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HSC Trial Examination 2005

Mathematics Extension 2

This paper must be kept under strict security and may only be used on or after the morning of Monday 8 August, 2005 as specified in the NEAP Examination Timetable.

General Instructions

Reading time 5 minutes.

Working time 3 hours.

Board-approved calculators may be used.

Write using blue or black pen.

A table of standard integrals is provided at the back of this paper.

All necessary working should be shown in every question.

Total marks – 120

Attempt questions 1–8.

All questions are of equal value.

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2005 HSC Mathematics Extension 2 Trial Examination.

Total marks 120
Attempt Questions 1–8
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\displaystyle\int_0^1 \frac{2}{\sqrt{1+3x}}\,dx$. **2**

(b) By using *integration by parts*, find $\displaystyle\int x^2 \ln 2x\,dx$. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \sin^3 2x\,dx$. **3**

(d) Using $t = \tan\frac{x}{2}$, find $\displaystyle\int \frac{dx}{1+\sin x}$. **4**

(e) (i) Find real constants A, B and C such that **2**

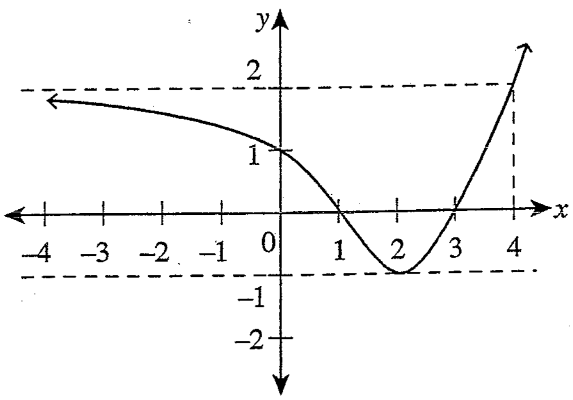
$$\frac{x+4}{x(x^2+4)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

(ii) Hence find $\displaystyle\int \frac{x+4}{x(x^2+4)}\,dx$. **2**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows the graph of $y = f(x)$.



On separate diagrams, sketch the following, showing essential features.

(i) $y = \dfrac{1}{f(x)}$ **2**

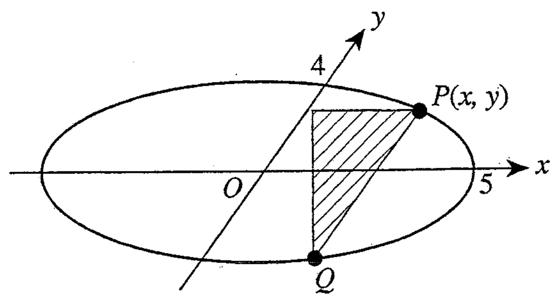
(ii) $y = f(x+2)$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \ln f(x)$ **2**

(b) Find the equation of the tangent to the curve $x^3 + y^3 - 3xy = 3$ at the point (1, 2). **3**

(c)



The base of a certain solid is the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.
Every cross-section perpendicular to the x-axis is an equilateral triangle. The shaded cross-section is thus an equilateral triangle with base PQ.

(i) Show that the shaded cross-sectional area is given by **1**

$$A = \sqrt{3}y^2.$$

(ii) Hence find the cross-sectional area as a function of x. **1**

(iii) Find the volume of the solid. **2**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Show that $(1+i)^3 = 2(i-1)$. **1**

(b) By evaluating, or otherwise, show that $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$ is a real number. **2**

(c) Draw on one Argand diagram the three loci:

(i) $|z-i| = 1$, **3**

$\arg(z-i) = \dfrac{\pi}{3}$,

$|z-i| = |z-3i|$.

(ii) Hence calculate the area of the intersection of the three loci: **1**

$$|z-i| \le 1, \quad \frac{\pi}{3} \le \arg(z-i) \le \frac{\pi}{2} \quad \text{and} \quad |z-i| \le |z-3i|.$$

(d) Let $p(z) = 2z^3 - 5z^2 + qz - 5$, where q is a real number.

(i) If $p(1-2i) = 0$, solve $p(z) = 0$. **2**

(ii) Hence determine the value of q if $p(1-2i) = 0$. **1**

(e) Let z be a complex number such that $z \neq 0$ and $z \neq 1$, and

$$\frac{z}{\bar{z}} = -\frac{z-1}{\bar{z}-1}$$

(i) Show that for any non-zero complex number z, **1**

$$\arg\left(\frac{z}{\bar{z}}\right) = 2\arg z.$$

(ii) Let z be a complex number such that $z \neq 0$ and $z \neq 1$, and **2**

$$\frac{z}{\bar{z}} = -\frac{z-1}{\bar{z}-1}.$$

Show that $\arg z = \arg(z-1) + \dfrac{\pi}{2}$ or $\arg z = \arg(z-1) - \dfrac{\pi}{2}$.

(iii) Hence sketch the locus of all points z that satisfy **2**

$$\frac{z}{\bar{z}} = -\frac{z-1}{\bar{z}-1}.$$

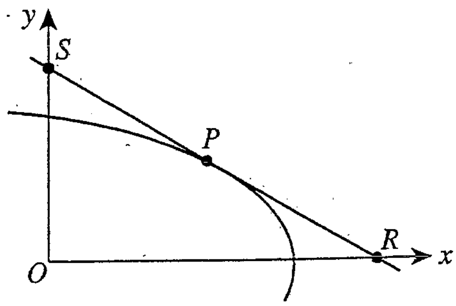
Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The point $P(\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta)$ lies on an ellipse.

(i) Find the equation of the tangent to the ellipse at P, where $0 < \theta < \frac{\pi}{2}$. **2**

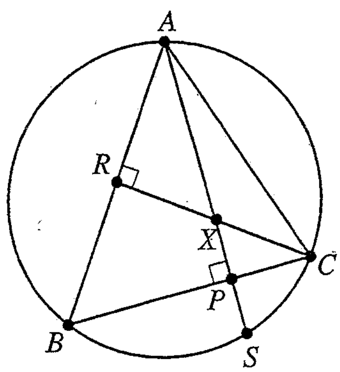
(ii) In the diagram below, the tangent to the ellipse at P intersects the x-axis at R and the y-axis at S.



(α) Show that the area of ΔORS is $\frac{4}{\sin 2\theta}$, where O is the origin. **1**

(β) Find the coordinates of P where this area is a minimum. **2**

(b) In the diagram below, X is the intersection of the altitudes of the triangle ABC. AP produced meets the circumscribed circle at S.



Copy the diagram onto your answer sheet.

(i) Show that $\angle RXA = \angle PSC$. **2**

(ii) Hence, or otherwise, prove that $XP = PS$. **2**

(c) (i) Show that the equation $x^3 + 13x - 16 = 0$ has exactly one real root, $x = \alpha$, and that $1 < \alpha < 2$. **2**

(ii) If $x = \beta$ is one of the non-real roots of the equation in part (i), show that **4**

$$-1 < \text{Re}(\beta) < -\frac{1}{2} \quad \text{and} \quad 2\sqrt{2} < |\beta| < 4.$$

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) Twelve pupils enter a competition. From the twelve pupils, two teams of five pupils are selected to compete against each other.

(i) How many different ways can the two teams be chosen? **2**

(ii) Bill, Paul and Patrick are a set of triplets amongst the twelve pupils. Find the probability that they will be chosen in the same team. **2**

(b) (i) Sketch the region containing all points that simultaneously satisfy the following: **2**

$$x \le 1, \quad y \ge 1 \quad \text{and} \quad y \le e^x.$$

(ii) The region in part (i) is rotated through one complete revolution about the x-axis. Use the method of cylindrical shells to show that the volume of the resulting solid is given by $\frac{\pi}{2}(e^2 - 3)$ cubic units. **4**

(c) A conical pendulum consists of a mass of M kg hanging at the end of a light string of length 1 metre attached from a fixed point O.

The mass rotates in a circle and moves with a period of S seconds.

The string makes a constant angle of θ to the vertical.

(i) Use a sketch to illustrate the forces acting on the mass. **1**

(ii) By resolving the forces acting on the mass, show that $S = 2\pi\sqrt{\frac{\cos\theta}{g}}$, where g is the acceleration due to gravity. **2**

(iii) The string can just support a stationary mass of $5M$ kg hanging vertically. **2**

Find the smallest period that the conical pendulum can have, also leaving your answer in terms of g.

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **2**

(ii) Deduce that $\int_0^{\frac{\pi}{4}} \frac{1-\sin 2x}{1+\sin 2x}dx = \int_0^{\frac{\pi}{4}} \tan^2 x\, dx$. **2**

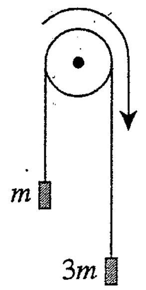
(iii) Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{1-\sin 2x}{1+\sin 2x}dx$. **2**

(b) Particles of mass $3m$ kg and m kg are connected by a light inextensible string which passes over a smooth fixed pulley, the string hanging vertically on each side.

The particles are released from rest and move under gravity.

The air resistance on each particle is kv newtons when the speed of the particle is v m s^{-1}.

Take the positive direction of motion as indicated by the arrow in the diagram below.



Let the tension in the string acting on the masses have a magnitude of T newtons.

(i) By resolving the forces on both particles, show that the equation of motion of the system is given by **2**

$$\frac{dv}{dt} = \frac{mg-kv}{2m}.$$

(ii) Hence find the terminal velocity of the system, stating your answer in terms of m, g and k. **1**

(iii) Prove that the time elapsed since the beginning of the motion is given by **3**

$$t = \frac{2m}{k}\log_e\left(\frac{mg}{mg-kv}\right).$$

(iv) If the bodies have attained a speed equal to half the terminal speed, show that the time elapsed is equal to **3**

$$\frac{V}{g}\log_e 4,$$

where V is the terminal speed.

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) With the aid of a diagram show that $\int_1^{\sqrt{u}} \frac{dx}{x} < \sqrt{u} - 1$ for $u > 1$. **1**

(ii) Hence show that $0 < \ln u < 2(\sqrt{u} - 1)$, for $u > 1$. **2**

(iii) Hence show that $\frac{\log u}{u} \to 0$, as $u \to \infty$. **1**

(b) (i) Show that $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$. **1**

(ii) Let α and β be the roots of the equation $z^2 \sin^2 \theta - z \sin 2\theta + 1 = 0$.

(α) Show that $\alpha + \beta = 2\cos\theta \operatorname{cosec}\theta$. **1**

(β) Show that $\alpha^2 + \beta^2 = 2\cos 2\theta \operatorname{cosec}^2\theta$. **1**

(γ) Hence by mathematical induction, or otherwise, prove that if n is a positive integer then **4**

$$\alpha^n + \beta^n = 2\cos n\theta \operatorname{cosec}^n\theta.$$

(c) Let p and q be non-zero real numbers such that $q(1 + p + q) < 0$.

(i) Show that the equation $x^2 + px + q = 0$ has exactly one real root in the interval $0 < x < 1$. **1**

(ii) Show that the equation $x^2 + px + q = 0$ has two distinct real roots, one positive and one negative. **3**

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) The line $y = 2x + c$ cuts the ellipse $x^2 + \frac{y^2}{16} = 1$ at the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(i) Show that the length of the chord PQ is $\sqrt{5}|x_1 - x_2|$. **2**

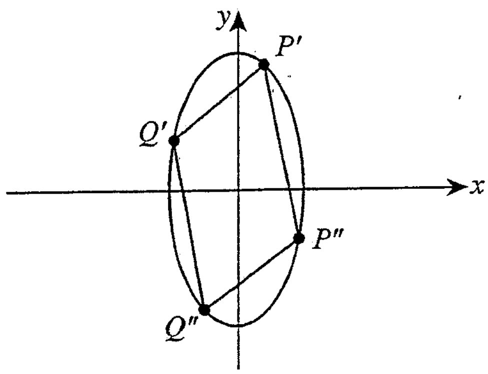
(ii) Show that x_1 and x_2 are the roots of the quadratic equation **4**

$$20x^2 + 4cx + (c^2 - 16) = 0$$

and hence find the two values of c such that the length of the chord PQ is $2\sqrt{2}$.

(iii) Let the two chords found in part (ii) above be $P'Q'$ and $P''Q''$.

$P'Q'Q''P''$ is a parallelogram as shown in the diagram below.



Find the area of the parallelogram $P'Q'Q''P''$. **2**

(b) (i) Show that $\sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$ and find a similar expression for $\cos\frac{\pi}{12}$. **2**

(ii) Expand $(x - iy)^3$. **1**

(iii) Hence or otherwise, find all real numbers x and y satisfying the following: **4**

$$\left.\begin{array}{l} x^3 - 3xy^2 = 1 \\ y^3 - 3x^2y = 1 \end{array}\right\}$$

Leave your answers in surd form.

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

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HSC Trial Examination 2005

Mathematics Extension 2

Solutions and marking guidelines

 PO Box 214 St Leonards NSW 2065 Tel: (02) 9438 1386 Fax: (02) 9438 1385

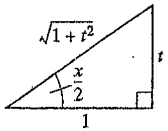
Question 1

	Sample answer	Syllabus outcomes and marking guide
(a)	$\int_0^1 \frac{2}{\sqrt{1+3x}}dx = \int_0^1 2(1+3x)^{-\frac{1}{2}}dx$ $= \left[2\times 2\times \frac{1}{3}(1+3x)^{\frac{1}{2}}\right]_0^1$ $= \frac{4}{3}(\sqrt{4}-\sqrt{1})$ $= \frac{4}{3}$	E8 • Correct solution.................... 2 • Appropriate substitution done correctly. OR • Correct modified primitive. OR • Equivalent merit but fails to get the correct solution.......................... 1
(b)	Let $u' = x^2$ and $v = \ln 2x$ $\int u'v\,dx = uv - \int uv'\,dx$, so $\int x^2 \ln 2x\,dx = \frac{1}{3}x^3\ln 2x - \int \frac{1}{3}x^3 \times \frac{1}{x}dx$ $= \frac{1}{3}x^3\ln 2x - \frac{1}{3}\int x^2 dx$ $= \frac{1}{3}x^3\ln 2x - \frac{1}{9}x^3 + c$	E8 • Correct primitive.................... 2 • Solution demonstrates understanding of the method of integration by parts but fails to get the correct primitive................ 1
(c)	$\sin^3 2x = \sin^2 2x \times \sin 2x = (1-\cos^2 2x)\sin 2x$ Hence $\int_0^{\frac{\pi}{6}} \sin^3 2x\,dx = \int_0^{\frac{\pi}{6}} (1-\cos^2 2x)\sin 2x\,dx$ $= \frac{1}{2}\int_{\frac{1}{2}}^{1} (1-u^2)\,du$ where $u = \cos 2x$ $= \frac{1}{2}\left[u - \frac{u^3}{3}\right]_{\frac{1}{2}}^{1}$ $= \frac{1}{2}\left[\left(1-\frac{1}{3}\right)-\left(\frac{1}{2}-\frac{1}{24}\right)\right]$ $= \frac{5}{48}$	E8 • Correct solution.................... 3 • Appropriate substitution done correctly. OR • Correct modified primitive. OR • Equivalent merit but fails to get the correct solution.......................... 2 • Recognises that $\sin^3 2x = (1-\cos^2 2x)\sin 2x$.......... 1

Question 1 (Continued)

Sample answer	Syllabus outcomes and marking guide
(d)	E8

(d) Let $t = \tan\frac{x}{2}$



Then $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

so $dx = \frac{2}{1+t^2}dt$

Also $1+\sin x = 1 + \frac{2t}{1+t^2}$

$= \frac{1+2t+t^2}{1+t^2}$

$= \frac{(1+t)^2}{1+t^2}$

Hence $\int \frac{dx}{1+\sin x} = \int \frac{1+t^2}{(1+t)^2} \times \frac{2dt}{1+t^2}$

$= 2\int (1+t)^{-2}dt$

$= 2 \times \frac{-1}{1+t} + c = \frac{-2}{1+t} + c$

$= \frac{-2}{1+\tan\frac{x}{2}} + c$

Syllabus outcomes and marking guide: E8

- Correct primitive. 4

- Appropriate substitution done correctly.
  OR
- Correct modified primitive.
  OR
- Equivalent merit but fails to get the correct primitive. 3

- Recognises that $1+\sin x = \frac{(1+t)^2}{1+t^2}$ and that $dx = \frac{2dt}{1+t^2}$ or equivalent merit 2

- Recognises that $1+\sin x = \frac{(1+t)^2}{1+t^2}$ or that $dx = \frac{2dt}{1+t^2}$. 1

(e) (i) $x+4 \equiv A(x^2+4) + x(Bx+C)$

Put $x = 0$

Then $4 = 4A$

so $A = 1$

Equating coefficients of x^2, $A+B=0$

so $B=-1$

Equating coefficients of x, $C=1$

Hence $A=1$, $B=-1$ and $C=1$

Syllabus outcomes and marking guide: E8

- Correct solutions. 2

- Applies method of partial fractions correctly but fails to get **all** the correct answers. . . 1

(ii) $\int \frac{x+4}{x(x^2+4)}dx = \int \left(\frac{1}{x} + \frac{-x+1}{x^2+4}\right)dx$

$= \int \frac{1}{x}dx - \frac{1}{2}\int \frac{2x}{x^2+4}dx + \int \frac{1}{x^2+4}dx$

$= \ln x - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

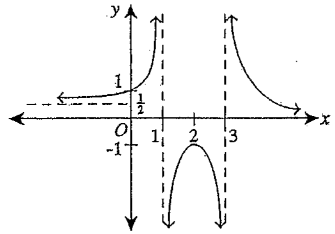
Syllabus outcomes and marking guide: E8

- Correct primitive. 2

- Makes reasonable progress after making the link to part (i) but fails to get the correct primitive. 1

Question 2

(a) (i)

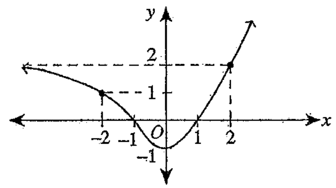


Syllabus outcomes and marking guide: E6

- Correct graph. 2

- Correct shape, but asymptotes and intercepts are not indicated or are incorrect.
  OR
- Equivalent merit. 1

(ii)

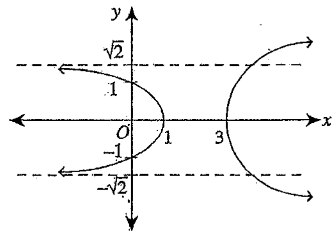


Syllabus outcomes and marking guide: E6

- Correct graph. 2

- Correct shape, but asymptotes and intercepts are not indicated or are incorrect.
  OR
- Equivalent merit. 1

(iii)



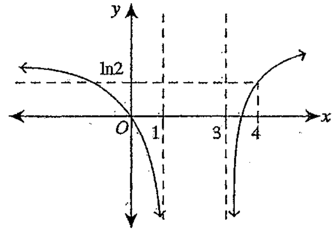
Syllabus outcomes and marking guide: E6

- Correct graph. 2

- Correct shape, but asymptotes and intercepts are not indicated or are incorrect.
  OR
- Equivalent merit. 1

Note: the concavity of the curve to the right of about $x = 4$ cannot be established.

(iv)

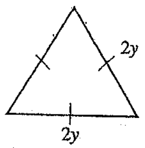


Syllabus outcomes and marking guide: E6

- Correct graph. 2

- Correct shape, but asymptotes and intercepts are not indicated or are incorrect.
  OR
- Equivalent merit. 1

Question 2 (Continued)

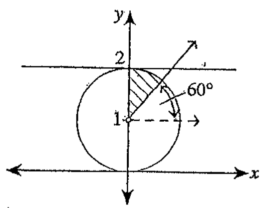
Sample answer	Syllabus outcomes and marking guide
(b) Given $x^3 + y^3 - 3xy = 3$. Differentiating both sides with respect to x, $3x^2 + 3y^2\frac{dy}{dx} - \left(3x\frac{dy}{dx} + 3y\right) = 0$ $\therefore \frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$ $\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$ At the point (1, 2) $\frac{dy}{dx} = \frac{6-3}{12-3}$ $= \frac{3}{9}$ $= \frac{1}{3}$ Hence the equation of the tangent is: $\frac{y-2}{x-1} = \frac{1}{3}$ $3y - 6 = x - 1$ $x - 3y + 5 = 0$	E6 • Correct answer. . . . . 3 • Finds the correct derivative and finds a linear equation . . . . 2 • Finds an equation with two basic errors . 1
(c) (i) The base of the cross-section is $2y$. Hence $A = \frac{1}{2} \times (2y)^2 \times \sin 60°$ $= \frac{1}{2} \times 4y^2 \times \frac{\sqrt{3}}{2}$ $= \sqrt{3}y^2$  	E7 • Correct answer. 1
(ii) Hence $A = \frac{16\sqrt{3}}{25}(25 - x^2)$ Solving the ellipse for y, $\frac{x^2}{25} + \frac{y^2}{16} = 1$ $\frac{y^2}{16} = 1 - \frac{x^2}{25}$ $\frac{y^2}{16} = \frac{25 - x^2}{25}$ $\therefore y^2 = \frac{16}{25}(25 - x^2)$	E7 • Finds the correct expression. 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

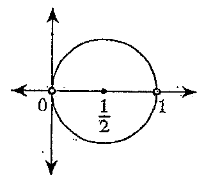
Question 2 (Continued)

Sample answer	Syllabus outcomes and marking guide
(iii) $V = \int_{-5}^{5} \frac{16\sqrt{3}}{25}(25 - x^2)\,dx$ $= \frac{32\sqrt{3}}{25}\int_{0}^{5}(25 - x^2)\,dx$ $= \frac{32\sqrt{3}}{25}\left[25x - \frac{x^3}{3}\right]_0^5$ $= \frac{32\sqrt{3}}{25}\left(125 - \frac{125}{3}\right)$ $= \frac{32\sqrt{3}}{25} \times 2 \times \frac{125}{3}$ $= \frac{320\sqrt{3}}{3}\text{u}^3$	E7 • Correct answer. . . . . 2 • Correct method with no more than one error. 1

Question 3

	Sample answer	Syllabus outcomes and marking guide
(a)	$(1+i)^3=(1+i)(1+i)^2$ $=(1+i)(2i)$ $=2(i-1)$, since $i^2=-1$	E3 • Correctly shows the result. 1
(b)	$\frac{1+2i}{3-4i}\times\frac{3+4i}{3+4i}+\frac{2-i}{5i}\times\frac{-i}{-i}=\frac{(1+2i)(3+4i)}{25}-\frac{1+2i}{5}$ $=\frac{1+2i}{25}(3+4i-5)$ $=\frac{1+2i}{25}(-2+4i)$ $=\frac{2(1+2i)(-1+2i)}{25}$ $=\frac{2\times -5}{25}$ $=-\frac{2}{5}$, which is a real number	E3 • Correctly shows the result. . . . . 2 • Substantially correct. 1
(c) (i)	 	E3 • Correct diagrams. . . . . 3 • One diagram incorrect. . . . . 2 • Two diagrams wrong. 1
(ii)	area $=\frac{1}{2}\times 1\times\frac{\pi}{6}$ $=\frac{\pi}{12}$ units^2	E3 • Correct solution. 1
(d) (i)	$p(z)=2z^3-5z^2+qz-5$ Since $p(1-2i)=0$, then $p(1+2i)=0$ [conjugate root theorem] Hence $1+2i+1-2i+\alpha=\frac{5}{2}$ [sum of roots] $\alpha=\frac{1}{2}$ Hence the solutions of $p(z)=0$ are $z=1+2i, 1-2i$ or $\frac{1}{2}$.	E4 • Correct solutions. . . . . 2 • Obtains the conjugate solution $1+2i$. . . . 1
(ii)	$p\left(\frac{1}{2}\right)=0$ from part (i). Hence $2\left(\frac{1}{2}\right)^3-5\left(\frac{1}{2}\right)^2+q\times\frac{1}{2}-5=0$ $\therefore \frac{1}{4}-\frac{5}{4}+\frac{q}{2}-5=0$ $\therefore \frac{q}{2}=6$ $\therefore q=12$	E4 • Correct value for q. 1

Question 3 (Continued)

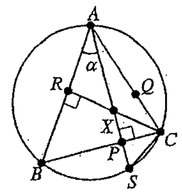
	Sample answer	Syllabus outcomes and marking guide
(e) (i)	$\arg\left(\frac{z}{\bar{z}}\right)=\arg z-\arg(\bar{z})$ $=\arg z-(-\arg z)$ $=2\arg z$	E3 • Correctly shows expression. 1
(ii)	Given $\frac{z}{\bar{z}}=-\frac{z-1}{\overline{z-1}}$ $\arg\left(\frac{z}{\bar{z}}\right)=\arg\left(-\frac{z-1}{\overline{z-1}}\right)$ $=\arg(-1)+\arg\left(\frac{z-1}{\overline{z-1}}\right)$ $2\arg z=\pi+2\arg(z-1)$ or $2\arg z=-\pi+2\arg(z-1)$ $\arg z=\arg(z-1)+\frac{\pi}{2}$ or $\arg z=\arg(z-1)-\frac{\pi}{2}$	E3 • Correctly shows both expressions . . . . . . 2 • Makes substantial progress. OR • Shows a link with (i) 1
(iii)	$\arg z-\arg(z-1)=\frac{\pi}{2}$ or $-\frac{\pi}{2}$  By the converse of the angle-in-a-semicircle theorem, this is the circle whose diameter is the interval joining $z=0$ and $z=1$. The points $z=0$ and $z=1$, however, are secluded, because $\arg 0$ is undefined.	E3 • Correct diagram of the locus. . . . . 2 • Draws a semicircle. OR • Substantial progress. OR • Fails to exclude the points $z=0$, 1 from the correct diagram. 1

Question 4

Sample answer	Syllabus outcomes and marking guide
(a) (i) $x=\sqrt{2}\cos\theta$ and $y=2\sqrt{2}\sin\theta$ Differentiating, $\frac{dx}{d\theta}=-\sqrt{2}\sin\theta$ and $\frac{dy}{d\theta}=2\sqrt{2}\cos\theta$ Hence $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$ $=2\sqrt{2}\cos\theta\times\frac{1}{-\sqrt{2}\sin\theta}$ $=\frac{-2\cos\theta}{\sin\theta}$ ∴ Equation of the tangent is given by: $y-2\sqrt{2}\sin\theta=\frac{-2}{\tan\theta}(x-\sqrt{2}\cos\theta)$ $\sin\theta(y-2\sqrt{2}\sin\theta)=-2\cos\theta(x-\sqrt{2}\cos\theta)$ $y\sin\theta-2\sqrt{2}\sin^2\theta=-2x\cos\theta+2\sqrt{2}\cos^2\theta$ $2x\cos\theta+y\sin\theta=2\sqrt{2}(\sin^2\theta+\cos^2\theta)$ $2x\cos\theta+y\sin\theta=2\sqrt{2}$ $\frac{x\cos\theta}{\sqrt{2}}+\frac{y\sin\theta}{2\sqrt{2}}=1$	E4 • Any form of the equation. . . . . 2 • Makes substantial progress towards the equation. 1
(ii) (α) S is the y-intercept, i.e. $\left(0,\frac{2\sqrt{2}}{\sin\theta}\right)$, and R is the x-intercept, i.e. $\left(\frac{\sqrt{2}}{\cos\theta},0\right)$. Area of $\Delta ORS=\frac{1}{2}\times\frac{\sqrt{2}}{\cos\theta}\times\frac{2\sqrt{2}}{\sin\theta}$ $=\frac{4}{2\sin\theta\cos\theta}$ $=\frac{4}{\sin2\theta}$	E4 • Correct solution. 1
(β) The minimum of the expression $\frac{4}{\sin2\theta}$ occurs when $\sin2\theta=1$, where $0<\theta<\frac{\pi}{2}$ That is, $2\theta=\frac{\pi}{2}$ $\theta=\frac{\pi}{4}$ Hence the area is a maximum when $P=\left(\sqrt{2}\cos\frac{\pi}{4},2\sqrt{2}\sin\theta\right)$ $=\left(\sqrt{2}\times\frac{1}{\sqrt{2}},2\sqrt{2}\times\frac{1}{\sqrt{2}}\right)$ $=(1,2)$	E4 • Correct solution. 2 • Correctly obtains the area but unable to find P. 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

Question 4 (Continued)

Sample answer	Syllabus outcomes and marking guide
(b) (i)  Let $\angle BAS=\alpha$ Then $\angle BCS=\alpha$ (angles on the same are BS) Hence $\angle RXA=\angle PSC=90°-\alpha$ (angle sums of ΔRXA and ΔPSC.	PE3 • Valid proof. . . . . 2 • Shows $\angle PCS=\angle RAX$. 1
(ii) $\angle CXP=90°-\alpha$ (vertically opposite angles) Hence $\Delta XPC\equiv\Delta SPC$ (AAS congruent test) so $XP=SP$ (matching sides of congruent triangles) $\angle RXA=\angle PXC$ (vertically opposite angles are equal). $\therefore\angle PXC=\angle PSC$ $\therefore\Delta XPC\equiv\Delta SPC$ (AAS test) $\therefore XP=PS$ (corresponding sides of congruent triangles)	PE3 • Valid proof. . . . . 2 • Substantially correct. 1
(c) (i) Let $P(x)=x^3+13x-16=0$ $P'(x)=3x^2+13$ Now $P'(x)>0$ for all x. ∴ It is always increasing. ∴ $P(x)=0$ has only one root. $P(1)=-2$ and $P(2)=8+26-16$ $=18$ ∴ The root α lies in the interval (1, 2), i.e. $1<\alpha<2$.	E4 • Valid proof. . . . . 2 • Reasonable approach, for example, finds $P(1)$ and $P(2)$ or shows $P'(x)>0$. 1

Question 4 **(Continued)**

Sample answer

(ii) If β is a non-real root of $P(x) = 0$,
then by the conjugate root theorem
$\overline{\beta}$ is also a root of $P(x) = 0$.

Now $\alpha + \beta + \overline{\beta} = 0$ (sum of roots)

$\alpha + 2\text{Re}(\beta) = 0$ (since $\beta + \overline{\beta} = 2\text{Re}(\beta)$)

$\text{Re}(\beta) = -\frac{\alpha}{2}$

Now $1 < \alpha < 2$

so $\frac{1}{2} < \frac{\alpha}{2} < 1$,

$-\frac{1}{2} > -\frac{\alpha}{2} > -1$

or $-1 < -\frac{\alpha}{2} < -\frac{1}{2}$

Hence $-1 < \text{Re}(\beta) < -\frac{1}{2}$

Also $\alpha \times \beta \times \overline{\beta} = 16$ (product of roots)

$\therefore \alpha|\beta|^2 = 16$ (since $\beta\overline{\beta} = |\beta|^2$)

$\alpha = \frac{16}{|\beta|^2}$

Now $1 < \alpha < 2$

so $1 < \frac{16}{|\beta|^2} < 2$

$\frac{1}{16} < \frac{1}{|\beta|^2} < \frac{1}{8}$

$16 > |\beta|^2 > 8$

$8 < |\beta|^2 < 16$

$2\sqrt{2} < |\beta| < 4$

Syllabus outcomes and marking guide

E4
- Is able to obtain **both** expressions correctly. 4
- Obtains **one** correctly and makes progress in the other. 3
- **One** answer correct. 2
- Some progress. 1

Question 5

Sample answer

(a) (i) Choose the first 5 in $\binom{12}{5}$ ways,

then the remaining 5 in $\binom{7}{5}$ ways.

But we have overcounted by a factor of 2, since the two terms could be chosen in either order.

Hence number of ways $= \frac{1}{2} \times \binom{12}{5} \times \binom{7}{5}$

$= 8316$

Syllabus outcomes and marking guide

PE3
- Correct answer. 2
- Substantial progress. 1

Sample answer

(ii) If Bill, Paul and Patrick are in the same team, then the remainder of their team can be chosen in $\binom{9}{2}$ ways, the other team in $\binom{7}{5}$ ways.

Hence number of ways $= \binom{9}{2} \times \binom{7}{5}$

$= 756$

Hence probability $= \frac{756}{8316} = \frac{1}{11}$

Alternative solution:

P(Bill is chosen) $= \frac{5}{6}$

If Bill is chosen, P(Paul is in same team) $= \frac{4}{11}$

If Bill and Paul are in the team,
P(Patrick is in same team) $= \frac{3}{10}$

Hence probability $= \frac{5}{6} \times \frac{4}{11} \times \frac{3}{10}$

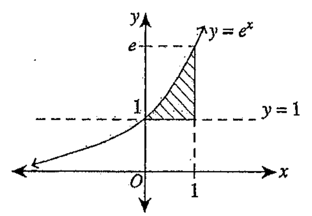
$= \frac{1}{11}$

Syllabus outcomes and marking guide

PE3
- Correct answer. 2
- Substantial progress. 1

Sample answer

(b) (i)



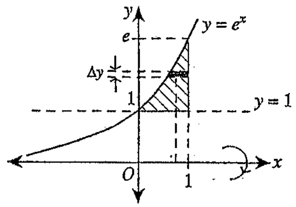
Syllabus outcomes and marking guide

E7
- Correct region sketched. 2
- Substantially correct with one error. 1

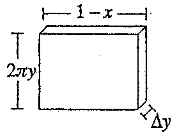
Question 5 (Continued)

Sample answer

(ii)



A shell of thickness Δy, height $1 - x$, radius y can be approximated by the rectangular prism below.



Let ΔV be the volume of this shell.

$$\therefore \Delta V \approx 2\pi y(1 - x)\Delta y$$

$$V = \lim_{\Delta y \to 0} \sum_{y=1}^{e} 2\pi y(1 - x)\Delta y$$

Hence $V = 2\pi \int_1^e y(1 - x)\,dy$

$$= 2\pi \int_1^e y(1 - \ln y)\,dy$$

[use integration by parts]

$$= 2\pi \left[\frac{y^2}{2}(1 - \ln y)\right]_1^e - \int_1^e \frac{y^2}{2} \times \frac{-1}{y}\,dy$$

$$= 2\pi\left[-\frac{1}{2} + \int_1^e \frac{y}{2}\,dy\right]$$

$$= 2\pi\left[-\frac{1}{2} + \frac{e^2}{4} - \frac{1}{4}\right]$$

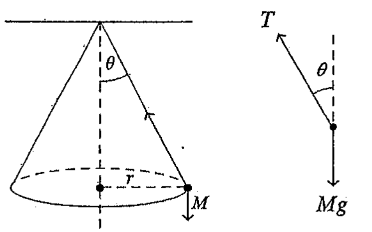
$$= \frac{\pi}{2}(e^2 - 3), \text{ as required}$$

Syllabus outcomes and marking guide

E7

- Correct solution 4
- Shows that $V = 2\pi \int_1^e y(1 - \ln y)\,dy$ or equivalent 3
- Writes down an integral for the volume using the correct height and radius 2
- Finds the height or radius of the shell 1

(c) (i)



T = tension in string, Mg = weight due to gravity

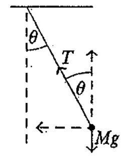
E5

- Correct diagram with all the forces 1

Question 5 (Continued)

Sample answer

(ii) Let the radius be r metres, and the angular velocity be ω radians per second.



Resolving vertically:

$T\cos\theta - Mg = 0$

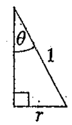
$T\cos\theta = Mg$ ...(1)

Resolving horizontally:

$T\sin\theta = M\omega^2 r$ ...(2)

(2) ÷ (1) $\quad \tan\theta = \dfrac{\omega^2 r}{g}$

$\left[\omega = \dfrac{2\pi}{S}\right] \quad \tan\theta = \dfrac{4\pi^2 \sin\theta}{gS^2}$



$[r = \sin\theta] \quad S^2 = \dfrac{4\pi^2 \sin\theta}{g\tan\theta}$

$$= \frac{4\pi^2 \cos\theta}{g}$$

$\therefore S = 2\pi\sqrt{\dfrac{\cos\theta}{g}}$ since S is positive

Syllabus outcomes and marking guide

E5

- Correct expression for S 2
- Attempts to resolve forces or equivalent merit 1

(iii) The smallest period will occur when $T = 5Mg$.

Hence from part (ii), $T\cos\theta = Mg$

$T \le 5Mg$ newtons

$\therefore T\cos\theta = Mg$ [from part (ii)]

$\therefore Mg \;\; T\cos\theta \le 5Mg\cos\theta$

i.e. $Mg \le 5Mg\cos\theta$

$\therefore \cos\theta \ge \dfrac{1}{5} \qquad \left[\dfrac{1}{5} \le \cos\theta < 1\right]$

From part (ii) $S = 2\pi\sqrt{\dfrac{\cos\theta}{g}}$

Hence the smallest period is

$$S = 2\pi\sqrt{\frac{1}{5g}}$$

$$= \frac{2\pi}{\sqrt{5g}} \text{ seconds}$$

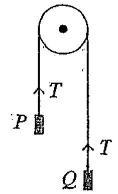
E5

- Correct solution 2
- Establishes that $\cos\theta = \frac{1}{5}$ or equivalent merit 1

Question 6

Sample answer	Syllabus outcomes and marking guide
(a) (i) We are required to show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ Let $u = a - x$ Then $du = -dx$ Hence $\int_0^a f(a-x)dx = -\int_a^0 f(u)du$ $= \int_0^a f(u)du$ $= \int_0^a f(x)dx$, as required	E8 • Correctly shown. .... 2 • Substantially correct. 1
(ii) $\int_0^{\frac{\pi}{4}} \frac{1-\sin 2x}{1+\sin 2x}dx = \int_0^{\frac{\pi}{4}} \frac{1-\sin 2\left(\frac{\pi}{4}-x\right)}{1+\sin 2\left(\frac{\pi}{4}-x\right)}dx$ $= \int_0^{\frac{\pi}{4}} \frac{1-\sin\left(\frac{\pi}{2}-2x\right)}{1+\sin\left(\frac{\pi}{2}-2x\right)}dx$ $= \int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{1+\cos 2x}dx$ $= \int_0^{\frac{\pi}{4}} \frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)}dx$ $= \int_0^{\frac{\pi}{4}} \frac{2\sin^2 x}{2\cos^2 x}dx$ $= \int_0^{\frac{\pi}{4}} \tan^2 x\,dx$	E8 • Correct answer. .... 2 • Applies the result from part (i). 1
(iii) $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)dx$ $= [\tan x - x]_0^{\frac{\pi}{4}}$ $= \tan\frac{\pi}{4} - \frac{\pi}{4} - (\tan 0 - 0)$ $= 1 - \frac{\pi}{4}$	E8 • Correctly evaluates the integral. .... 2 • Correctly substitutes into a primitive. OR • Correct primitive and a mistake in the substitution. 1

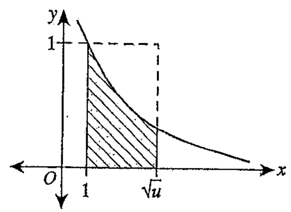
Question 6 (Continued)

Sample answer	Syllabus outcomes and marking guide
(b) (i) At P, $m\ddot{x} = T - kv - mg$ ...(1) At Q, $3m\ddot{x} = -T + 3mg - kv$ ...(2) Adding (1) and (2): $4m\ddot{x} = 2mg - 2kv$ $\ddot{x} = \frac{mg-kv}{2m}$ $\frac{dv}{dt} = \frac{mg-kv}{2m}$ 	E5 • Correct expression. .... 2 • Obtains the correct forces at P and Q. ... 1
(ii) Let the terminal velocity be v_T For terminal velocity: Let $\frac{dv}{dt} \to 0$ i.e. $\frac{mg-kv}{2m} = 0$ $mg = kv$ $v_T = \frac{mg}{k}$ (where $v_T = V$) $\therefore V = \frac{mg}{k}$	E5 • Correct answer. 1
(iii) From (1): $\frac{dt}{dv} = \frac{2m}{mg-kv}$ $t = \frac{-2m}{k}\ln(mg-kv) + c$ When $t = 0$, $v = 0$. $\therefore 0 = \frac{-2m}{k}\ln mg + c$ $c = \frac{2m}{k}\ln mg$ $t = \frac{-2m}{k}\ln(mg-kv) + \frac{2m}{k}\ln mg$ $t = \frac{2m}{k}\ln\frac{mg}{mg-kv}$	E5 • Correct expression. .... 3 • Substantially correct. .... 2 • Makes some progress. 1

Question 6 (Continued)

Sample answer	Syllabus outcomes and marking guide
(iv) Put v equal to half the terminal velocity. That is, $v = \frac{mg}{2k}$. Then $t = \frac{2m}{k}\ln\frac{mg}{mg - k \times \frac{mg}{2k}}$ $= \frac{2m}{k}\ln\frac{mg}{mg - \frac{1}{2}mg}$ $= \frac{2m}{k}\ln 2$ $= \frac{2v_T}{g}\ln 2$ $= \frac{v}{g}\ln 4$	E5 • Correct answer. .... 3 • Substantially correct. .... 2 • Makes some progress. 1

Question 7

Sample answer	Syllabus outcomes and marking guide
(a) (i)  $\int_1^{\sqrt{u}} \frac{dx}{x}$ = shaded area < area of upper rectangle at $x = 1$ $= (\sqrt{u} - 1) \times 1$ $= \sqrt{u} - 1$	PE3, H8 • Correctly shows the inequality. 1
(ii) Clearly $\int_1^{\sqrt{u}} \frac{dx}{x} > 0$, since the curve is above the x-axis. Hence $0 < \int_1^{\sqrt{u}} \frac{dx}{x} < \sqrt{u} - 1$ $0 < [\ln x]_1^{\sqrt{u}} < \sqrt{u} - 1$ $0 < \ln(\sqrt{u}) < \sqrt{u} - 1$ $\quad [\ln\sqrt{u} = \frac{1}{2}\ln u]$ $0 < \frac{1}{2}\ln u < \sqrt{u} - 1$ $0 < \ln u < 2(\sqrt{u} - 1)$, as required	PE3, H8 • Correctly shows the inequality. .... 2 • Substantial progress linking from part (i) but fails to complete the proof.............. 1
(iii) Since $0 < \ln u < 2(\sqrt{u} - 1)$, $\therefore 0 < \frac{\log u}{u} < \frac{2(\sqrt{u} - 1)}{u}$. Now $\frac{2(\sqrt{u} - 1)}{u} \to 0$ as $u \to \infty$. Hence $\frac{\log u}{u} \to 0$ as $v \to \infty$. That is $\lim_{u \to \infty} \frac{\log u}{u} = 0$.	PE3, H8 • Correctly shows the limit.............. 1
(b) (i) $\cos(A + B) + \cos(A - B)$ $= \cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B$ $= 2\cos A\cos B$	PE3 • Correctly shows expression............ 1

Question 7 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(ii) (α) $\alpha+\beta=\dfrac{\sin 2\theta}{\sin^2\theta}$ $=\dfrac{2\sin\theta\cos\theta}{\sin^2\theta}$ $=\dfrac{2\cos\theta}{\sin\theta}$ $=2\cos\theta\operatorname{cosec}\theta$	E4 • A correct expression for $\alpha+\beta$. 1
(β) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ $=(2\cos\theta\operatorname{cosec}\theta)^2-2\operatorname{cosec}^2\theta$ $=2\cos^2\theta\operatorname{cosec}^2\theta-2\operatorname{cosec}^2\theta$ $=(2\cos^2\theta-1)\operatorname{cosec}^2\theta$ $=\cos 2\theta\operatorname{cosec}^2\theta$	E4 • A correct expression for $\alpha^2+\beta^2$. 1
(γ) From (ii) and (iii), the formula is true for $n=1$ and $n=2$. Assume it is true for all n in the interval $2<n\le k$. That is, $\alpha^k+\beta^k=2\cos k\theta\operatorname{cosec}^k\theta$ for $2<n\le k$. e now have the result true for $n=k+1$. That is, we have $\alpha^{k+1}+\beta^{k+1}=2\cos(k+1)\theta\operatorname{cosec}^{k+1}\theta$ Multiply both sides of $z^2\sin^2\theta-z\sin 2\theta+1=0$ by z^{k-1}, substitute α and β, and then add. $(\alpha^{k+1}+\beta^{k+1})\sin^2\theta-(\alpha^k+\beta^k)\sin 2\theta+\alpha^{k-1}+\beta^{k-1}=0$ so $(\alpha^{k+1}+\beta^{k+1})\sin^2\theta$ $=(2\cos k\theta\operatorname{cosec}^k\theta)\sin 2\theta-2\cos(k-1)\theta\operatorname{cosec}^{k-1}\theta$ $=4\cos k\theta\cos\theta\operatorname{cosec}^{k-1}\theta-2\cos(k-1)\theta\operatorname{cosec}^{k-1}\theta$ so $\alpha^{k+1}+\beta^{k+1}$ $=2\operatorname{cosec}^{k+1}\theta[2\cos k\theta\cos\theta-\cos(k-1)\theta]$ $=2\operatorname{cosec}^{k+1}\theta[\cos(k+1)\theta+\cos(k-1)\theta$ $-\cos(k-1)\theta]$ from part (i) $=2\operatorname{cosec}^{k+1}\theta\cos(k+1)\theta$ Hence the formula is true for $n=k+1$. So by the principle of mathematical induction $\alpha^n+\beta^n=2\operatorname{cosec}^n\theta\cos n\theta$ for integers $n\ge 1$	HE2, E4 • Proves the inductive step. .... 4 • Makes considerable progress in proving the inductive step, but fails to finish. .... 3 • Shows the identity $(\alpha^{k+1}+\beta^{k+1})\sin^2\theta$ $-(\alpha^k+\beta^k)\sin 2\theta+(\alpha^{k+1}+\beta^{k+1})=0$ .... 2 • Attempts to relate the case $n=k+1$ to the case $n=k$. OR • multiplies by z^{k-1} but fails to continue... 1
(c) (i) Let $u(x)=x^2+px+q$ Then $u(0)\times u(1)=q(1+p+q)$ <0 Hence $u(x)$, being a quadratic, has exactly one zero between 0 and 1.	E4 • Correctly shows the result. 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

Question 7 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(ii) Let $v(x)=\dfrac{1}{x+2}+\dfrac{p}{x+1}+\dfrac{q}{x}$, where $x\ne 0$, $x\ne -1$ and $x\ne -2$. Put $v(x)=0$. Then $x(x+1)+px(x+2)+q(x+1)(x+2)=0$ $\therefore x^2(1+p+q)+x(1+2p+3q)+2q=0$ Now product of roots $=\dfrac{2q}{1+p+q}$ $=\dfrac{2}{(1+p+q)^2}\times q(1+p+q)$ <0, since $q(1+p+q)<0$ Hence $v(x)$ has two roots, one positive and one negative. Note: Once the product is negative, there is no need to prove separately that the roots are distinct. (Note also that if the roots are complex, then they are conjugates, and $(a+ib)(a-ib)=a^2+b^2>0$.)	E4 • Correctly shows both results. .... 3 • Correctly shows one result and substantial progress with the other. .... 2 • Correct expression for the discriminant or equivalent merit. 1

Question 8

Sample answer / **Syllabus outcomes and marking guide**

(a) (i)

$$PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (x_1 - x_2)^2 + [(2x_1 + c) - (2x_2 + c)]^2$$
$$= (x_1 - x_2)^2 + [2(x_1 - x_2)]^2$$
$$= 5(x_1 - x_2)^2$$

Hence $PQ = \sqrt{5} \times |x_1 - x_2|$

E4
- Correct answer. 2
- Substantially correct.
  OR
  A correct approach. 1

(ii) $16x^2 + y^2 = 16$ ... (1)

$y = 2x + c$ ... (2)

Substituting (2) into (1),

$$16x^2 + 4x^2 + 4cx + c^2 = 6$$
$$20x^2 + 4cx + c^2 - 16 = 0$$

Hence $x_1 + x_2 = -\frac{4c}{20}$

$$= -\frac{c}{5}$$

and $x_1 x_2 = \frac{c^2 - 16}{20}$

Hence $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2$

$$= \frac{c^2}{25} - \frac{c^2 - 16}{5}$$
$$= \frac{c^2 - 5c^2 - 80}{25}$$
$$= \frac{4(20 - c^2)}{25}$$

Put $PQ = 2\sqrt{2}$

$$PQ^2 = 8$$
$$5 \times \frac{4(20 - c^2)}{25} = 8$$
$$20 - c^2 = 10$$
$$10 = c^2$$
$$c = \sqrt{10} \text{ or } -\sqrt{10}$$

Alternative solution:

To find x_1 and x_2, solve simultaneously

$x^2 + \frac{y^2}{16} = 1$ ...(1)

and $y = 2x + c$ ...(2)

Substitute (2) into (1):

$$x^2 + \frac{(2x + c)^2}{16} = 1$$
$$16x^2 + 4x^2 + 4xc + c^2 = 16$$

Giving $20x^2 + 4xc + (c^2 - 16) = 0$ ...(A)

E4
- Correct solution.. 4
- Able to obtain the equation but not the correct values of c. 3
- Able to make substantial progress 2
- Able to make some progress 1

Question 8 **(Continued)**

Sample answer / **Syllabus outcomes and marking guide**

Solving (A) gives x_1 and x_2:

$$x = \frac{-4c \pm \sqrt{16c^2 - 80(c^2 - 16)}}{40}$$

$$|x_1 - x_2| = \left| \frac{-4c + \sqrt{1280 - 64c^2}}{40} - \frac{-4c - \sqrt{1280 - 64c^2}}{40} \right|$$

$$= \left| \frac{\sqrt{1280 - 64c^2}}{20} \right|$$

Let $\sqrt{5} \times \frac{\sqrt{1280 - 64c^2}}{20} = 2\sqrt{2}$

$$\sqrt{5} \times \frac{8\sqrt{20 - c^2}}{20} = 2\sqrt{2}$$
$$\sqrt{5} \times \sqrt{20 - c^2} = 5\sqrt{2}$$
$$5(20 - c^2) = 50$$
$$20 - c^2 = 10$$
$$c^2 = 10$$
$$c = \pm\sqrt{10}$$

(iii) The area of the parallelogram $P'Q'Q''P''$ is given by lb where $l = P'Q'$ and b is the perpendicular distance between $P'Q'$ and $Q''P''$.

Now, $l = 2\sqrt{2}$ (given)

and $b = \left| \frac{2 \times 0 - 1 \times -\sqrt{10} - \sqrt{10}}{\sqrt{(2)^2 + (1)^2}} \right|$

$$= \frac{2\sqrt{10}}{\sqrt{5}}$$
$$= 2\sqrt{2}$$

$\therefore$ Area $= (2\sqrt{2})^2$

$$= 8 \text{ m}^2$$

E4
- Correct answer. 2
- Uses a valid method. 1

Question 8 (Continued)

Sample answer	Syllabus outcomes and marking guide
(b) (i) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ $= \sin\frac{\pi}{3} \times \cos\frac{\pi}{4} - \sin\frac{\pi}{4} \times \cos\frac{\pi}{3}$ $= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$ $= \frac{\sqrt{3}-1}{2\sqrt{2}}$ $= \frac{\sqrt{6}-\sqrt{2}}{4}$ Similarly $\cos\frac{\pi}{12} = \cos\frac{\pi}{3} \times \cos\frac{\pi}{4} - \sin\frac{\pi}{3} \times \sin\frac{\pi}{4}$ $= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$ $= \frac{1+\sqrt{3}}{2\sqrt{2}}$ $= \frac{\sqrt{2}+\sqrt{6}}{2}$	PE2 • Correct expressions. . . . . 2 • One correct. 1
(ii) $(x - iy)^3 = x^3 - 3x^2(iy) + 3x(iy)^2 - (iy)^3$ $= x^3 - 3ix^2y - 3xy^2 + iy^3$ $= x^3 - 3xy^2 + i(y^3 - 3x^2y)$	E3 • Correct expansion. 1

Question 8 (Continued)

Sample answer	Syllabus outcomes and marking guide
(iii) By part (ii), $(x - iy)^3 = x^3 - 3y^2 + i(y^3 - 3x^2y)$ $= 1 + i$, by the given identities, $= \sqrt{2}\text{cis}\frac{\pi}{4},\ \sqrt{2}\text{cis}\frac{9\pi}{4}$ or $\sqrt{2}\text{cis}\frac{17\pi}{4}$ $\therefore x - iy = 2^{\frac{1}{6}}\text{cis}\frac{\pi}{12},\ 2^{\frac{1}{6}}\text{cis}\frac{3\pi}{4}$ or $2^{\frac{1}{6}}\text{cis}\frac{17\pi}{12}$ We now apply part (i). In the first case, $x - iy = 2^{\frac{1}{6}}\text{cis}\frac{\pi}{12}$ $= 2^{\frac{1}{6}}\left(\frac{\sqrt{6}+\sqrt{2}}{4} + i\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right)$ so $x = 2^{\frac{1}{6}}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$ and $y = -2^{\frac{1}{6}}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$. In the second case, $x - iy = 2^{\frac{1}{6}}\cos\frac{3\pi}{4}$ $= 2^{\frac{1}{6}}\cos\left(\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$. $\therefore x = -2^{\frac{1}{6}} \times \frac{1}{\sqrt{2}} \quad y = -2^{\frac{1}{6}} \times \frac{1}{\sqrt{2}}$ In the third case, $x = -2^{-\frac{1}{3}}$ and $y = -2^{-\frac{1}{3}}$ $x - iy = 2^{\frac{1}{6}}\text{cis}\frac{17\pi}{12}$ $= 2^{\frac{1}{6}}\text{cis}\left(-\frac{7\pi}{12}\right)$ $= 2^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$ $= 2^{\frac{1}{6}}\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right)$ $= 2^{\frac{1}{6}}\left(-\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right)$ $= 2^{\frac{1}{6}}\left(-\cos\left(\frac{\pi}{2} - \frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{2} - \frac{\pi}{12}\right)\right)$ $= 2^{\frac{1}{6}}\left(-\sin\frac{\pi}{12} - i\cos\frac{\pi}{12}\right)$ $= 2^{\frac{1}{6}}\left(-\frac{\sqrt{6}-\sqrt{2}}{4} - i \times \frac{\sqrt{6}-\sqrt{2}}{4}\right)$ so $x = -2^{\frac{1}{6}}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$ and $y = 2^{\frac{1}{6}}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$	E3 • Finds all the solutions. . . . . 4 • Finds two of the solutions. . . . . 3 • Makes substantial progress. . . . . 2 • Makes some progress. 1