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HSC Trial Examination 2006

# Mathematics Extension 1

This paper must be kept under strict security and may only be used on or after the afternoon of Wednesday 9 August, 2006 as specified in the Neap Examination Timetable

**General Instructions**

Reading time 5 minutes

Working time 2 hours

Write using black or blue pen

Board-approved calculators may be used

A table of standard integrals is provided at the back of this paper

All necessary working should be shown in every question

**Total marks – 84**

Attempt questions 1–7

All questions are of equal value

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2006 HSC Mathematics Extension 1 Examination.

Total marks 84
Attempt Questions 1–7
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

**Question 1** (12 marks) Use a SEPARATE writing booklet.

(a) Find $\frac{d}{dx}\cos^{-1}5x$. **2**

(b) Calculate the size of the acute angle between the lines $y = 2x - 7$ and $3x + 2y = 5$. (Answer correct to the nearest degree). **2**

(c) Determine the coordinates of the point $P$ that divides the interval joining the points $(1, 3)$ and $(4, 9)$ externally in the ratio of $2 : 5$. **2**

(d) Solve $\frac{2}{x-5} \geq 3$. **3**

(e) Use the substitution $u = \cos x$ to find $\int 6\sin x(1 - \cos x)^3 dx$. **3**

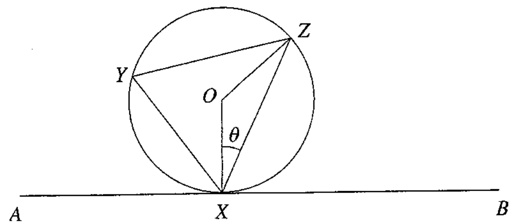





Marks

**Question 2** (12 marks) Use a SEPARATE writing booklet.

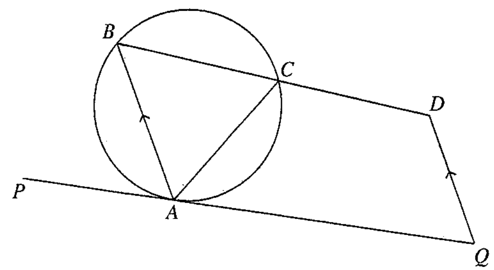
(a) **2**




In the diagram, $O$ is the centre of the circle that passes through the points $Y$, $Z$ and $X$. Line $AB$ is a tangent to the circle at $X$ and $\angle OXZ = \theta$.

Prove $\angle ZXB = \angle XYZ$.

(b) **3**




In the diagram points $A$, $B$ and $C$ lie on the circle. Line $PQ$ is a tangent to the circle at $A$. Line $QD$ is parallel to $AB$, meeting $BC$ produced at $D$.

Prove that $ACDQ$ is a cyclic quadrilateral.

(c) (i) Show that there is a root to the equation $x^3 + x = 9$ between $x = 1.5$ and $x = 2$. **1**

(ii) Starting with $x_1 = 2$ as the first approximation to the root of $x^3 + x = 9$, use one application of Newton's method to find a better approximation to the root. Express your answer in simplest, rational form. **2**

(d) Use mathematical induction to prove that $4 \times 2^n + 3^{3n}$ is divisible by 5 for all integers $n$, $n \geq 0$. **4**

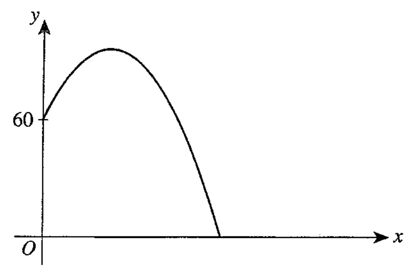
Marks

**Question 3** (12 marks) Use a SEPARATE writing booklet.

(a) What is the coefficient of $x^3$ in the expansion of $(2x-5)^7$? **2**

(b) The polynomial $P(x) = 2x^3 + 6x^2 - 8$ has three roots: 1, –2 and $\alpha$. Determine the value of $\alpha$, and explain the geometrical significance of this value. **2**

(c) A ball is projected from the top of a 60 m vertical cliff with a velocity of 10 m/s at an angle of 30° above the horizontal. Take the origin as (0, 0). Assume g = 10 m/s$^2$.




(i) Show that $x = 5\sqrt{3}t$ and $y = -5t^2 + 5t + 60$. **3**

(ii) Find the maximum height of the ball above the ground. **2**

(iii) Find the time that elapses before the ball hits the ground. **1**

(iv) Find the Cartesian equation of the trajectory of the ball. **2**






Marks

**Question 4** (12 marks) Use a SEPARATE writing booklet.

(a) I have 5 notes: one \$5 note, one \$10 note, one \$20 note, one \$50 note and one \$100 note. By choosing the \$5 note and the \$20 note I can make a total of \$25. **2**

How many different sets of money with a value greater than \$0 can I make by choosing any or all of the notes?

(b) Karen is a member of a 9-player softball team.

(i) In how many ways can they bat if Karen bats in the 9th position? **1**

(ii) There are two left-handers in the team. If the batting order is randomly selected, what is the probability that the left-handers will be in the 1st and 9th positions? **1**

(iii) Every time they bat, each player has a probability of $\frac{1}{4}$ of getting out. What is the probability that, at most, 2 of the first 5 batters will get out? **2**

(c) By expanding the expression $(x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)$ or otherwise, determine the number of real solutions to the equation $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$. **2**

(d) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Show that the coordinates of the mid-point, $M$, of the chord $PQ$ are $\left[a(p+q), \frac{a}{2}(p^2+q^2)\right]$. **1**

(ii) The chord $PQ$ is a focal chord, i.e. $pq = -1$. **3**

Find the equation of the locus of $M$ and describe the locus of $M$ geometrically.

Marks

**Question 5** (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int 2\sin^2 4x\,dx$. **2**

(b) (i) Prove $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d^2x}{dt^2}$. **1**

(ii) The speed, $v$ cm/s, of a particle moving along the $x$-axis is given by $v^2 = 72 - 12x - 4x^2$. **2**

Show that the motion is simple harmonic.

(iii) Find the period and the amplitude of the motion. **3**

(c) An ice cube tray is filled with water at a temperature of 18°C and placed in a freezer that has a constant temperature of –19°C. The cooling rate of the water is proportional to the difference between the temperature of the freezer and the temperature of the water, $T$.

That is, $T$ satisfies the equations

$$\frac{dT}{dt} = -k(T + 19) \text{ and } T = -19 + Ae^{-kt}.$$

(i) Show that $A = 37$. **1**

(ii) After 5 minutes in the freezer the temperature of the water is 3°C. Find the time for the water to reach –18.9°C. Answer correct to the nearest minute. **3**

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Marks

**Question 6** (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_{\sqrt{3}}^{3} \frac{12}{x^2 + 9}\,dx$. **3**

(b) Prove $\frac{\sin 4\theta}{\cos^2\theta - \sin^2\theta} = 4\sin\theta\cos\theta$. **2**

(c) (i) Use the formula ${}^nC_r = \frac{n!}{r!(n-r)!}$ to simplify ${}^{n+1}C_8 \times {}^8C_{n-3}$. **2**

Express your answer in the form of $\frac{A!}{B!C!D!}$.

(ii) Hence, or otherwise, find the *number* of different values $n$ can take. **2**

(d) Find all solutions to the equation **3**

$$\sin\theta - \cos\theta = \sqrt{2} \text{ for } 0 \le \theta \le 2\pi.$$

Marks

**Question 7** (12 marks) Use a SEPARATE writing booklet.

(a) The volume, $V$ m$^3$, of usable wood in a tree of radius $R$ metres can be modelled using the formula

$$\log_e V = 3\log_e 2R - 0.81$$

(i) Determine the value of $e^{0.81}$ correct to 3 significant figures. **1**

(ii) Using your answer to part (i), and working with 3 significant figure accuracy, show that the formula $\log_e V = 3\log_e 2R - 0.81$ can be expressed as $V = \frac{32R^3}{9}$. **3**

(iii) The radius of the tree is increasing at a rate of 0.02 m/year. At what rate is the usable volume of the wood in the tree increasing when the radius of the tree is 1.2 m? Answer in m$^3$/year to 2 significant figure accuracy. **2**

(b) (i) Show that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$. **2**

(ii) Given $\sin^{-1}\left(-\frac{2}{3}\right) - \cos^{-1}\left(-\frac{2}{3}\right) = k$. **4**

By starting with expressions for $\sin^{-1}(-x)$ and $\cos^{-1}(-x)$, or otherwise, find an expression for $\cos^{-1}\left(\frac{2}{3}\right)$ in terms of $k$.

**End of paper**






**STANDARD INTEGRALS**

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln(x + \sqrt{x^2-a^2}), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x + \sqrt{x^2+a^2})$

Note: $\ln x = \log_e x, \quad x > 0$

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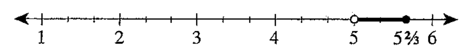
# Mathematics Extension 1

## Solutions and marking guidelines





### Question 1

| | Sample answer | Syllabus outcomes and marking guide |
|---|---|---|
| (a) | $\frac{d}{dx}(\cos^{-1}5x) = \frac{-1}{\sqrt{1-(5x)^2}} \times 5$<br>$= \frac{-5}{\sqrt{1-25x^2}}$ | HE4<br>• Correct answer in any form . . . . . . . 2<br>• Uses formula for $\frac{d}{d\theta}(\cos^{-1}\theta)$ . . . . . . . . 1 |
| (b) | $m_1 = 2,\ m_2 = \frac{-3}{2}$<br>$\tan\theta = \left\lvert \frac{2+\frac{3}{2}}{1-\left(2\times\frac{3}{2}\right)} \right\rvert$<br>$\theta = 60^\circ$ to nearest degree | PE2<br>• Correct answer, ignore accuracy . . . . . 2<br>• Uses formula for angle between two lines<br>OR<br>• Finds both gradients correctly . . . . . . . . . 1 |
| (c) | $\left(\frac{1\times5-2\times4}{3}, \frac{3\times5-2\times9}{3}\right)$<br>$= (-1, -1)$ | PE2<br>• Correct numerical expression . . . . . . . . 2<br>• $\left(\frac{13}{7}, \frac{33}{7}\right)$ from internal division<br>OR<br>• Makes some progress with external division. . . . . . . . . . . . . . . . . . . . . . . . 1 |
| (d) | $\frac{2}{x-5} \geq 3,\ x \neq +5$<br>$(x-5)\times2 - 3(x-5)^2 \geq 0$<br>$(x-5)(17-3x) \geq 0$<br><br>$5 < x \leq 5\frac{2}{3}$ | PE3<br>• Correct answer; accept answer displayed on the number line . . . . . . . . . . . . . . . . . 3<br>• $5 \leq x \leq 5\frac{2}{3}$<br>OR<br>Significant progress . . . . . . . . . . . . . . . 2<br>• $x \neq 5$<br>OR<br>• Progress with a correct method . . . . . . . 1 |
| (e) | $u = \cos x$<br>$du = -\sin x\,dx$<br>$I = \int -6(1-u)^3 du$<br>$= \frac{6}{4}(1-u)^4 + c$<br>$= \frac{3}{2}(1-\cos x)^4 + c$ | HE6<br>• Correct answer, with or without + $c$. . . . . 3<br>• A correct expression in terms of $u$<br>OR<br>• A correct answer in $x$ from a non-trivial incorrect integral in $u$. . . . . . . . . . . . . . . . 2<br>• A correct expression for $du$ . . . . . . . . . . . 1 |

**Question 2**

**Sample answer** | **Syllabus outcomes and marking guide**

(a) $OX \perp AB$ (angle between tangent and radius at the point of contact is 90°)

$\angle ZXB = 90 - \theta$

$XO = OZ$ (equal radii)

$\angle OZX = \theta$ (base angles of an isosceles triangle are equal)

$\angle XOZ = 180° - 2\theta$ (angles in a triangle add to 180°)

$\angle XYZ = 90° - \theta$ (angle at the centre is twice the angle at the circumference standing on the same arc)

$\therefore \angle ZXB = \angle XYZ$ (both equal to $90 - \theta$)

PE2 PE3 PE6
- Clear, logical proof with reasons . . . . . . . 2
- Proof without reasons
  OR
- Two relevant facts with reasons . . . . . . . 1

(b) Let $\angle PAB = \theta$

$\therefore \angle BCA = \theta$ (alternate segment theorem)

$\angle BAP = \angle DQA = \theta$ (corresponding angles BA || DQ)

$\therefore \angle BCA = \angle AQD$ (both $\theta$)

$\therefore ACDQ$ is a cyclic quadrilateral as the exterior angle is equal to the opposite interior angle

PE2 PE3 PE6
- Clear logical proof with reasons . . . . . . . 3
- Logical proof without adequate reasons
  OR
- Significant progress towards a valid proof. . . . . . . . . . . . . . . . . . . . . . . . . . 2
- One relevant fact with a supporting reason. . . . . . . . . . . . . . . . . . . . . . . . . . 1

(c) (i) $f(x) = x^3 + x - 9$

| $x$ | 1.5 | 2 |
|---|---|---|
| $f(x)$ | $-4.125$ | 1 |

$f(x)$ is continuous and has a change of sign in $1.5 < x < 2$

$\therefore$ there is a root in $1.5 < x < 2$

PE3
- Correct answer. . . . . . . . . . . . . . . . . . . . . . 1

(ii) $f(x) = x^3 + x - 9$

$f(x_1) = f(2)$

$= 1$

$f'(x_1) = 3(2)^2 + 1$

$= 13$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2 - \frac{1}{13}$$

$$= 1\frac{12}{13}$$

PE3
- Correct answer. . . . . . . . . . . . . . . . . . . . . 2
- Correct answer not in simplest rational form
  OR
- Correct use of Newton's method with either an algebraic or arithmetical error. . . . . . . 1

 TENME1_SS_06.FM 



**Question 2** **(Continued)**

**Sample answer** | **Syllabus outcomes and marking guide**

(d) To prove $4 \times 2^n + 3^{3n}$ is divisible by 5 for integers $n \geq 0$:

- Test for $n = 0$.

  $4 \times 2^0 + 3^0 = 5$

  $\therefore$ true for $n = 0$.

- Assume $4 \times 2^k + 3^{3k}$ is divisible by 5 for intergers $k \geq 0$.

  $\therefore 4 \times 2^k + 3^{3k} = 5M$ for an integer $M$.

- Required to prove $4 \times 2^{k+1} + 3^{3(k+1)} = 5J$, for an integer $J$.

  Proof

  $4 \times 2^{k+1} + 3^{3(k+1)}$

  $= 4 \times 2 \times 2^k + 3^3 \times 3^{3k}$

  $= 8 \times 2^k + 3^3 \times [5M - 4 \times 2^k]$

  $= 2^k[8 - 27 \times 4] + 5M \times 3^3$

  $= -100 \times 2^k + 5M \times 3^3$

  $= 5[-20 \times 2^k + M \times 3^3]$

  As $M$ is an integer, $K$ is an integer, then $[-20 \times 2^k + M \times 3^3]$ is an integer also

  $\therefore 4 \times 2^{k+1} + 3^{3(k+1)} = 5J$, for an integer $J$.

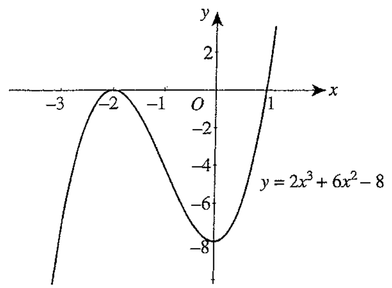
  $\therefore$ The expression is divisible by 5.

- If the expression is divisible by 5 for a value of $n$, then it is divisible by 5 for the following integral value of $n$. It has been shown to be divisible by 5 for $n = 0$ and so is divisible by 5 for $n = 1$, $n = 2$ etc.

HE2
- Correct proof including all steps . . . . . . . 4
- Significant progress towards a proof
  OR
- A 'correct' proof with a minor fault . . . . 3
- Some progress
  OR
- A 'correct' proof with errors . . . . . . . . . 2
- One part of the proof correct . . . . . . . . . . 1

 TENME1_SS_06.FM 

**Question 3**

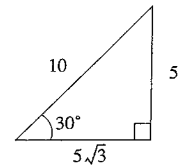
| Sample answer | Syllabus outcomes and marking guide |
|---|---|
| (a) $(2x-7)^7$<br>Term $= {}^7C_3(2x)^3(-5)^4$<br>$= {}^7C_3 \times 8 \times 625x^3$<br>Coefficient $= 2^3 \times 5^4 \times {}^7C_3$ or 175 000 | PE3<br>• Correct answer: either $2^3 \times 5^4 \times {}^7C_3$ or 175 000 or equivalent. . . . . . . . . . . . . . 2<br>• Correct term in $x^3$<br>OR<br>• Significant progress . . . . . . . . . . . . . 1 |
| (b) Product of roots $= \frac{-d}{a}$<br>$1 \times -2 \times \alpha = \frac{8}{2}$<br>$\therefore \alpha = -2$<br>$\therefore$ two roots are the same<br><br>The $x$-axis is a tangent to the polynomial at $x = -2$. | PE3 HE7<br>• $\alpha = -2$ and a correct geometrical significance . . . . . . . . . . . . . . 2<br>• $\alpha = -2$<br>OR<br>• Correct geometrical significance from an incorrect value of $\alpha$ . . . . . . . . . . . . . 1 |







**Question 3 (Continued)**

| Sample answer | Syllabus outcomes and marking guide |
|---|---|
| (c) (i) $\ddot{x} = 0$<br>$\dot{x} = C_1$<br><br>$\dot{x} = 5\sqrt{3}$ from initial conditions<br>Starts at $x = 0$, $\therefore$ $0 = 0 + C_2$<br>$\therefore$ $x = 5\sqrt{3}t$<br>$\ddot{y} = -10$<br>$\dot{y} = -10t + C_3$<br>When $t = 0$, $\dot{y} = 5 \Rightarrow C_3 = 5$<br>$\dot{y} = -10t + 5$<br>$y = -5t^2 + 5t + C_4$<br>When $t = 0$, $y = 60 \Rightarrow C_4 = 60$<br>$y = -5t^2 + 5t + 60$ | HE3<br>• Correct derivations. . . . . . . . . . . . . . . . . 3<br>• Significant progress . . . . . . . . . . . . . . . . 2<br>• Some progress . . . . . . . . . . . . . . . . . 1 |
| (ii) Max. height when $\dot{y} = 0$<br>$-10t + 5 = 0$<br>$\therefore$ $t = \frac{1}{2}$<br>$\therefore$ Max. height $= -5\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) + 60$<br>$= 61.25$ m | PE3<br>• Correct answer. . . . . . . . . . . . . . . . . . 2<br>• $t = \frac{1}{2}$<br>OR<br>• The height from their value of $t$ for $\dot{y} = 0$ . . . . . . . . . . . . . . . . . . . . . . . . 1 |
| (iii) Ball hits ground when $y = 0$.<br>$-5t^2 + 5t + 60 = 0$<br>$t^2 - t - 12 = 0$<br>$(t-4)(t+3) = 0$<br>Since $t \geq 0$, $t = 4$ seconds. | HE3<br>• Correct answer. . . . . . . . . . . . . . . . . . . . . 1 |
| (iv) $x = 5\sqrt{3}t \Rightarrow t = \frac{x}{5\sqrt{3}}$<br>$y = -5\left(\frac{x}{5\sqrt{3}}\right)^2 + 5\left(\frac{x}{5\sqrt{3}}\right) + 60$<br>$= -\frac{x^2}{15} + \frac{x}{\sqrt{3}} + 60$ | HE3<br>• Correct Cartesian equation. . . . . . . . . . . 2<br>• $t = \frac{x}{5\sqrt{3}}$<br>OR<br>• Other progress towards a solution . . . . . . 1 |

**Question 4**

| | Sample answer | Syllabus outcomes and marking guide |
|---|---|---|
| (a) | I can take notes:<br>1 at a time $^5C_1 = 5$<br>2 at a time $^5C_2 = 10$<br>3 at a time $^5C_3 = 10$<br>4 at a time $^5C_4 = 5$<br>5 at a time $^5C_5 = 1$<br>Total $= 31$ | HE6<br>• Correct answer ............ 2<br>• Attempts to use appropriate strategy .... 1 |
| (b) | (i) 8! | PE3<br>• Correct answer .................... 1 |
| | (ii) \| 2 \| 7 \| 6 \| 5 \| 4 \| 3 \| 2 \| 1 \| 1 \|<br>Number of ways left-handers can be in 1st and 9th positions $= 2 \times 7!$<br>Probability $= \dfrac{2 \times 7!}{9!}$ | HE3<br>• Correct answer in any form,<br>e.g. $\dfrac{2\times 7!}{9!}$, $\dfrac{10\ 080}{9!}$, $\dfrac{1}{36}$ ............ 1 |
| | (iii) Binomial expression = (out + not out)$^5$<br>Pr(out) $= \frac{1}{4}$ Pr(not out) $= \frac{3}{4}$<br>Need (0 out + 1 out + 2 out)<br>$= {^5C_0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^5 + {^5C_1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^4 + {^5C_2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3$<br>$= \frac{3^5}{4^5} + \frac{5 \times 3^4}{4^5} + \frac{5 \times 4 \times 3^3}{2 \times 4^5}$<br>$= \frac{243 + 405 + 270}{1024}$<br>$= \frac{459}{512}$ | HE3<br>• Correct numerical expression ......... 2<br>• Correct approach using binomial probability<br>OR<br>• Tree diagram ..................... 1 |
| (c) | $(x-1)(x^5 + x^4 + x^3 + x^2 + x + 1) = x^6 - 1$<br>$x^6 - 1 = 0$ has 2 real solutions $x = -1, +1$<br>$(x-1)(x^5 + x^4 + x^3 + x^2 + x + 1) = x^6 - 1$<br>has 2 real solutions $x = -1, +1$<br>$\therefore$ $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$<br>has only 1 real solution, $x = -1$ | HE7<br>• Correct expansion and 1 real solution ... 2<br>• Correct expansion .................. 1 |



HSC Mathematics Extension 1 Trial Examination Solutions and marking guidelines

**Question 4** **(Continued)**

| | Sample answer | Syllabus outcomes and marking guide |
|---|---|---|
| (d) | (i) $P(2ap, ap^2)$ $Q(2aq, aq^2)$<br>$M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$<br>$M = \left(a(p+q), \frac{a}{2}(p^2 + q^2)\right)$ | PE3<br>• Correct demonstration ............... 1 |
| | (ii) $x = a(p+q)$ $y = \left(\frac{a}{2}(p^2 + q^2)\right)$<br>$y = \frac{a}{2}\{(p+q)^2 - 2pq\}$<br>$y = \frac{a}{2}\{(p+q)^2 + 2\}$<br>$y = \frac{a}{2}\left(\frac{x^2}{a^2} + 2\right)$<br>$y = \frac{x^2}{2a} + a$<br>$x^2 = 2a(y - a)$<br>which is a parabola with vertex at $(0, a)$<br>and focal length $\frac{1}{2}a$ | PE3 HE7<br>• Correct equation and geometrical description ...................... 3<br>• Correct equation.................... 2<br>• Progress towards the solution ......... 1 |

**Question 5**

| | Sample answer | Syllabus outcomes and marking guide |
|---|---|---|
| (a) | $\int 2\sin^2 4x\,dx = 2\times\frac{1}{2}\times\int(1-\cos 8x)dx$<br>$= x - \frac{1}{8}\sin 8x + C$ | HE6<br>• Correct answer (with or without the $C$) . . 2<br>• Attempts to make use of double angle ($2\theta$) results . . . . . . . . . . 1 |
| (b) (i) | $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}\times 2\times v\times\frac{dv}{dx}$<br>$= v\frac{dv}{dx}$<br>$= \frac{dx}{dt}\times\frac{dv}{dx}$<br>$= \frac{dv}{dt}$<br>$= \frac{d^2x}{dt^2}$ | HE5<br>• A correct proof . . . . . . . . . . . . . . . . . . 1 |
| (ii) | $v^2 = 72 - 12x - 4x^2$<br>$\frac{1}{2}v^2 = 36 - 6x - 2x^2$<br>$\ddot{x} = -6 - 4x$<br>$= -4\left(x + 1\frac{1}{2}\right)$<br>∴ It is SHM as it is in the form<br>$\ddot{x} = -n^2x$ | HE3<br>• Correct solution . . . . . . . . . . . . . . . . . . 2<br>• Use of $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ . . . . . . . . . . . . . . 1 |
| (iii) | $n = 2 \Rightarrow$ period $= \pi$<br>Centre of motion of $x = -1\frac{1}{2}$.<br>It stops when $x^2 + 3x - 18 = 0$<br>when $(x-3)(x+6) = 0$<br>It stops at $x = +3$ and $x = -6$<br>From $-6$ to $+3$ is 9 units.<br>$2\times$ amplitude $= 9$<br>∴ amplitude $= 4\frac{1}{2}$ | HE5<br>• Period $= \pi$, AND amplitude $= 4\frac{1}{2}$ . . . . . 3<br>• Either period $= \pi$<br>OR<br>• Amplitude $= 4\frac{1}{2}$ . . . . . . . . . . . . . . . . 2<br>• Some progress toward a solution, e.g. puts $v = 0$<br>OR<br>• Finds centre of motion . . . . . . . . . . . . . . 1 |





**Question 5** (Continued)

| | Sample answer | Syllabus outcomes and marking guide |
|---|---|---|
| (c) (i) | When $t = 0$, $T = 18$.<br>∴ $18 = -19 + Ae^0$<br>$A = 18 + 19$<br>$A = 37$ | HE3<br>• $18 = -19 + Ae^0$ or equivalent . . . . . . . . 1 |
| (ii) | $t = 5 \quad T = 3$<br>$3 = -19 + 37e^{-5k}$<br>$e^{-5k} = \frac{22}{37}$<br>$k = -\frac{1}{5}\log_e\frac{22}{37}$<br>When $T = -18.9$, $t = ?$<br>$-18.9 = -19 + 37e^{-kt}$<br>$\Rightarrow 370e^{-kt} = 1$<br>∴ $t = -\frac{\log_e\left(\frac{1}{370}\right)}{k}$<br>$t = 57$ minutes (correct to the nearest minute) | HE3<br>• Correct answer (ignore rounding of final answer) . . . . . . . . . . . . . . . . . . 3<br>• Finds the correct value of $k$<br>i.e. $k = -\frac{1}{3}\log_e\frac{22}{37}$, or 0.17329 . . . . . . . . 2<br>• Makes some progress (e.g. attempts to find the value of $k$) . . . . . . . . . . . . . . . 1 |

**Question 6**

| Sample answer | Syllabus outcomes and marking guide |
|---|---|
| (a) $\int_{\sqrt{3}}^{3} \frac{12}{x^2+9}dx = \left[\frac{12}{3}\tan^{-1}\frac{x}{3}\right]_{\sqrt{3}}^{3}$ <br> $= 4\left[\tan^{-1}1 - \tan^{-1}\frac{1}{\sqrt{3}}\right]$ <br> $= 4\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$ <br> $= \frac{\pi}{3}$ | HE6 <br> • Correct value $4\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$ .............. 3 <br> • $4\tan^{-1}\frac{x}{3}$ or equivalent <br> OR <br> • Correct evaluation of their integration (provided it is non-trivial) ........... 2 <br> • An expression of the form $A\tan^{-1}\frac{x}{B}$, or $A\tan^{-1}Bx$ .............. 1 |
| (b) LHS $= \frac{2\sin 2\theta\cos 2\theta}{\cos 2\theta}$ <br> $= 2\sin 2\theta$ <br> $= 2 \times 2\sin\theta\cos\theta$ <br> $= 4\sin\theta\cos\theta$ <br> = RHS | PE2 <br> • A correct proof.................... 2 <br> • Uses either of the sin $2\theta$ or cos $2\theta$ results correctly .......................... 1 |
| (c) (i) ${}^{n+1}C_8 \times {}^8C_{n-3}$ <br> $= \frac{(n+1)!}{8! \times (n+1-8)!} \times \frac{8!}{(n-3)! \times [8-(n-3)]!}$ <br> $= \frac{(n+1)!}{(n-7)!(n-3)!(11-n)!}$ | PE3, HE3 <br> • Correct answer .................... 2 <br> • Makes significant progress towards the solution.......................... 1 |
| (ii) $n+1 \geq 0 \quad (n-7) \geq 0$ <br> $n-3 \geq 0 \quad (11-n) \geq 0$ <br> $n \geq -1,\ n \geq 7,\ n \geq 3,\ n \leq 11$ <br> $\therefore\ n \geq 7$ and $n \leq 11$ <br> i.e. $n = 7, 8, 9, 10$ or $11$ <br> i.e. $n$ can take 5 values | PE3, HE3 <br> • Correct answer.................... 2 <br> • The concept that $7 \leq n \leq 11$ .......... 1 |





**Question 6 (Continued)**

| Sample answer | Syllabus outcomes and marking guide |
|---|---|
| (d) Let $\alpha$ be acute. <br> $\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta = 1$ <br> Let $\frac{1}{\sqrt{2}} = \cos\alpha \qquad \alpha = \frac{\pi}{4}$ <br> $\sin\left(\theta - \frac{\pi}{4}\right) = 1 \qquad \therefore\ \theta - \frac{\pi}{4} = \frac{\pi}{2}$ <br> In $0 \leq \theta \leq 2\pi \qquad \therefore\ \theta = \frac{3\pi}{4}$ <br> **Alternative solution:** <br> $\sin\theta - \cos\theta = \sqrt{2}$ <br> Let $t = \tan\frac{\theta}{2}$, $\therefore\ \sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$ <br> $\therefore\ \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = \sqrt{2}$ <br> $2t - 1 + t^2 = \sqrt{2} + \sqrt{2}t^2$ <br> $(\sqrt{2}-1)t^2 - 2t + \sqrt{2} + 1 = 0$ <br> $t = \frac{2 \pm \sqrt{4 - 4(\sqrt{2}-1)(\sqrt{2}+1)}}{2(\sqrt{2}-1)}$ <br> $= \frac{2 \pm \sqrt{0}}{2(\sqrt{2}-1)}$ <br> $= \frac{1}{\sqrt{2}-1}$ <br> $= \sqrt{2}+1$ <br> $\therefore\ \tan\frac{\theta}{2} = \sqrt{2}+1$ <br> $\therefore\ \frac{\theta}{2} = 67.5°$ or $247.5°$ <br> $\therefore\ \theta = 135°$ or Not in domain <br> $\therefore\ \theta = \frac{3\pi}{4}$ in $0 \leq \theta \leq 2\pi$ | H5 <br> • Correct solution by any method........ 3 <br> • Makes significant progress towards the solution......................... 2 <br> • Attempts to use an appropriate method .. 1 |

**Question 7**

| Sample answer | Syllabus outcomes and marking guide |
|---|---|
| (a) (i) $e^{0.81} = 2.25$ (to 3 significant figures) | H3<br>• Correct value with 3 significant figures . . 1 |
| (ii) $\log_e V = \log_e 8R^3 - 0.81$<br>but $e^{0.81} = 2.25$<br>$\log_e e^{0.81} = \log_e 2.25$<br>$\Rightarrow 0.81 = \log_e 2.25$<br>$\log_e V = \log_e 8R^3 - \log_e 2.25$<br>$\log_e V = \dfrac{\log_e 8R^3}{2.25}$<br>$V = \dfrac{8R^3}{2.25} \times \dfrac{4}{4}$<br>$V = \dfrac{32R^3}{9}$ | H3, H4, HE7<br>• Correct answer – any method . . . . . . . . 3<br>• Makes significant progress . . . . . . . . . . 2<br>• Makes some progress. . . . . . . . . . . . 1 |
| (iii) $\dfrac{dR}{dt} = 0.02$ metres/year<br>$\dfrac{dV}{dt} = \dfrac{dV}{dR} \times \dfrac{dR}{dt}$<br>$\dfrac{dV}{dR} = \dfrac{32 \times 3R^2}{9}$<br>$\therefore \dfrac{dV}{dt} = \dfrac{32 \times 3 \times (1.2)^2 \times 0.02}{9}$<br>$\dfrac{dV}{dt} = 0.3072$ or $0.31$ m³/year | HE5<br>• Correct answer – ignore rounding . . . . . 2<br>• $\dfrac{dV}{dt} = \dfrac{dV}{dR} \times \dfrac{dR}{dt}$ or similar correct equation<br>OR<br>• $\dfrac{dR}{dt} = 0.02$ . . . . . . . . . . . . . . . . . . . . . . 1 |





**Question 7**

| Sample answer | Syllabus outcomes and marking guide |
|---|---|
| (b) (i) $\dfrac{d}{dx}(\sin^{-1}x + \cos^{-1}x)$<br>$= \left(\dfrac{1}{\sqrt{1-x^2}} + \dfrac{-1}{\sqrt{1-x^2}}\right)$<br>$= 0$<br>$\therefore \sin^{-1}x + \cos^{-1}x$ has a constant value<br>Let $x = 0$.<br>$\Rightarrow \sin^{-1}0 + \cos^{-1}0 = \dfrac{\pi}{2}$<br>$\therefore \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ | HE4<br>• A correct demonstration. . . . . . . . . . . . . 2<br>• $\dfrac{d}{dx}(\sin^{-1}x + \cos^{-1}x) = 0$<br>OR<br>• Evaluation of $\sin^{-1}x + \cos^{-1}x$ . . . . . . . . 1 |
| (ii) $\sin^{-1}(-x) = -\sin^{-1}x$<br>$\cos^{-1}(-x) = \pi - \cos^{-1}x$<br>$\sin^{-1}\left(-\dfrac{2}{3}\right) - \cos^{-1}\left(-\dfrac{2}{3}\right) = k$<br>$\therefore -\sin^{-1}\left(\dfrac{2}{3}\right) - \left[\pi - \cos^{-1}\left(\dfrac{2}{3}\right)\right] = k$<br>$\Rightarrow \cos^{-1}\left(\dfrac{2}{3}\right) = k + \pi + \sin^{-1}\left(\dfrac{2}{3}\right)$<br>From (i):<br>$\sin^{-1}\left(\dfrac{2}{3}\right) = -\left[-\cos^{-1}\left(\dfrac{2}{3}\right) + \dfrac{\pi}{2}\right]$<br>$\therefore \cos^{-1}\left(\dfrac{2}{3}\right) = k + \pi + \left[\dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{2}{3}\right)\right]$<br>$2\cos^{-1}\left(\dfrac{2}{3}\right) = k + \pi + \dfrac{\pi}{2}$<br>$\Rightarrow 4\cos^{-1}\left(\dfrac{2}{3}\right) = 2k + 3\pi$<br>$\therefore \cos^{-1}\left(\dfrac{2}{3}\right) = \dfrac{2k + 3\pi}{4}$ | HE4, HE7<br>• Correct value in any form . . . . . . . . . . . 4<br>• Substantial progress as well as values for $\sin^{-1}(-x) + \cos^{-1}(-x)$ . . . . . . . . . . . . . 3<br>• Both expressions for $\sin^{-1}(-x)$ and $\cos^{-1}(-x)$. . . . . . . . . . . . . . . . . . . . 2<br>• Either $\sin^{-1}(-x) = -\sin^{-1}x$<br>OR<br>• $\cos^{-1}(-x) = \pi - \cos^{-1}x$ . . . . . . . . . . . 1 |