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HSC Trial Examination 2001

Mathematics Extension 2

This paper must be kept under strict security and may only be used on or after the morning of Tuesday 7 August, 2001, as specified in the NEAP Examination Timetable

General Instructions

Reading time 5 minutes

Working time 3 hours

Write using blue or black pen.

Board-approved calculators may be used.

A table of standard integrals is provided on page 14.

All necessary working should be shown in every question.

Examination structure

Total marks 120

Attempt all questions

All questions are of equal value

QUESTION 1. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^3 \frac{x\,dx}{\sqrt{16+x^2}}$. **3**

(b) Find $\displaystyle\int \frac{dx}{x^2+6x+13}$. **2**

(c) Find $\displaystyle\int xe^{-x}\,dx$. **2**

(d) Find $\displaystyle\int \cos^3\theta\,d\theta$. **3**

(e) (i) Find constants A, B and C such that **3**

$$\frac{x^2-4x-1}{(1+2x)(1+x^2)} \equiv \frac{A}{1+2x}+\frac{Bx+C}{1+x^2}.$$

(ii) Hence find $\displaystyle\int \frac{x^2-4x-1}{(1+2x)(1+x^2)}\,dx$. **2**

QUESTION 2. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that $z = 1 + i$ and $w = -3$, find, in the form $x + iy$:

(i) wz^2, **1**

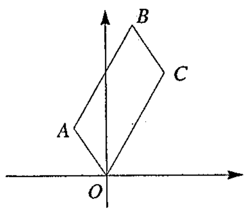
(ii) $\dfrac{z}{z+w}$. **2**

(b) Using de Moivre's theorem, simplify $(-1 - i\sqrt{3})^{-10}$, expressing the answer in the form $x + iy$. **3**

(c) Sketch the region described by the following: **2**

$$|z| < 2 \text{ and } \frac{2\pi}{3} \le \arg z \le \frac{5\pi}{6}.$$

(d) **3**



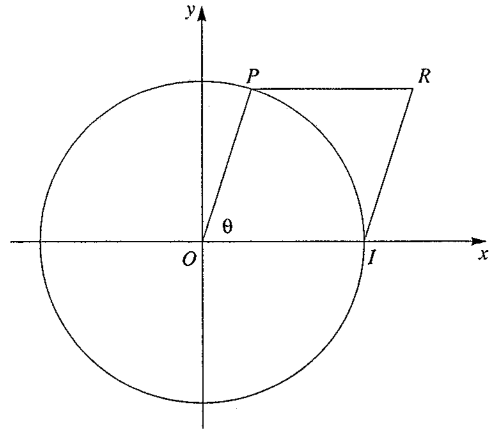
In the diagram above, $OABC$ is a parallelogram with $OA = \frac{1}{2}OC$.

The point A represents the complex number $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

If $\angle AOC = 60°$, what complex number does C represent?

QUESTION 2. (Cont.) **Marks**

(e) In the Argand diagram below, P represents $\cos\theta + i\sin\theta$, I represents the number $1 + 0i$, and R represents the number $z = 1 + \cos\theta + i\sin\theta$.



(i) Using the properties of the rhombus, or otherwise, show that z can be expressed as **2**

$$z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$$

(ii) Hence show that $\frac{1}{z} = \frac{1}{2} - \frac{i}{2}\tan\frac{\theta}{2}$. **2**

QUESTION 3. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $1, \omega, \omega^2$ be the three cube roots of unity.

(i) Show that:

α. $\omega^3 = 1$, **1**

β. $1 + \omega + \omega^2 = 0$. **1**

(ii) If $1, \omega, \omega^2$ are the roots of $x^3 + ax^2 + bx + c = 0$, find a, b and c. **3**

(b) Find the acute angle between the tangent to $x^3 + y^3 = 1$ at $x = 1$ and the line $y = x$. **4**

(c) On separate number planes, draw graphs of the following functions, showing essential features.

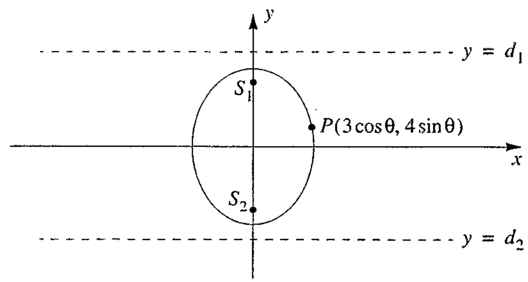
(i) $y = \dfrac{x+1}{x-1}$ **2**

(ii) $y = \sqrt{\dfrac{x+1}{x-1}}$ **2**

(iii) $y = \ln\left(\dfrac{x+1}{x-1}\right)$ **2**

QUESTION 4. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram above shows an ellipse with parametric equation

$$x = 3\cos\theta$$
$$y = 4\sin\theta$$

(i) Write down the cartesian equation of the ellipse. **1**

(ii) Find the coordinates of the foci S_1 and S_2. **2**

(iii) Find the equation of the directrices $y = d_1$ and $y = d_2$. **2**

(iv) By using a characterisation of an ellipse as a locus, or otherwise, show that $S_1P + S_2P = 8$. **2**

(b) In a series of five games played by two equally matched teams, team A and team B, the team that wins three games first is the champion.

(i) If team B wins the first two games, what is the probability that team A is the champion? **1**

(ii) If team A has won the first game, what is the probability that team A is the champion? **2**

(c) (i) Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. **2**

(ii) Hence or otherwise find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. **2**

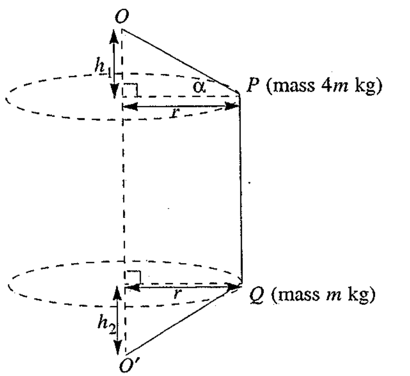
(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. You do not need to give the coordinates of turning points or inflections. **1**

QUESTION 5. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle moves in a circle of radius r, with a constant speed $r\omega$. **1**

Write down the magnitude and direction of its acceleration.

(b)



The diagram above shows two particles, P and Q, of masses $4m$ kg and m kg respectively, which are attached to a light inextensible string. The ends of the strings are attached to fixed points O and O'. O is vertically above O'.

The particles P and Q move in horizontal circles, of equal radius r metres, about OO', with the same constant angular velocity ω, so that Q always remains vertically below P.

The depth of P below the level of O is h_1 and the height of Q above the level of O' is h_2.

The angle that OP makes with the horizontal is α.

(i) Let the tension in the string PQ be T newtons and the tension in the string OP be T_1 newtons. **2**

By drawing a force diagram and resolving the forces acting on P, show that

$$T_1 \sin\alpha = 4mg + T$$

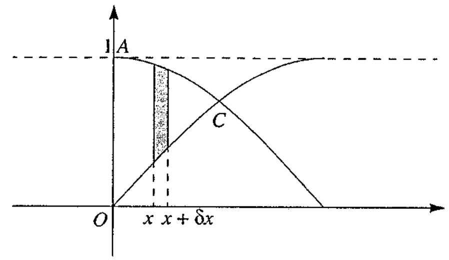
$$T_1 \cos\alpha = 4m\omega^2 r$$

(ii) Hence show that $h_1 = \dfrac{4mg + T}{4m\omega^2}$. **2**

(iii) Hence show that $(4h_1 - h_2)\omega^2 = 5g$. **3**

QUESTION 5. (Cont.) **Marks**

(c) The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$. The graph of $y = \cos x$ meets the y axis at A, and the C is the first point of intersection of the two graphs to the right of the y axis.



The region OAC is to be rotated about the line $y = 1$.

(i) Write down the coordinates of the point C. **1**

(ii) The shaded strip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resulting slice is given by **2**

$$\delta V = \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\delta x.$$

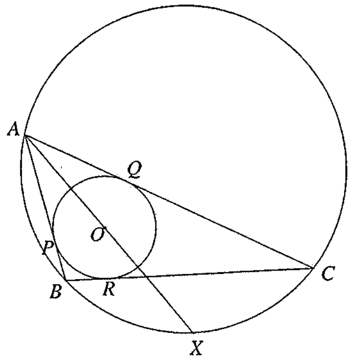
(iii) Hence evaluate the total volume when the region OAC is rotated about the line $y = 1$. **4**

QUESTION 6. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the diagram below, ABC is a triangle.

The incircle tangent to all three sides has centre O, and touches the sides AB, AC and BC at P, Q and R respectively.

The circumcircle through A, B and C meets the line AO produced at X.



(i) Show that $\angle CBX = \angle CAX$. **1**

(ii) Use congruence to prove that $\angle OBA = \angle OBC$. **2**

(iii) Prove that ΔXBO is an isosceles triangle. **3**

(iv) Prove that $BX = XC$. **1**

(b) (i) α. Differentiate $y = \log_e(1+x)$, and hence draw $y = x$ and $y = \log_e(1+x)$ on one set of axis. **1**

β. Using this graph, explain why **1**

$$\log_e(1+x) < x, \text{ for all } x > 0.$$

(ii) α. Differentiate $y = \frac{x}{1+x}$, and hence draw $y = \frac{x}{1+x}$ and $y = \log_e(1+x)$ on one set of axis. **1**

β. Using this graph, explain why **1**

$$\frac{x}{1+x} < \log_e(1+x), \text{ for all } x > 0.$$

(iii) Use the inequalities of parts (i) and (ii) to show that **4**

$$\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \frac{1}{2}\log_e 2.$$

QUESTION 7. (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $\frac{t^n}{1+t^2} = t^{n-2} - \frac{t^{n-2}}{1+t^2}$. **1**

(ii) Let $I_n = \int \frac{t^n}{1+t^2}\,dt$. **1**

Show that $I_n = \frac{t^{n-1}}{n-1} - I_{n-2}$, $n \geq 2$.

(iii) Show that $\int_0^1 \frac{t^6}{1+t^2}\,dt = \frac{13}{15} - \frac{\pi}{4}$. **3**

(b) Consider the rectangular hyperbola $xy = 4$.

(i) Show that the gradient of the tangent at the point $P\left(2p, \frac{2}{p}\right)$ is $-\frac{1}{p^2}$. **1**

(ii) Show that the normal at P is given by $p^3x - py = 2(p^4 - 1)$. **1**

(iii) This normal meets the hyperbola again at $Q\left(2q, \frac{2}{q}\right)$. **3**

By considering the product of the roots of the equation formed by the intersection of $xy = 4$ and $p^3x - py = 2(p^4 - 1)$, or otherwise, prove that $p^3q = -1$.

(iv) Hence, or otherwise, find the equation of the chord that is a normal at both ends of the chord. **2**

QUESTION 7. (Cont.) Marks

(c) You may assume that, for all positive real numbers a and b,

$$\sqrt{ab} \le \frac{a+b}{2}.$$

(i) Show that for all positive integers n, **1**

$$^nC_0 + {^nC_1} + \ldots + {^nC_n} = 2^n.$$

(ii) Prove that for all positive integers n, **2**

$$\left(\sqrt{^nC_1} + \sqrt{^nC_2} + \ldots + \sqrt{^nC_n}\right)^2 \le n(2^n - 1).$$

You may use the identity

$$(x_1 + x_2 + \ldots + x_n)^2 = (x_1^2 + x_2^2 + \ldots + x_n^2) + \sum_{i<j} 2x_i x_j.$$

QUESTION 8. (15 marks) Use a SEPARATE writing booklet. Marks

(a) At a dinner party there are twelve people, consisting of the six State Premiers and their partners. Each couple was representing one of the six States: New South Wales, Victoria, Western Australia, South Australia, Tasmania and Queensland.

(i) The dinner took place at a circular table. Find how many seating arrangements are possible if:

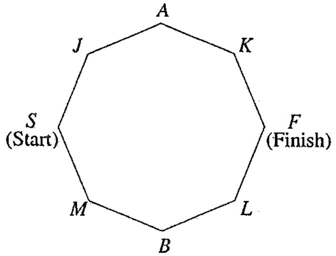
α. there are no restrictions, **1**

β. the males and females are in alternate positions. **1**

(ii) A committee of six is to be formed from the Premiers and their partners, where not more than one State can have two representatives. How many such committees are possible? **2**

QUESTION 8. (Cont.) Marks

(b) The diagram below shows a regular octagon $SJAKFLBM$. A frog starts jumping at vertex S, hoping to reach a pool at the opposite vertex F. From any vertex of the octagon except F, the frog may jump to either of the two adjacent vertices. But when it reaches the vertex F, it stops jumping and swims away.



A path of n jumps is thus a sequence $(V_0, V_1, \ldots, V_n)$ of vertices such that:

For $i = 0, 1, \ldots, n-1$, the vertex V_i is distinct from F.

For $i = 0, 1, \ldots, n-1$ the vertices V_i and V_{i+1} are adjacent.

Let x_n be the number of distinct paths of exactly n jumps starting at S and ending at F.

(i) Explain why $x_1 = x_2 = x_3 = 0$ and $x_4 = 2$. **2**

(ii) Explain why $x_n = 0$ if n is an odd positive integer. **1**

Now let y_n be the number of distinct paths of exactly n jumps starting at A and ending at F. (Notice that y_n is also the number of distinct paths of exactly n jumps starting at B and ending at F.)

(iii) By considering the position of the frog one jump away from S, show that **2**

$$x_n = 2x_{n-2} + 2y_{n-2}, \text{ for all integers } n \geq 2.$$

(iv) By considering the position of the frog one jump away from A, show that **2**

$$y_n = 2y_{n-2} + x_{n-2}, \text{ for all integers } n \geq 4.$$

(v) Hence show that $x_{2m} = 4x_{2m-2} - 2x_{2m-4}$, for all integers $m \geq 3$. **1**

(vi) Hence, or otherwise, prove that $x_{2n} = \frac{1}{\sqrt{2}}(\alpha^{n-1} - \beta^{n-1})$, for all integers $n \geq 1$, where $\alpha = 2 + \sqrt{2}$ and $\beta = 2 - \sqrt{2}$. **3**

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Suggested Solutions

QUESTION 1

(a) $\int_0^3 \frac{x\,dx}{\sqrt{16+x^2}} = \int_{16}^{25} \frac{\frac{1}{2}\,du}{\sqrt{u}}$

Let $u = 16 + x^2$
$du = 2x\,dx$ ✓

$= \frac{1}{2}\int_{16}^{25} u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{16}^{25}$ ✓

$= \left[\sqrt{u}\right]_{16}^{25}$

$= \sqrt{25} - \sqrt{16}$

$= 5 - 4$

$= 1$ ✓

(b) $\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{(x+3)^2+4}$ ✓

$= \frac{1}{2}\tan^{-1}\frac{x+3}{2} + c$ ✓

(c) $\int xe^{-x}dx = \int x\frac{d}{dx}(-e^{-x})dx$

$= -xe^{-x} - \int 1(-e^{-x})dx$ ✓

$= -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + c$ ✓

(d) $\int \cos^3\theta\, d\theta = \int \cos^2\theta\cos\theta\, d\theta$ ✓

Let $u = \sin\theta$
$du = \cos\theta\, d\theta$

$= \int (1-\sin^2\theta)\cos\theta\, d\theta$

$= \int (1-u^2)du$ ✓

$= u - \frac{1}{3}u^3 + c$

$= \sin\theta - \frac{1}{3}\sin^3\theta + c$ ✓

(e) (i) Let $\frac{x^2-4x-1}{(1+2x)(1+x^2)} \equiv \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$.

Then $x^2-4x-1 \equiv A(1+x^2) + (1+2x)(Bx+C)$ ✓

$x^2-4x-1 \equiv A + Ax^2 + Bx + C + 2Bx^2 + 2Cx$

Equating coefficients of like terms,

$1 = A + 2B$ [Eq. 1]

$-4 = B + 2C$ [Eq. 2]

$-1 = A + C$ [Eq. 3]

Multiply Eq. 2 by -2:

$8 = -2B - 4C$ [Eq. 2a]

Eq. 1 + Eq. 2a:

$9 = A - 4C$ [Eq. 4]

Eq. 3 – Eq. 4:

$-10 = 5C$

$C = -2$

Substitute C into Eq. 3:

$-1 = A - 2$

$A = 1$

Substitute A into Eq. 1:

$1 = 1 + 2B$

$B = 0$ ✓✓

(ii) $\int \frac{x^2-4x-1}{(1+2x)(1+x^2)}dx = \int\left(\frac{1}{1+2x} + \frac{-2}{1+x^2}\right)dx$

$= \frac{1}{2}\ln|1+2x| - 2\tan^{-1}x + c$ ✓✓

QUESTION 2

(a) (i) $wz^2 = -3(1+i)^2$

$= -3(1-1+2i)$

$= -6i$ ✓

(ii) $\dfrac{z}{z+w} = \dfrac{1+i}{-2+i} \times \dfrac{-2-i}{-2-i}$

$= \dfrac{-(1+i)(2+i)}{5}$ ✓

$= \dfrac{-(1+3i)}{5}$

$= -\dfrac{1}{5} - \dfrac{3i}{5}$ ✓

(b) $-1 - i\sqrt{3} = 2\text{cis}\left(-\dfrac{2\pi}{3}\right)$ ✓

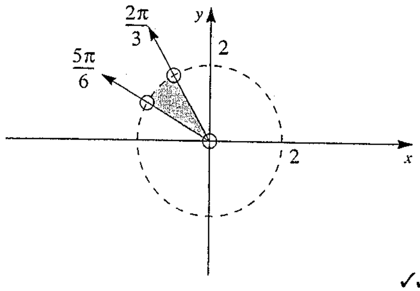
Hence $\left(2\text{cis}\left(-\dfrac{2\pi}{3}\right)\right)^{-10} = 2^{-10}\text{cis}\left(\dfrac{20\pi}{3}\right)$ ✓

$= 2^{-10}\text{cis}\left(\dfrac{2\pi}{3}\right)$

$= \dfrac{1}{1024}\left(-\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}\right)$

$= -\dfrac{1}{2048} + \dfrac{i\sqrt{3}}{2048}$ ✓

(c)



✓✓

(d) To rotate $\overrightarrow{OA}$ by $-60°$, we need to multiply by $\text{cis}\left(-\dfrac{\pi}{3}\right)$ ✓.

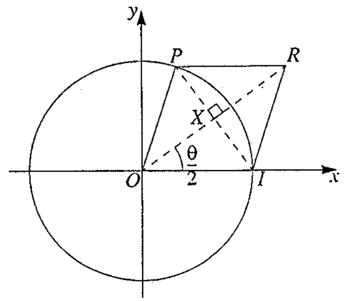
Thus $\overrightarrow{OC} = 2 \times \overrightarrow{OA} \times \text{cis}\left(-\dfrac{\pi}{3}\right)$

$= 2 \times \text{cis}\left(\dfrac{2\pi}{3}\right) \times \text{cis}\left(-\dfrac{\pi}{3}\right)$

$= 2\text{cis}\left(\dfrac{\pi}{3}\right)$ ✓

$= 1 + i\sqrt{3}$ ✓

(e) (i) Draw in the diagonals, noting that they bisect one another at right angles, and also that $\angle ROI = \dfrac{\theta}{2}$.



So $z = |\overrightarrow{OR}|\text{cis}\dfrac{\theta}{2}$ ✓

But $|\overrightarrow{OR}| = 2|\overrightarrow{OX}|$

and $|\overrightarrow{OX}| = \cos\dfrac{\theta}{2}$

so $|\overrightarrow{OR}| = 2\cos\dfrac{\theta}{2}$ ✓

Hence $z = 2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)$

(ii) $\dfrac{1}{z} = z^{-1} = \left(2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)\right)^{-1}$

$= \dfrac{1}{2\cos\dfrac{\theta}{2}}\left(\cos\dfrac{\theta}{2} - i\sin\dfrac{\theta}{2}\right)$ (de Moivre's theorem) ✓

$= \dfrac{1}{2}\left(\dfrac{\cos\dfrac{\theta}{2} - i\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}}\right)$

$= \dfrac{1}{2}\left(1 - \dfrac{i\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}}\right)$

$= \dfrac{1}{2}\left(1 - i\tan\dfrac{\theta}{2}\right)$ ✓

$= \dfrac{1}{2} - \dfrac{i}{2}\tan\dfrac{\theta}{2}$

QUESTION 3

(a) (i) α. Since ω is a root of $z^3 = 1$, it follows that $\omega^3 = 1$. ✓

β. For the cubic equation $z^3 - 1 = 0$,

$$\text{sum of roots} = -\frac{\text{coefficient of } z^2}{\text{coefficient of } z^3}$$

$$= -\frac{0}{1} = 0$$

Hence $1 + \omega + \omega^2 = 0$ ✓

(ii) Given $x^3 + ax^2 + bx + c = 0$ with roots $1, \omega, \omega^2$

First, $\sum \alpha = 1 + \omega + \omega^2 = -a$

$0 = -a$

$a = 0$ ✓

Secondly, $\sum \alpha\beta = \omega + \omega^2 + \omega^3 = b$

$\omega + \omega^2 + 1 = b$

$b = 0$ ✓

Thirdly, $\alpha\beta\gamma = 1 \times \omega \times \omega^2 = -c$

$\omega^3 = -c$

$c = -1$ ✓

(b) Given $x^3 + y^3 = 1$

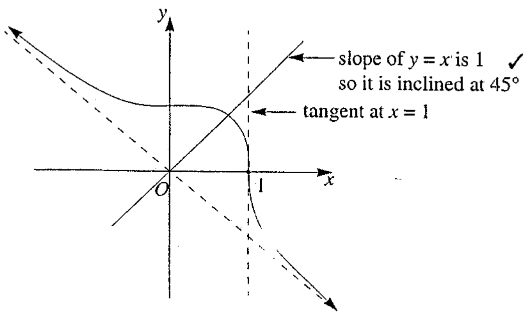
$3x^2 + 3y^2\frac{dy}{dx} = 0$ ✓

$\frac{dy}{dx} = -\frac{x^2}{y^2}$ ✓

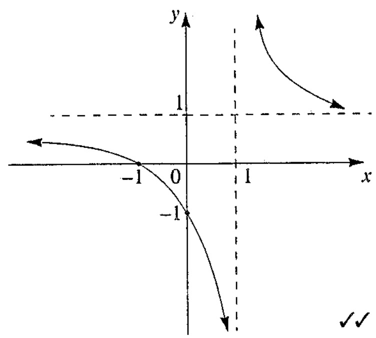
At $x = 1$, $y = 0$

so the tangent is vertical at $(1, 0)$ ✓

Hence the angle between the tangent at $x = 1$ and the line $y = x$ is 45°.



(c) (i)



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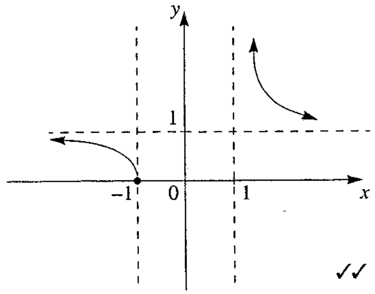
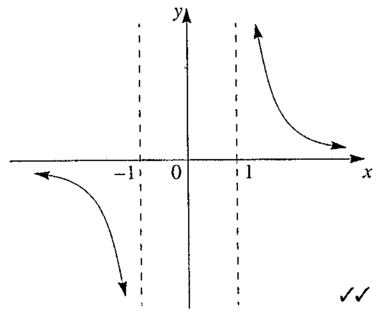
- vertical asymptote at $x = 1$
- horizontal asymptote at $y = 1$
- x intercept at $x = -1$
- y intercept at $y = -1$

(ii)



✓✓

(iii)



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QUESTION 4

(a) (i) $x = 3\cos\theta,\ y = 4\sin\theta$

$\frac{x}{3} = \cos\theta$

$\frac{y}{4} = \sin\theta$

Now $\cos^2\theta + \sin^2\theta = 1$

so $\frac{x^2}{9} + \frac{y^2}{16} = 1$ ✓

(ii) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

The foci are $S_1(0, ae)$ and $S_2(0, -ae)$.

Here $a = 4, b = 3$

Hence $e^2 = 1 - \left(\frac{b}{a}\right)^2$

$= 1 - \frac{9}{16}$

$= \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$ ✓

Thus $S_1 = (0, \sqrt{7})$

$S_2 = (0, -\sqrt{7})$ ✓

(iii) The upper directrix is $y = \frac{a}{e}$.

$y = \frac{4}{\frac{\sqrt{7}}{4}}$

$y = \frac{16}{\sqrt{7}}$ ✓

The lower directrix is $y = -\frac{a}{e}$.

$y = \frac{-4}{\frac{\sqrt{7}}{4}}$

$y = -\frac{16}{\sqrt{7}}$ ✓

(iv) $S_1P = ePM_1$ where PM_1 is the distance from P to the line $y = d_1$

$PM_1 = d_1 - y$

$PM_2 = y - d_2$ ✓

Hence $S_1P + S_2P = e(PM_1 + PM_2)$

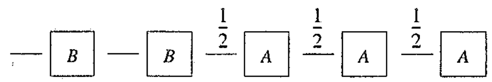
$= (d_1 - d_2)$

$= \frac{\sqrt{7}}{4}\left(\frac{32}{\sqrt{7}}\right)$

$= 8$ ✓

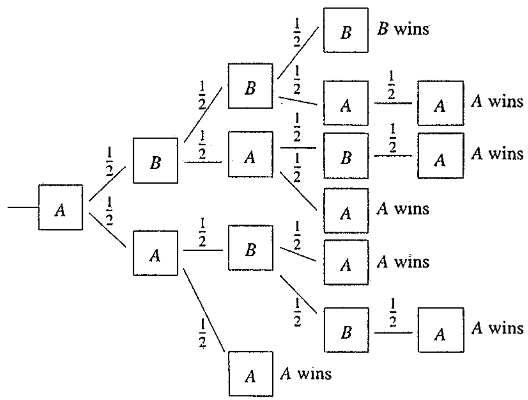
(b) The probability that A wins any game is $\frac{1}{2}$.

(i) If team B wins the first two games:



$P(A \text{ wins}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ ✓

(ii) If team A has won the first game:



$$\text{P}(A\,\text{wins}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}$$ ✓

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{4} = \frac{11}{16}$$ ✓

(c) (i) For $P(x)$ to have a zero with multiplicity of 3, we can write $P(x)$ as follows:

$P(x) = (x-\alpha)^3 Q(x)$, where $Q(\alpha) \neq 0$ ✓

Differentiating, $P'(x) = (x-\alpha)^3 Q'(x) + 3(x-\alpha)^2 Q(x)$ (product rule)

$= (x-\alpha)^2[(x-\alpha)Q'(x) + 3Q(x)]$

$= (x-\alpha)^2 R(x)$, where $R(\alpha) = 3Q(\alpha) \neq 0$ ✓

So $P'(x)$ has a zero of multiplicity 2.

(ii) Let $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$ and let $x = \alpha$ be the zero of multiplicity 3.

Differentiating, $P'(x) = 32x^3 - 75x^2 + 54x - 11$

and $P''(x) = 96x^2 - 150x^2 + 54$

$= 6(16x^2 - 25x^2 + 9)$

$= 6(x-1)(16x-9)$

So the zeros of $P''(x)$ are $x = 1$ and $x = \frac{9}{16}$. ✓

Testing $x = 1$, $P(1) = 0$ and $P'(1) = 0$, so $P(x) = (x-1)^3 Q(x)$

Let $x = \beta$ be the other zero.

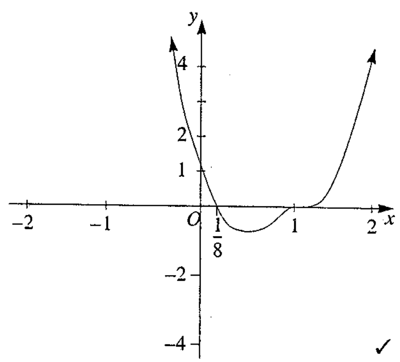
Then $\alpha + \alpha + \alpha + \beta = \frac{25}{8}$

$\beta = \frac{25}{8} - 3$

$= \frac{1}{8}$

So the zeroes of $8x^4 - 25x^3 + 27x^2 - 11x + 1$ are $x = 1, 1, 1, \frac{1}{8}$ ✓

(iii)

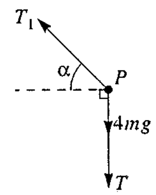


✓

QUESTION 5

(a) Its acceleration has magnitude $\omega^2 r$ and is directed towards the centre of the circle. ✓

(b) (i) Forces at P:



Resolving vertically: $T_1 \sin\alpha = 4\text{ mg} + T$ ✓

Resolving horizontally: $T_1 \cos\alpha = 4m\omega^2 r$ ✓

(ii) From (i), $\frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{4mg + T}{4m\omega^2 r}$

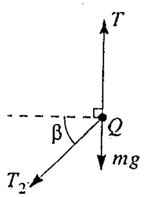
$\tan\alpha = \frac{4mg + T}{4m\omega^2 r}$

Now $\tan\alpha = \frac{h_1}{r}$ ✓

so $\frac{h_1}{r} = \frac{4mg + T}{4m\omega^2 r}$

$h_1 = \frac{4mg + T}{4m\omega^2}$ ✓

(iii) Forces at Q:



Let T_2 be the tension in the string $O'Q$ and let β be the angle that $O'Q$ makes with the horizontal.

Resolving vertically, $T_2 \sin\beta = T - mg$ [Eq. 1]

Resolving horizontally, $T_2 \cos\beta = m\omega^2 r$ [Eq. 2] ✓

Dividing Eq. 1 by Eq. 2:

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} = \frac{T - mg}{m\omega^2 r}$$

$$\tan\beta = \frac{T - mg}{m\omega^2 r}$$

Now $\tan\beta = \dfrac{h_2}{r}$

$$\text{so } \frac{h_2}{r} = \frac{T - mg}{m\omega^2 r}$$

$$h_2 = \frac{T - mg}{m\omega^2} \quad ✓$$

$$\text{Hence } 4h_1 - h_2 = \frac{4mg + T}{m\omega^2} - \frac{T - mg}{m\omega^2}$$

$$= \frac{4mg + T - (T - mg)}{m\omega^2}$$

$$= \frac{5g}{\omega^2}$$

$$(4h_1 - h_2)\omega^2 = 5g \quad ✓$$

(c) (i) Put $\cos x = \sin x$

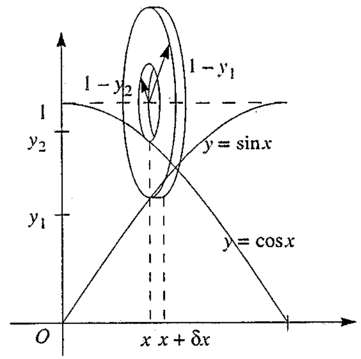
$$\Rightarrow 1 = \frac{\sin x}{\cos x}$$

Then $\tan x = 1$

$$x = \frac{\pi}{4}$$

Hence C is the point $\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$ ✓

(ii)



$$\delta V = \pi[(1 - y_1)^2 - (1 - y_2)^2]\delta x \quad ✓$$

$$= \pi[(1 - \sin x)^2 - (1 - \cos x)^2]\delta x$$

$$= \pi[1 - 2\sin x + \sin^2 x - (1 - 2\cos x + \cos^2 x)]\delta x$$

$$= \pi[2\cos x - 2\sin x + \sin^2 x - \cos^2 x]\delta x \quad ✓$$

(iii)
$$V = \lim_{\delta x \to 0} \sum_{x=0}^{\frac{\pi}{4}} \pi[2\cos x - 2\sin x + \sin^2 x - \cos^2 x]\delta x$$

$$= \pi\int_0^{\frac{\pi}{4}} (2\cos x - 2\sin x - \cos 2x)\, dx \quad ✓$$

$$= \pi\left[2\sin x + 2\cos x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} \quad ✓$$

$$= \pi\left[2 \times \frac{1}{\sqrt{2}} + 2 \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times 1 - \left(0 + 2 - \frac{1}{2} \times 0\right)\right] \quad ✓$$

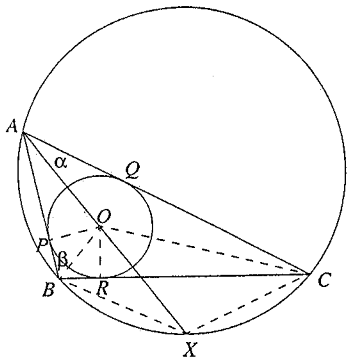
$$= \pi\left[\sqrt{2} + \sqrt{2} - \frac{1}{2} - 2\right]$$

$$= \pi\left[2\sqrt{2} - \frac{5}{2}\right] \quad ✓$$

$$= \frac{\pi}{2}[4\sqrt{2} - 5] \text{ cubic units}$$

QUESTION 6

(a)



(i) $\therefore \angle CBX = \angle CAX$ (angles on the same arc CX) ✓

(ii) In the triangles POB and ROB:

1. $OB = OB$ (common)
2. $OP = OR$ (radii)
3. $\angle OPB = \angle ORB = 90°$ (radius and tangent)

so $\Delta POB \equiv \Delta ROB$ (RHS) ✓

Hence $\angle OBA = \angle OBC$ (corresponding angles of congruent triangles) ✓

(iii) Let $\angle OBA = \angle OBC = \beta$ and $\angle CAX = \angle CBX = \alpha$.

Then by a similar proof to (ii), $\angle BAX = \alpha$. ✓

Hence $\angle BOX = \alpha + \beta$ (exterior angle of ΔABO). ✓

But $\angle OBX = \alpha + \beta$ (adjacent angles),

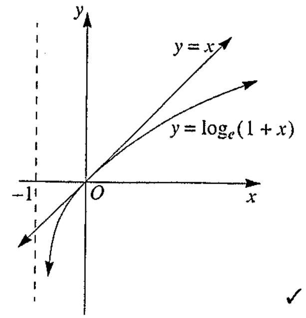
so $BX = OX$ (opposite angles in ΔOBX are equal). ✓

(iv) Similarly, $CX = OX$.

Hence $BX = CX$. ✓

(b) (i) α. $y = \log_e(1 + x)$

$\frac{dy}{dx} = \frac{1}{1 + x}$



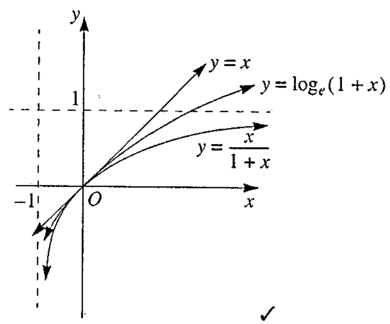
✓

β. When $x = 0$, $\frac{dy}{dx} = 1$, so $y = x$ is a tangent at $(0, 0)$.

Since $y = \log_e(1 + x)$ is concave down, it follows that its graph is below the line $y = x$ for $x > 0$. ✓

(ii) α. $y = \frac{x}{1 + x}$

Using the quotient rule, $\frac{dy}{dx} = \frac{1}{(1 + x)^2}$.



✓

β. When $x = 0$, $\frac{dy}{dx} = 1$, so $y = x$ is a tangent to both curves at $(0, 0)$.

But for $x > 0$, the gradient function of $y = \frac{x}{1 + x}$ is less than the gradient function of $y = \log_e(1 + x)$, because $\frac{1}{(1 + x)^2} < \frac{1}{1 + x}$ for $x > 0$.

Hence the graph of $y = \frac{x}{1 + x}$ is always below the graph of $y = \log_e(1 + x)$ for $x > 0$. ✓

(iii) From (i) and (ii), $\frac{x}{1+x} < \log_e(1+x) < x$ for all $x > 0$.

Hence $\frac{x}{(1+x)(1+x^2)} < \frac{\log_e(1+x)}{1+x^2} < \frac{x}{1+x^2}$ for all $x > 0$

and so $\int_0^1 \frac{x}{(1+x)(1+x^2)}\,dx < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \int_0^1 \frac{x}{1+x^2}\,dx$ for all $x > 0$. ✓

Now $\int_0^1 \frac{x}{1+x^2}\,dx = \left[\frac{1}{2}\log_e(x^2+1)\right]_0^1$

$= \frac{1}{2}\log_e 2$ ✓

Also, $\int_0^1 \frac{x}{(1+x)(1+x^2)}\,dx = \int_0^1 \left(-\frac{1}{2(x+1)} + \frac{1+x}{2(x^2+1)}\right)dx$ (partial fractions)

$= \left[-\frac{1}{2}\log_e(1+x)\right]_0^1 + \left[\frac{1}{4}\log_e(x^2+1)\right]_0^1 + \left[\frac{1}{2}\tan^{-1}x\right]_0^1$ ✓

$= -\frac{1}{2}\log_e 2 + \frac{1}{4}\log_e 2 + \frac{1}{2}\tan^{-1} 1$

$= \frac{\pi}{8} - \frac{1}{4}\log_e 2$

Hence $\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \frac{1}{2}\log_e 2$ for all $x > 0$. ✓

QUESTION 7

(a) (i) RHS $= t^{n-2} - \frac{t^{n-2}}{1+t^2}$

$= \frac{(1+t^2)t^{n-2} - t^{n-2}}{1+t^2}$

$= \frac{t^{n-2} + t^n - t^{n-2}}{1+t^2}$

$= \frac{t^n}{1+t^2}$

= LHS ✓

(ii) $I_n = \int \frac{t^n}{1+t^2}dt$

$= \int \left(t^{n-2} - \frac{t^{n-2}}{1+t^2}\right)dt$

$= \frac{t^{n-1}}{n-1} - \int \frac{t^{n-2}}{1+t^2}dt$

$= \frac{t^{n-1}}{n-1} - I_{n-2}$ ✓

(iii) Let $J_n = \int_0^1 \frac{t^n}{1+t^2}dt$

Then $J_n = \left[\frac{t^{n-1}}{n-1}\right]_0^1 - J_{n-2}$

$= \frac{1}{n-1} - J_{n-2}$ ✓

Hence $J_6 = \frac{1}{5} - J_4$

$= \frac{1}{5} - \frac{1}{3} + J_2$

$= \frac{1}{5} - \frac{1}{3} + 1 - J_0$ ✓

But $J_0 = \int_0^1 \frac{1}{1+t^2}dt$

$= \left[\tan^{-1}t\right]_0^1 = \frac{\pi}{4}$

Hence $J_6 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$

$= \frac{13}{15} - \frac{\pi}{4}$ ✓

(b) (i) $xy = 4$

$y = 4x^{-1}$

$\frac{dy}{dx} = -4x^{-2}$

So the slope at $P\left(2p, \frac{2}{p}\right)$ is $-4 \times \frac{1}{(2p)^2} = -\frac{1}{p^2}$ ✓

(ii) From (i), the normal has gradient p^2

So the normal is $y - \frac{2}{p} = p^2(x - 2p)$

$py - 2 = p^3(x - 2p)$

$py - 2 = p^3x - 2p^4$

$p^3x - py = 2(p^4 - 1)$ ✓

(iii) Solve $xy = 4$ [Eq. 1] and $p^3x - py = 2(p^4 - 1)$ [Eq. 2] simultaneously by substitution.

From Eq. 1, $y = \frac{4}{x}$

Substitute y into Eq. 2: $p^3x - \frac{4p}{x} = 2(p^4 - 1)$

$p^3x^2 - 4p = 2x(p^4 - 1)$

$p^3x^2 - 2x(p^4 - 1) - 4p = 0$ [Eq. 3] ✓

Now product of the roots in Eq. 3 is $-\frac{4p}{p^3} = -\frac{4}{p^2}$ ✓

Also, the roots of Eq. 3 are $2p$ and $2q$,

so $2p \times 2q = -\frac{4}{p^2}$

$p^3 = -\frac{1}{q}$

$p^3q = -1$ ✓

(iv) For the chord to be a normal at both ends, we can equate the gradients of the normals to the two branches of the hyperbola,

that is, $p^2 = q^2$ [Eq. 4]

and $p^3q = -1$ [Eq. 5]

From Eq. 4 and Eq. 5, $p = 1$ and $q = -1$ or $p = -1$ and $q = 1$. ✓

Substituting $p = 1$ into the equation for the normal (from (ii)),

$(1)^3x - (1)y = 2(1^4 - 1)$

$x - y = 0$

$y = x$ ✓

(c) (i) The numbers ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ are defined by $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_n$.

Substituting $x = 1$, $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$ ✓

(ii) Using the given identity,

$\text{LHS} = {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n + \sum_{i<j} 2\sqrt{{}^nC_i}\sqrt{{}^nC_j}$.

Using (i) and the AM/GM inequality,

$\text{LHS} < 2^n - 1 + \sum_{i<j} 2\sqrt{{}^nC_i}\sqrt{{}^nC_j}$ ✓

$= 2^n - 1 + (n - 1)({}^nC_1 + {}^nC_2 + \ldots + {}^nC_n)$

$= 2^n - 1 + (n - 1)(2^n - 1)$

$= n(2^n - 1)$, as required. ✓

QUESTION 8

(a) (i) α. Number of arrangements when there are no restrictions $= 11!$ ✓

$= 39916800$

β. The males and females are in alternate positions.
Sit a person down. There are $5!$ ways of seating the remaining members of the same sex.
Then there are $6!$ ways of seating the opposite sex.
So the total number of ways $= 5! \times 6!$ ways. ✓

(ii) Two cases:

(1) If one state has two representatives, number of ways $= \binom{6}{4} \times 2^5 = 480$ ✓

(2) If no state has two representatives, number of ways $= 2^6 = 64$ ✓

Hence total number of ways $= 480 + 64 = 544$

(b) (i) A path from S to F must be at least 4 jumps, ✓
so $x_1 = x_2 = x_3 = 0$.
There are two ways to get from S to F with only four jumps: via J or via M, ✓
so $x_4 = 2$.

(ii) After two jumps, the frog is on its original vertex or two vertices away. Repeating this, after an even number of jumps, the frog is an even number of vertices away. ✓
So $x_n = 0$ when n is odd.

(iii) 1 jump from S is J or M, then after 1 more jump, the frog is either at

(a) S or A or (b) S or B

Then for the remaining $(n-2)$ jumps to F, there are:

(a) x_{n-2} or y_{n-2} paths or (b) x_{n-2} or y_{n-2} paths ✓

Thus $x_n = 2x_{n-2} + 2y_{n-2}$ ✓

(iv) 1 jump from A is either K or J, then after 1 more jump, the frog is either at

(a) A or S or (b) A or F

Since we are finding a relation for y_n for $n > 2$, this eliminates the point F.
Then for the remaining $n > 2$ jumps, the frog takes

(a) x_{n-2} or y_{n-2} or (b) y_{n-2} ✓

Thus $y_n = 2y_{n-2} + x_{n-2}$ for $n > 2$ ✓

(v) $x_n = 2x_{n-2} + 2y_{n-2}$ [Eq. 1]

$y_n = 2y_{n-2} + x_{n-2}$ [Eq. 2]

Eq. 1 – Eq. 2:

$x_n - y_n = x_{n-2}$ [Eq. 3]

$2x_{n-2} - 2y_{n-2} = 2x_{n-4}$ [Eq. 4]

Substitute $2y_{n-2}$ from Eq. 1 into Eq. 4:

$2x_{n-2} - (x_n - 2x_{n-2}) = 2x_{n-4}$

$\therefore x_n = 4x_{n-2} - 2x_{n-4}$

Substitute $n = 2m$:

$\therefore x_{2m} = 4x_{2m-2} - 2x_{2m-4}$ ✓

(vi) When $n = 1$,

LHS $= x_2 = 0$

RHS $= \frac{1}{\sqrt{2}}(\alpha^0 - \beta^0) = 0$ ✓

Therefore the formula is true for $n = 1$.

When $n = 2$,

LHS $= x_4 = 2$

RHS $= \frac{1}{\sqrt{2}}(2 + \sqrt{2} - 2 + \sqrt{2}) = 2$

Therefore the formula is true for $n = 2$.

Assume the formula is true for $n = k$ and for $n = k + 1$, where $k \geq 1$.

That is, assume $x_{2k} = \frac{1}{\sqrt{2}}(\alpha^{k-1} - \beta^{k-1})$ and $x_{2k+2} = \frac{1}{\sqrt{2}}(\alpha^k - \beta^k)$.

We need to prove the formula true when $n = k + 2$.

That is, we need to prove $x_{2k+4} = \frac{1}{\sqrt{2}}(\alpha^{k+1} - \beta^{k+1})$ ✓

$$
\begin{aligned}
\text{LHS} &= x_{2k+4} \\
&= 4x_{2k+4} - 2x_{2k} \quad \text{(from (v))} \\
&= \frac{4}{\sqrt{2}}(\alpha^k - \beta^k) - \frac{2}{\sqrt{2}}(\alpha^{k-1} - \beta^{k-1}) \\
&= 2\sqrt{2}(\alpha^k - \beta^k) - \sqrt{2}(\alpha^{k-1} - \beta^{k-1}) \\
&= \sqrt{2}\alpha^{k-1}(2\alpha - 1) - \sqrt{2}\beta^{k-1}(2\beta - 1) \\
&= \sqrt{2}\alpha^{k-1}[2(2+\sqrt{2}) - 1] - \sqrt{2}\beta^{k-1}[2(2-\sqrt{2}) - 1] \\
&= \sqrt{2}\alpha^{k-1}(3 + 2\sqrt{2}) - \sqrt{2}\beta^{k-1}(3 - 2\sqrt{2}) \\
&= \sqrt{2}\alpha^{k-1}\left(\frac{6+4\sqrt{2}}{2}\right) - \sqrt{2}\beta^{k-1}\left(\frac{6-4\sqrt{2}}{2}\right) \\
&= \sqrt{2}\alpha^{k-1}\left(\frac{(2+\sqrt{2})^2}{2}\right) - \sqrt{2}\beta^{k-1}\left(\frac{(2-\sqrt{2})^2}{2}\right) \\
&= \frac{\sqrt{2}\alpha^{k-1}\alpha^2}{2} - \frac{\sqrt{2}\beta^{k-1}\beta^2}{2} \\
&= \frac{1}{\sqrt{2}}(\alpha^{k+1} - \beta^{k+1}) \\
&= \text{RHS} \quad ✓
\end{aligned}
$$

Therefore the formula is true for $n = k + 2$, whenever it is true for $n = k$ and for $n = k + 1$.
So by the principle of mathematical induction, the formula is true for all integers $n \geq 1$.