + 2\ln 3 - 2\ln 2$ $= \ln 3$	E8 • Finds two correct integrations AND • Correctly shows four subsequent expressions following the substitution of limits ... 2 • One of the above ... 1

Question 1 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(d) Let $t = \tan\frac{\theta}{2}$ $\therefore \cos\theta = \frac{1-t^2}{1+t^2}$ and $d\theta = \frac{2dt}{1+t^2}$ When $\theta = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$ $\theta = 0$, $t = 0$ $\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+4\cos\theta} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$ $= \int_0^1 \frac{2dt}{5(1+t^2)+4(1-t^2)}$ $= \int_0^1 \frac{2dt}{9+t^2}$ $= \left[\frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)\right]_0^1$ $= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$	E8 • Finds correct exchange of limits AND • Correctly substitutes into integrand AND • Finds $\int_0^1 \frac{2dt}{9+t^2}$ or CFPA AND • Reaches correct subsequent evaluation of integral, as long as it has not been simplified by previous errors 4
	• Any three of the above............... 3
	• Any two of the above................. 2
	• Any one of the above................. 1

Question 2

Sample answer	Syllabus outcomes and marking guide
(a) (i) 	E6 • Finds correct reflection of graph on the RHS of the y-axis, in the y-axis 1
(ii) 	E6 • Correctly shows all three sections of the curve, with correct asymptotic behaviour 3
	• Correctly shows two sections of the curve, with correct asymptotic behaviour 2
	• Correctly shows one section of the curve, with correct asymptotic behaviour 1
(iii) 	E6 • Correctly sketches both sections of the curve and clearly indicates at least one of $(a, 1)$ or $(b, 1)$....................... 2
	• Either correctly sketches both sections of the curve OR clearly indicates at least one of $(a, 1)$ or $(b, 1)$)..................... 1

Question 2 **(Continued)**

Sample answer	Syllabus outcomes and marking guide

(b) (i)

$$\begin{array}{r|l} & x^2 \qquad -1 \\ \hline x^2+1 & x^4 \qquad\quad +1 \\ & x^4 \; +x^2 \\ \hline & \quad -x^2 \\ & \quad -x^2 \quad -1 \\ \hline & \qquad\qquad 2 \end{array}$$

$$\therefore \frac{x^4+1}{x^2+1} \equiv x^2 - 1 + \frac{2}{x^2+1}$$

OR, alternatively

$$\frac{x^4+1}{x^2+1}$$
$$= \frac{x^4-1+2}{x^2+1}$$
$$= \frac{(x^2-1)(x^2+1)}{x^2+1} + \frac{2}{x^2+1}$$
$$= x^2 - 1 + \frac{2}{x^2+1}$$

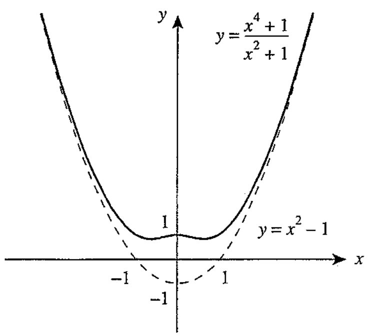
E4
- Reaches final expression of $x^2 - 1 + \frac{2}{x^2+1}$

OR
- Finds $m = 0$ and $n = 2$ 1

(ii) Since it is an even function, $\frac{x^4+1}{x^2+1}$ approaches $x^2 - 1$ from above as $x \to \pm\infty$.

E6
- States the approach of $x^2 - 1$ 1

(iii)



E6
- Shows correct asymptotic behaviour of $y = x^2 - 1$ on both sides of the y-axis and shows correct shape near (0, 1) i.e. maximum turning point at (0, 1) and two relative minima on both sides 2
- Either shows correct asymptotic behaviour of $y = x^2 - 1$ on both sides of the y-axis OR shows correct shape near (0, 1) 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

Question 2 **(Continued)**

Sample answer	Syllabus outcomes and marking guide

(c) (i) To find the polynomial with roots α^2, β^2 and γ^2, make the substitution $x \to x^{\frac{1}{2}}$ in $x^3 - 2x^2 + 3x + 1 = 0$.

$$x^{\frac{3}{2}} - 2x + 3x^{\frac{1}{2}} + 1 = 0$$
$$\left(x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\right)^2 = (2x-1)^2$$
$$x^3 + 6x^2 + 9x = 4x^2 - 4x + 1$$
$$x^3 + 2x^2 + 13x - 1 = 0$$

So $x^3 + 2x^2 + 13x - 1 = 0$ is the polynomial equation with roots α^2, β^2 and γ^2.

E4
- Substitutes $x^{\frac{1}{2}}$ or $\sqrt{x}$ into the equation for x and reaches correct answer 2
- Gives correct answer 1

(ii)
$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$= 4 - 6$$
$$= -2$$

$$\alpha^3 - 2\alpha^2 + 3\alpha + 1 = 0 \qquad (1)$$
$$\beta^3 - 2\beta^2 + 3\beta + 1 = 0 \qquad (2)$$
$$\gamma^3 - 2\gamma^2 + 3\gamma + 1 = 0 \qquad (3)$$

Adding (1), (2) and (3) gives

$$\alpha^3+\beta^3+\gamma^3 = 2(\alpha^2+\beta^2+\gamma^2) - 3(\alpha+\beta+\gamma) - 3$$
$$= -4 - 6 - 3$$
$$= -13$$

E4
- Finds $\alpha^2+\beta^2+\gamma^2 = -2$

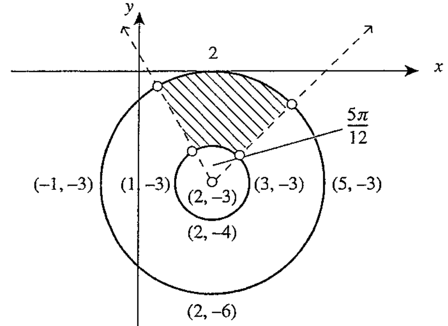
AND
- Sums the three equations to get the expression for $\alpha^3+\beta^3+\gamma^3$

AND
- Finds correct answer (or correct answer from incorrect value of $\alpha^2+\beta^2+\gamma^2$) 3
- Any two of the above................ 2
- One of the above 1

Question 3

	Sample answer	Syllabus outcomes and marking guide
(a)	(i) $z = 4 - 3i$ and $w = 2 + i$ $z + 3w = 4 - 3i + 6 + 3i$ $= 10$	E3 • Correct answer 1
	(ii) $\frac{z}{w} = \frac{4+3i}{2+i} \times \frac{2-i}{2-i}$ $= \frac{11+2i}{5}$ $= \frac{11}{5} + \frac{2i}{5}$	E3 • Correct answer 1
(b)	(i) $\bar{\alpha} = -\sqrt{3} - i$ and $\beta = 1 - i$ $\lvert\bar{\alpha}\rvert = 2$, $\arg(\bar{\alpha}) = -\frac{5\pi}{6}$, $\lvert\beta\rvert = \sqrt{2}$, $\arg(\beta) = -\frac{\pi}{4}$ $\bar{\alpha} = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$ $\beta = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$	E3 • Correct expressions for both α and β . . 2 • Correct expression for either α or β . . . 1
	(ii) $\bar{\alpha}\beta = 2\sqrt{2}\left(\cos\left(-\frac{5\pi}{6} - \frac{\pi}{4}\right) + i\sin\left(-\frac{5\pi}{6} - \frac{\pi}{4}\right)\right)$ $= 2\sqrt{2}\left(\cos\left(-\frac{13\pi}{12}\right) + i\sin\left(-\frac{13\pi}{12}\right)\right)$ $= 2\sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$	E3 • Correct solution OR • Correct solution for values of $\bar{\alpha}$ and β given in (i) . . . . 2 • Correct modulus with argument either incorrect or not expressed as principal argument 1
	(iii) $\bar{\alpha}\beta = (-\sqrt{3} - i)(1 - i)$ $= -1 - \sqrt{3} + i(\sqrt{3} - 1)$ Equating real and imaginary parts for $\bar{\alpha}\beta$ gives $\cos\frac{11\pi}{12} = \frac{-1-\sqrt{3}}{2\sqrt{2}}$ and $\sin\frac{11\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$. $\therefore \tan\frac{11\pi}{12} = -\frac{\sqrt{3}-1}{\sqrt{3}+1}$ $= -\frac{4-2\sqrt{3}}{2}$ $= \sqrt{3} - 2$	E3 • Finds $\bar{\alpha}\beta$ in the form $x + iy$, equates real and imaginary parts of the two forms of $\bar{\alpha}\beta$ and correctly evaluates $\tan\frac{11\pi}{12}$ . . . . 2 • Uses a correct process with no more than one error 1

Question 3 (Continued)

	Sample answer	Syllabus outcomes and marking guide
(c)	$1 \le \lvert z - 2 + 3i\rvert \le 3$ and $\frac{\pi}{4} < \arg(z - 2 + 3i) < \frac{2\pi}{3}$ 	E3 • Completely correct sketch, including marking open dots, dotted lines, the centre of the circles, the required region, and clearly indicating the radii of the circles and the size of the angle between the rays . . 3 • Correct sketch with no more than one of the above elements missing . . . . 2 • Either the circles or lines in the correct position with the corresponding region marked 1
(d)	(i) $w = x + iy$ and $w = \frac{2+z}{i-z}$, where $z \ne i$ and $\lvert z\rvert = 1$. $w = \frac{2+z}{i-z}$ $iw - wz = 2 + z$ $z(1 + w) = iw - 2$ $z = \frac{iw-2}{1+w}$	E3 • Correctly rearranges to find the desired expression 1
	(ii) $\lvert z\rvert = \left\lvert\frac{iw-2}{1+w}\right\rvert$ $1 = \left\lvert\frac{iw-2}{1+w}\right\rvert$ since $\lvert z\rvert = 1$ $\lvert 1 + w\rvert = \lvert iw - 2\rvert$ We then multiply the right-hand side by i as this does not change the magnitude of the real and imaginary parts. $\lvert w + 1\rvert = \lvert w + 2i\rvert$	E2, E3 • Substitutes 1 for $\lvert z\rvert$, multiplies by $-i$ to find the desired result and explains the validity of this action 1
	(iii) $\lvert w - (-1)\rvert = \lvert w - (-2i)\rvert$ Hence the locus of w is the perpendicular bisector of the interval joining $(-1, 0)$ and $(0, -2)$. $(x + 1)^2 + y^2 = x^2 + (y + 2)^2$ $x^2 + 2x + 1 + y^2 = x^2 + y^2 + 4y + 4$ $2x - 4y - 3 = 0$ is the equation of the locus.	E2, E3 • Correct equation and description of the locus . . . . 2 • Substantial progress towards solution . . . 1

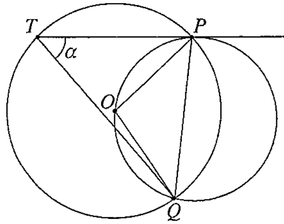
Question 4

		Sample answer	Syllabus outcomes and marking guide
(a)	(i)	$\sin(A+B)-\sin(A-B)$ $=\sin A\cos B+\cos A\sin B-[\sin A\cos B-\sin A\cos B]$ $=2\sin A\cos B$	E2 • Correct answer 1
	(ii)	$\int_0^{\frac{\pi}{4}}\sin 5x\cos 2x\,dx$ $=\frac{1}{2}\int_0^{\frac{\pi}{4}}(\sin 7x+\sin 3x)\,dx$ $=-\frac{1}{2}\left[\frac{1}{7}\cos 7x+\frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{4}}$ $=-\frac{1}{2}\left(\frac{1}{7}\times\frac{1}{\sqrt{2}}-\frac{1}{3}\times\frac{1}{\sqrt{2}}\right)$ $=\frac{2}{21\sqrt{2}}$ $=\frac{\sqrt{2}}{21}$	E2 • Correct substitution, integration and evaluation . . . . 2 • Correct substitution and integration 1
(b)	(i)	At P, $x=a\sec\theta$, $y=b\tan\theta$ $\frac{dx}{d\theta}=a\sec\theta\tan\theta$, $\frac{dy}{d\theta}=b\sec^2\theta$ $\frac{dy}{dx}=\frac{b\sec\theta}{a\tan\theta}$ For the tangent PQ, $y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$ $ay\tan\theta-ab\tan^2\theta=bx\sec\theta-ab\sec^2\theta$ $bx\sec\theta-ay\tan\theta=ab(\sec^2\theta-\tan^2\theta)$ $bx\sec\theta-ay\tan\theta=ab$ $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$	E4 • Finds $\frac{dy}{dx}$ and correctly shows the equation to be $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$ . . . . 2 • Correct proof 1

Question 4 (Continued)

		Sample answer	Syllabus outcomes and marking guide
(b)	(ii)	To find the coordinates of Q, substitute $y=-\frac{b}{a}x$ into $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$ $\frac{x\sec\theta}{a}+\frac{bx}{a}\times\frac{\tan\theta}{b}=1$ $\frac{x\sec\theta+x\tan\theta}{a}=1$ $x=\frac{a}{\sec\theta+\tan\theta}$ $y=-\frac{b}{a}\times\frac{a}{\sec\theta+\tan\theta}$ $=\frac{-b}{\sec\theta+\tan\theta}$ Q has coordinates $\left(\frac{a}{\sec\theta+\tan\theta},\frac{-b}{\sec\theta+\tan\theta}\right)$.	E3, E4 • Substitutes $y=-\frac{b}{a}x$ into the equation of the tangent to find the x-coordinate of Q and then uses this value to find the y-coordinate of Q. . . . . 2 • Correct process with one error 1
	(iii)	The coordinates of X are $\left(\frac{a}{\sec\theta+\tan\theta},0\right)$. The coordinates of Y are $\left(0,\frac{-b}{\sec\theta+\tan\theta}\right)$. The coordinates of P are $(a\sec\theta, b\tan\theta)$. $m_{XY}=\frac{b}{a}$, $m_{PY}=\frac{b\tan\theta}{a\sec\theta-\frac{a}{\sec\theta+\tan\theta}}$ $=\frac{b\tan\theta(\sec\theta+\tan\theta)}{a(\sec^2\theta+\sec\theta\tan\theta-1)}$ $=\frac{b(\tan^2\theta+\sec\theta\tan\theta)}{a(\tan^2\theta+\sec\theta\tan\theta)}$ $=\frac{b}{a}$ $\therefore m_{XY}=m_{PX}$, so P, X and Y are collinear. Hence, P lies on the line XY.	E2, E3 • Finds the correct coordinates of X and Y and the correct gradient of PX and XY and draws the correct conclusion. OR • Correctly shows that P lies on XY by another method . . . . 3 • Finds the correct coordinates of X and Y and the correct gradient of either PX or PY . . . 2 • Finds the correct coordinates of X and Y . . 1

Question 4 (Continued)

	Sample answer	Syllabus outcomes and marking guide
(c) (i)	 $\angle POQ = 2\angle PTQ = 2\alpha$ (angle at centre is twice angle at circumference) $\angle QPR = \angle POQ = 2\alpha$ (angle between tangent and chord equals angle in the alternate segment) In ΔPTQ $\angle TQP + \angle PTQ = \angle QPR$ (exterior angle equals opposite interior angles of triangle) $\angle TQP = \angle PTQ$ ΔPTQ is isosceles since two angles are equal.	E2 • Finds $\angle POQ$, $\angle QPR$ and $\angle TQP = \angle PTQ$ with full reasoning. . . . . 3 • Correctly uses two reasons but unable to correctly show the triangle as isosceles . . . . 2 • Finds one angle with correct reasoning . . 1
(ii)	ΔPOQ is isosceles as OP and OQ are radii of the same circle and $\angle OPQ = \angle OQP$. $\therefore\ \angle PQO = \frac{1}{2}(180 - 2\alpha)$ (angle sum of isosceles Δ) $= 90 - \alpha$ In ΔPQR, $\angle PRQ = \angle PQO = 90 - \alpha$ (PQ bisects $\angle RQO$) $\angle PRQ = 180 - 2\alpha - (90 - \alpha)$ (angle sum of triangle) $= 90 - \alpha = \angle PQR$ $\therefore \Delta PQR$ is isosceles with $PR = PQ$. But $PQ = PT$ because from (i) ΔPTQ is isosceles. $\therefore TP = PR$ and P is the midpoint of RT.	E2 • Showing $\angle PRQ = \angle PQR$ with full reasoning and hence explaining why P is the midpoint of RT. . . . . 2 • Finding the size of $\angle PQO$ with full reasoning 1

Question 5

	Sample answer	Syllabus outcomes and marking guide
(a)	Let P_n be the probability of Galois winning the duel by firing his n^{th} shot. Then $P_1 = \frac{2}{5}$ $P_2 = \frac{2}{5} \times \left(\frac{3}{5} \times \frac{2}{5}\right) = \frac{2}{5} \times \frac{6}{25}$ $P_3 = \frac{2}{5} \times \left(\frac{3}{5} \times \frac{2}{5}\right) \times \left(\frac{3}{5} \times \frac{2}{5}\right) = \frac{2}{5} \times \left(\frac{6}{25}\right)^2$ ... $P_n = \frac{2}{5} \times \left(\frac{3}{5} \times \frac{2}{5}\right) \times \ldots \times \left(\frac{3}{5} \times \frac{2}{5}\right) = \frac{2}{5} \times \left(\frac{6}{25}\right)^{n-1}$ The probability of his survival would then be $P_{\text{survival}} = P_1 + P_2 + \ldots + P_n + \ldots$ $= \frac{2}{5}\left[1 + \left(\frac{6}{25}\right) + \left(\frac{6}{25}\right)^2 + \ldots\right]$ $= \frac{2}{5} \times \left(\frac{1}{1 - \frac{6}{25}}\right)$ $= \frac{10}{19} = 53\%$	E4, E8 • Presents a correct series, correctly evaluates its sum and gives the correct answer as any of a fraction, decimal or percentage . . . . 3 • Presents a correct series. . . . . 2 • Attempts to present an appropriate sequence of probabilities 1
(b) (i)	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ $\Rightarrow y^2 = b^2 + \frac{x^2}{a^2}$ $y = \pm\sqrt{b^2 + \frac{x^2}{a^2}}$ $= \pm\frac{b}{a}\sqrt{a^2 + x^2}$ The area above the x-axis then resolves as $A_{\text{above}} = \int_{-a}^{a} y\,dx$ $= \frac{b}{a}\int_{-a}^{a} \sqrt{a^2 + x^2}\,dx$ $= \frac{2b}{a}\int_0^a \sqrt{a^2 + x^2}\,dx$ (by symmetry) The total area then resolves as $A_{\text{total}} = \frac{4b}{a}\int_0^a \sqrt{a^2 + x^2}\,dx$ (again by symmetry)	E4, E6 • Demonstrates the required result via a logically consistent and comprehensive analysis. . . . . 3 • Determines that the area above the x-axis is $\frac{2b}{a}\int_0^a \sqrt{a^2 + x^2}\,dx$ . . . . 2 • Correctly finds $y = \pm\frac{b}{a}\sqrt{a^2 + x^2}$ 1

Question 5 **(Continued)**

Sample answer

(b) (ii) $A = \frac{4b}{a}\int_0^a \sqrt{a^2+x^2}\,dx$

Now let $x = a\tan\theta$

$\frac{dx}{d\theta} = a\sec^2\theta$

$dx = a\sec^2\theta\, d\theta$

Limits: $x = a \rightarrow \theta = \frac{\pi}{4}$

$x = 0 \rightarrow \theta = 0$

$$A = \frac{4b}{a}\int_0^{\frac{\pi}{4}} \sqrt{a^2 + a^2\tan^2\theta}\, a\sec^2\theta\, d\theta$$

$$= 4b\int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2\theta}\, a\sec^2\theta\, d\theta$$

$$= 4ab\int_0^{\frac{\pi}{4}} \sec\theta\sec^2\theta\, d\theta$$

$$= 4ab\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$$

$$= 4ab\left[\frac{\sec\theta\tan\theta}{2} + \frac{\ln(\sec\theta + \tan\theta)}{2}\right]_0^{\frac{\pi}{4}}$$

$$= 2ab[\sqrt{2} + \ln(\sqrt{2}+1)]$$

Syllabus outcomes and marking guide

E4, E8

- Demonstrates the required result via a logically consistent and comprehensive analysis. 3
- Obtains an expression for the area as $4ab\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$ 2
- Correctly performs the substitution $x = a\tan\theta$ to obtain the integrand $\frac{4b}{a}\int_0^{\frac{\pi}{4}} \sqrt{a^2 + a^2\tan^2\theta}\, a\sec^2\theta\, d\theta$ 1

Sample answer

(iii) $V = V_{\text{above axis}} + V_{\text{below axis}}$

$= 2V_{\text{above axis}}$

$$= 2\int_0^a 2\pi xy\,dy$$

$$= 4\pi\int_0^a x\frac{b}{a}\sqrt{a^2+x^2}\,dx$$

$$= \frac{4\pi b}{a}\int_0^a x\sqrt{a^2+x^2}\,dx$$

$$= \frac{4\pi b}{a}\left[\frac{(a^2+x^2)^{\frac{3}{2}}}{3}\right]_0^a$$

$$= \frac{4\pi b}{3a}\left[(a^2+a^2)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}}\right]$$

$$= \frac{4\pi b}{3a}[2\sqrt{2}a^3 - a^3]$$

$$= \frac{4\pi ba^2}{3}[2\sqrt{2} - 1]$$

Syllabus outcomes and marking guide

E4, E7

- Demonstrates the correct result via a logically consistent and comprehensive analysis. 3
- Correctly integrates to obtain $\frac{4\pi b}{a}\left[\frac{(a^2+x^2)^{\frac{3}{2}}}{3}\right]_0^a$ 2
- Takes symmetry into account to find an integral of the form $2\int_0^a 2\pi xy\,dy$ 1

Question 5 **(Continued)**

Sample answer

(c) The side length of the pyramid's base is w.

Hence the area of the base is w^2.

The pyramid's height is h.

The area of the base of a thin horizontal slab at height y is

$$A_{\text{slab}} = \left[\frac{w}{h}(h-y)\right]^2$$

$$V = \int_0^h A_{\text{slab}}\,dy$$

$$= \int_0^h \left[\frac{w}{h}(h-y)\right]^2 dy$$

$$= \left(\frac{w}{h}\right)^2 \int_0^h (h^2 - 2hy + y^2)\,dy$$

$$= \left(\frac{w}{h}\right)^2 \left[h^2y - hy^2 + \frac{y^3}{3}\right]_0^h$$

$$= \left(\frac{w}{h}\right)^2 \left[h^3 - h^3 + \frac{h^3}{3}\right]$$

$$= \left(\frac{w}{h}\right)^2 \left[\frac{h^3}{3}\right]$$

$$= \frac{w^2h}{3}$$

Syllabus outcomes and marking guide

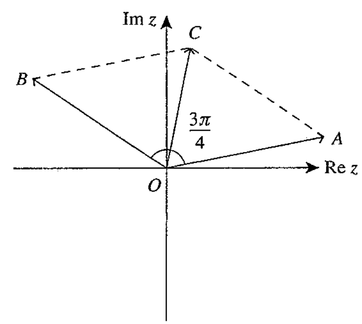
E6, E7, E9

- Demonstrates the required result via a logically consistent and comprehensive analysis. 3
- Presents an integral of the form $\int_0^h A_{\text{slab}}\,dy$ or $\int_0^h \left[\frac{w}{h}(h-y)\right]^2 dy$ and makes some attempt to integrate it 2
- Recognises that the area of a horizontal slab is given by $\left[\frac{w}{h}(h-y)\right]^2$ 1

Question 6

Sample answer	Syllabus outcomes and marking guide
(a) (i) $xy = c^2$, the coordinates of P are $\left(cp, \frac{c}{p}\right)$ and the coordinates of Q are $\left(cq, \frac{c}{q}\right)$. $m_{PQ} = \dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq}$ $= \dfrac{c(q-p)}{cpq(p-q)}$ $= -\dfrac{1}{pq}$ $y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$ $pqy - cq = -x + cp$ $x + pqy = c(p+q)$	E3, E4 • Completely correct process . . . . 2 • Finds the correct gradient 1
(ii) $x + pqy = c(p+q)$ At M, $y = 0$ and $x = c(p+q)$ At N, $x = 0$ and $y = \frac{c(p+q)}{pq}$ $PN^2 = c^2p^2 + \left(\frac{c}{p} - \frac{c(p+q)}{pq}\right)^2$ $= c^2p^2 + \left(\frac{cq - cp - cq}{pq}\right)^2$ $= c^2p^2 + \frac{c^2}{q^2}$ $QM^2 = (cq - c(p+q))^2 + \frac{c^2}{q^2}$ $= c^2p^2 + \frac{c^2}{q^2}$ $= PN^2$ $\therefore PN = QM$	E2, E4 • Finds the coordinates of both M and N, finds both distances correctly and shows that they are equal . . . . 3 • Finds the coordinates of both M and N and finds one distance correctly . . . . 2 • Finds the coordinates of both M and N . . 1

Question 6 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(b) $t^2 - 2t + 2 = 0$ and $\cot\theta = x + 1$ $t = \frac{2 \pm \sqrt{4-8}}{2}$ $t = 1 \pm i$ $\alpha = 1 + i$ and $\beta = 1 - i$ $\dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \dfrac{(x+1+i)^n - (x+1-i)^n}{1+i-(1-i)}$ $= \dfrac{(\cot\theta + i)^n - (\cot\theta - i)^n}{2i}$ $= \dfrac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{2i\sin^n\theta}$ $= \dfrac{\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta}{2i\sin^n\theta}$ $= \dfrac{\sin n\theta}{\sin^n\theta}$	E2, E3, E4 • Completely correct proof. . . . . 4 • Expresses $\dfrac{(\cot\theta+i)^n - (\cot\theta-i)^n}{2i}$ in terms of sin and cos . . . . 3 • Solves the quadratic equation to find α and β and then substitutes them into the fraction to give $\dfrac{(\cot\theta+i)^n - (\cot\theta-i)^n}{2i}$ . . . . . . 2 • Finds α and β by solving the quadratic equation 1
(c) (i) $v = \left(-\frac{1}{\sqrt{2}}\right)(1-i)u$ Let $u = x + iy$ $\|u\| = \sqrt{x^2 + y^2}$ $v = -\frac{1}{\sqrt{2}}(1-i)(x+iy)$ $\|v\| = \left\|-\frac{1}{\sqrt{2}}(1-i)\right\| \times \|u\|$ $\left\|-\frac{1}{\sqrt{2}}(1-i)\right\| = 1$ $\arg\left(-\frac{1}{\sqrt{2}}(1-i)\right) = \frac{3\pi}{4}$ $\therefore \arg v = \arg u + \frac{3\pi}{4}$ 	E2, E3, E9 • Shows that $\|u\| = \|v\|$ and $\arg v = \arg u + \frac{3\pi}{4}$ AND • Clearly marks u, v, $u+v$, the size of $\angle AOB$ and $AO = BO$ on a diagram . . . . 3 • Shows that $\|u\| = \|v\|$ and clearly marks u, v, $u+v$, and $AO = BO$ on a diagram OR • Shows that $\arg v = \arg u + \frac{3\pi}{4}$ and clearly marks u, v and $u+v$ on a diagram . . . . . 2 • Shows that $\|u\| = \|v\|$ and clearly marks u, v and $AO = BO$ on a diagram 1

Question 6 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(c) (ii) $\overline{OC}$ represents $u+v$ and $\overline{BA}$ represents $u-v$ Let $m = OA = OB = AC = BC$. $AB^2 = OB^2 + OA^2 - 2\times OA\times OB\cos\frac{3\pi}{4}$ $= 2m^2 + \sqrt{2}m^2$ $= \sqrt{2}m^2(\sqrt{2}+1)$ $\angle AOB + \angle OAC = \pi$ (co-interior angles, $AC \parallel OB$) $\angle OAC = \frac{\pi}{4}$ $OC^2 = OA^2 + AC^2 - 2\times OA\times AC\cos\frac{3\pi}{4}$ $= 2m^2 - \sqrt{2}m^2$ $= \sqrt{2}m^2(\sqrt{2}-1)$ $\frac{\lvert u-v\rvert^2}{\lvert u+v\rvert^2} = \frac{\sqrt{2}m^2(\sqrt{2}+1)}{\sqrt{2}m^2(\sqrt{2}-1)}$ $= \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$ $= 3+2\sqrt{2}$	E2, E3 • Finds the distances AB and OC, recognises that they represent $u-v$ and $u+v$ and hence proves the result. . . . . 3 • States that $OA = OB = AC = BC$ and that $\overline{OC}$ and $\overline{BA}$ represent $u+v$ and $u-v$ respectively, and uses ΔOAB to find the length AB . . . . 2 • States that $OA = OB = AC = BC$ and that $\overline{OC}$ and $\overline{BA}$ represent $u+v$ and $u-v$ respectively 1

Question 7

Sample answer	Syllabus outcomes and marking guide
(a) $\frac{dv}{dt} = g - kv$ $= 0$ (when $v = v_t$) $\Rightarrow g = kv_t$ $v_t = \frac{g}{k}$	E5 • Gives a valid equation of motion, notes that $\frac{dv_t}{dt} = 0$ and obtains the required result . . 2 • Writes the equation of motion as $\frac{dv}{dt} = g - kv$ 1
(b) (i) $\frac{dv}{dt} = -g - kv$ $\int \frac{dv}{g+kv} = \int -dt$ $\ln(g+kv) = -kt + c$ Substituting in the initial conditions $t = 0$ and $v = v_t$ gives $\ln\left(\frac{g+kv}{g+kv_t}\right) = -kt$ $e^{-kt} = \frac{g+kv}{g+kv_t}$ $= \frac{v+\frac{g}{k}}{v_t+\frac{g}{k}}$ $= \frac{v+v_t}{2v_t}$ $v = 2v_te^{-kt} - v_t$ $= v_t(2e^{-kt}-1)$	E5 • Obtains the required result via a logically correct and comprehensive method . . . . . 3 • Obtains an exponential equation of the form $e^{-kt} = \frac{g+kv}{g+kv_t}$ . . . . 2 • Formulates and evaluates an integral to obtain a logarithmic equation of the form $\ln(g+kv) = kt + c$ 1
(ii) $\frac{dx}{dt} = v$ $= v_t(2e^{-kt}-1)$ $x = \int v_t(2e^{-kt}-1)dt$ $= v_t\int(2e^{-kt}-1)dt$ $= v_t\left[\frac{2e^{-kt}}{-k} - t + c\right]$ Substituting in the initial conditions $x = 0$ and $t = 0$ gives $0 = v_t\left[\frac{2}{-k} + c\right]$ $\Rightarrow c = \frac{2}{k}$ $\Rightarrow x = v_t\left[\frac{2e^{-kt}}{-k} - t + \frac{2}{k}\right]$ $= \frac{v_t}{k}[2 - kt - 2e^{-kt}]$	E5 • Obtains the correct result via a logically correct and comprehensive method . . . . . 3 • Formulates a differential equation of motion and solves the corresponding integral to find $\frac{dx}{dt} = v_t\left[\frac{2e^{-kt}}{-k} - t + c\right]$ . . . . 2 • Formulates a differential equation and performs a correct substitution for v to find $\frac{dx}{dt} = v_t(2e^{-kt}-1)$ 1

Question 7 (Continued)

Sample answer	Syllabus outcomes and marking guide
(c) When the particle is at its maximum height, $v = 0$. $v = v_t(2e^{-kt} - 1) = 0$ $2e^{-kt} - 1 = 0$ $kt = \ln 2$ If we define x_{max} and t_{max} as the maximum height and the time when the particle reaches the maximum height respectively, then $x_{max} = \frac{v_t}{k}(2 - kt_{max} - 2e^{-kt_{max}})$ $= \frac{v_t}{k}\left(2 - \frac{\ln 2}{t_{max}}t_{max} - 2e^{-\frac{\ln 2}{t_{max}}t_{max}}\right)$ $= \frac{v_t}{k}(2 - \ln 2 - 1)$ $= \frac{v_t}{k}(1 - \ln 2)$	E5 • Obtains a correct solution via a logically correct and comprehensive method ..... 3 • Defines t_{max} or an equivalent appropriately and formulates an equation of the form $x_{max} = \frac{v_t}{k}(2 - kt_{max} - 2e^{-kt_{max}})$ .... 2 • Observes that, at x_{max}, $v = 0$, and then finds $2e^{-kt} - 1 = 0$ or $kt = \ln 2$ 1
(d) (i) $v_t = \frac{g}{k}$ $\Rightarrow k = \frac{g}{v_t}$ $= \frac{10 \text{ m s}^{-2}}{1.8 \text{ m s}^{-1}}$ $= 5.6 \text{ s}^{-1}$	E5 • Correctly substitutes for g and v_t to find correct value of k 1
(ii) $x_{max} = \frac{v_t}{k}(1 - \ln 2)$ $= \frac{g}{k^2}(1 - \ln 2)$ $= \frac{10 \text{ m s}^{-2}}{(5.6 \text{ s}^{-1})^2}(1 - \ln 2)$ $= 0.3189(1 - \ln 2)$ m $= 0.3189 \times 0.3069$ m $= 0.098$ m $= 9.8$ cm	E5, E9 • Correctly substitutes the value of g and the value of k found in (b)(iv)(α) to obtain an answer consistent with (b)(iv)(α)....... 2 • Recognises the need to use the equation shown for x_{max} and attempts some form of substitution 1
(iii) The net acceleration is asymmetric with respect to velocity. For an ascending particle $\frac{dv}{dt} = -g - kv$, whereas for a descending particle $\frac{dv}{dt} = g - kv$. Essentially friction slows the descent of the falling particle (acting against gravity) but also slows the ascent of the rising particle (acting with gravity).	E5, E9 • Presents an argument showing the asymmetry of the frictional force on the upward and downward journeys........ 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

Question 8

Sample answer	Syllabus outcomes and marking guide
(a) (i) $I_n = \int_0^{\frac{\pi}{4}} \sec^n x\,dx$, where $n = 0, 1, 2, \ldots$ $I_{n+2} = \int_0^{\frac{\pi}{4}} \sec^2 x \sec^n x\,dx$, $= [\tan x \sec^n x]_0^{\frac{\pi}{4}} - n\int_0^{\frac{\pi}{4}} \tan x \sec x \tan x \sec^{n-1} x\,dx$ $= (\sqrt{2})^n - n\int_0^{\frac{\pi}{4}} \tan^2 x \sec^n x\,dx$ $= 2^{\frac{n}{2}} - n\int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\sec^n x\,dx$ $= 2^{\frac{n}{2}} - n\int_0^{\frac{\pi}{4}} (\sec^{n+2} x - \sec^n x)dx$ $= 2^{\frac{n}{2}} - nI_{n+2} + nI_n$ $I_{n+2} + nI_{n+2} = 2^{\frac{n}{2}} + nI_n$ $I_{n+2} = \frac{2^{\frac{n}{2}}}{n+1} + \frac{n}{n+1}I_n$	E2, E8, E9 • Completely correct solution.......... 3 • Arrives correctly at $(\sqrt{2})^n - n\int_0^{\frac{\pi}{4}} \tan^2 x \sec^n x\,dx$ .... 2 • Expresses $\sec^{n+2} x$ as $\sec^2 x \sec^n x$ and integrates correctly 1
(ii) $I_0 = [x]_0^{\frac{\pi}{4}}$ $= \frac{\pi}{4}$ $I_2 = \int_0^{\frac{\pi}{4}} \sec^2 x\,dx$ $= [\tan x]_0^{\frac{\pi}{4}}$ $= 1$ Using the answer for (i), with $n = 2$: $I_4 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$ Using the answer for (i), with $n = 4$: $I_6 = \frac{4}{5} + \frac{4}{5} \times \frac{4}{3} = \frac{28}{15}$	E8 • Uses values of I_4 and I_2 to find the value of I_6. .... 2 • Correctly evaluates I_0 and I_2 1
(iii) $\frac{d}{dx}\ln(\sec x + \tan x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$ $= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$ $= \sec x$	H3 • Correct answer..................... 1

Question 8 (Continued)

Sample answer	Syllabus outcomes and marking guide
(b) (ii) Prove $x_n^2 - 3y_n^2 = 1$ for all positive integers, n. $n = 1$ LHS $= 2^2 - 3(1)^2$ (from definition) $= 1 =$ RHS So the result is true for $n = 1$ Assume the result is true for $n = k$, i.e. $x_k^2 - 3y_k^2 = 1$. Prove the result true for $n = k + 1$, if true for $n = k$, $n = k + 1$. LHS $= x_{k+1}^2 - 3y_{k+1}^2$ $= (x_{k+1} + \sqrt{3}y_{k+1})(x_{k+1} - \sqrt{3}y_{k+1})$ $= (x_k + \sqrt{3}y_k)^2(x_k - \sqrt{3}y_k)^2$ $= (x_k^2 - 3y_k^2)^2$ $= 1$ (from assumption) So if the result is true for $n = k$, it is also true for $n = k + 1$. The result is true for $n = 1$, so it is true for $n = 2$, $n = 3$, $n = 4$ and so on. Hence, by mathematical induction the result is true for all positive integers, n.	E2 • Completely correct proof . . . . 3 • Using definitions to show the result is true for $n = 1$ and using the definitions to reach LHS $= (x_{k+1} + \sqrt{3}y_{k+1})(x_{k+1} - \sqrt{3}y_{k+1})$ . . . . 2 • Using the definitions to show the result is true for $n = 1$ 1
(iii) $x_n^2 - 3y_n^2 = 1$ $\frac{x_n^2}{y_n^2} = \frac{1}{y_n^2} + 3$ $\frac{x_n}{y_n} = \sqrt{3 + \frac{1}{y_n^2}}$ x_n and y_n are positive integers $y_n \to \infty$, $\frac{x_n}{y_n} \to \sqrt{3}$ from above $x_n^2 - 3y_n^2 = 1$ $3y_n^2 = x_n^2 - 1$ $\frac{3y_n^2}{x_n^2} = 1 - \frac{1}{x_n^2}$ $\frac{\sqrt{3}y_n}{x_n} = \sqrt{1 - \frac{1}{x(n)^2}}$ $\frac{3y_n}{x_n} = \sqrt{3 - \frac{3}{x(n)^2}}$ $x_n \to \infty$, $\frac{3y_n}{x_n} \to \sqrt{3}$ from below Therefore both limits tend towards the value $\sqrt{3}$, the first from above and the second from below.	E2 • Completely correct procedure . . . . 3 • Finding expressions for both limits without the value of the limit and incomplete reasoning . . . . 2 • Finding expression for one limit with correct reasoning 1

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Question 8 (Continued)

Sample answer	Syllabus outcomes and marking guide
(a) (iv) $I_1 = [\ln(\sec x + \tan x)]_0^{\frac{\pi}{4}}$ $= \ln(\sqrt{2} + 1)$ Using the answer for (i), with $n = 1$: $I_3 = \frac{\sqrt{2}}{2} + \frac{1}{2}I_1$ $= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2} + 1)$ Using the answer for (i), with $n = 3$: $I_5 = \frac{2^{\frac{3}{2}}}{4} + \frac{3}{4}I_3$ $= \frac{2\sqrt{2}}{4} + \frac{3}{4}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2} + 1)\right]$ $= \frac{7\sqrt{2}}{8} + \frac{3}{8}\ln(\sqrt{2} + 1)$	E8 • Correctly uses the values of I_1 and I_3 to find I_5 (this need not be expanded) . . . . 2 • Correctly evaluates I_1. 1
(b) (i) For $x_{n+1} + \sqrt{3}y_{n+1} = (x_n + \sqrt{3}y_n)^2$ $= x_n^2 + 2\sqrt{3}x_n y_n + 3y_n^2$ and $x_{n+1} = x_n^2 + 3y_n^2$ $y_{n+1} = 2x_n y_n$ For $x_{n+1} - \sqrt{3}y_{n+1} = (x_n - \sqrt{3}y_n)^2$ $= x_n^2 - 2\sqrt{3}x_n y_n + 3y_n^2$ and $x_{n+1} = x_n^2 + 3y_n^2$ $y_{n+1} = 2x_n y_n$ Therefore, $x_{n+1} - \sqrt{3}y_{n+1} = (x_n - \sqrt{3}y_n)^2$ and $x_{n+1} + \sqrt{3}y_{n+1} = (x_n + \sqrt{3}y_n)^2$, together with $x_1 = 2$ and $y_1 = 1$ are equivalent definitions.	E2 • Correct expansion of both expression and equating of rational and irrational parts to show equivalence 1