Neap.
advancing education

HSC Trial Examination 2006

Mathematics Extension 2

This paper must be kept under strict security and may only be used on or after the morning of Tuesday 8 August, 2006 as specified in the Neap Examination Timetable

General instructions

Reading time – 5 minutes.

Working time – 3 hours.

Board-approved calculators may be used.

Write using blue or black pen.

A table of standard integrals is provided at the back of this paper.

All necessary working should be shown in every question.

Total marks – 120

Attempt questions 1–8.

All questions are of equal value.

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2006 HSC Mathematics Extension 2 examination.

Neap Trial Exams are licensed to be photocopied or placed on the school intranet and used only within the confines of the school purchasing them, for the purpose of examining that school's students only. They may not be otherwise reproduced or distributed. The copyright of Neap Trial Exams remains with Neap. No Neap Trial Exam or any part thereof is to be issued or passed on by any person to any party inclusive of other schools, non-practising teachers, coaching colleges, tutors, parents, students, publishing agencies or websites without the express written consent of Neap.
Copyright © 2006 Neap ABN 49 910 906 643 PO Box 214 St Leonards NSW 2065 Tel: (02) 9438 1386 Fax: (02) 9438 1385 TENME2_QA_06.FM

Total marks 120
Attempt Questions 1–8
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

Consider the function $g(x) = 4\sqrt{x} - 2x$.

(a) Write down the domain of $g(x)$. 1

(b) Find the x intercepts of the graph of $y = g(x)$. 1

(c) Show that the curve $y = g(x)$ is concave downwards for all $x > 0$. 1

(d) Find the coordinates of the stationary point and determine its nature. 1

(e) Sketch the graph of $y = g(x)$, clearly showing all essential features. 1

(f) Hence, by consideration of the graph of $y = g(x)$, sketch each of the following on separate diagrams, showing all essential features:

(i) $y = |g(x)|$ 2

(ii) $y = g(x-2)$ 2

(iii) $y = g(|x|)$ 2

(iv) $|y| = g(x)$ 2

(v) $y = \dfrac{1}{g(x)}$. 2

 TENME2_QA_06.FM Copyright © 2006 Neap

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Simplify $\dfrac{\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)}{\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}}$, giving your answer in the form $a + bi$. 2

(b) Find real numbers p and q such that 3

$$(p + qi)(1 - 2i) = (1 - p) - qi.$$

(c) By considering the complex number $z = x + iy$ in the Argand plane, sketch the locus of the following on separate Argand diagrams.

(i) $\arg z = \frac{\pi}{3}$ 1

(ii) $\arg \bar{z} = \frac{\pi}{3}$ 1

(iii) $\arg(-z) = \frac{\pi}{3}$ 1

(d) Let $w = r(\cos\phi + i\sin\phi)$ where ϕ is an acute angle.
With the aid of a suitable diagram, or otherwise:

(i) show that the distance between w and $\overline{w}$ in the complex plane is $2r\sin\phi$ 2

(ii) find $(w + \overline{w})$. 1

(e) If $z = x + iy$ is any complex number, use only a labelled sketch to illustrate that 2

$$|z - 2| = |\bar{z} - 2|.$$

(f) Sketch the region of the complex plane for which the complex number $z = x + iy$ has a positive real part and $|z + 3i| \le 2$. 2

Copyright © 2006 Neap TENME2_QA_06.FM

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) If p, q, and r are the roots of the equation $x^3 + 4x^2 - 3x + 1 = 0$, find the equation whose roots are **3**

$$\frac{1}{p}, \frac{1}{q} \text{ and } \frac{1}{r}.$$

(b) Find the roots of $3x^3 - 26x^2 + 52x - 24 = 0$, given that the roots are in geometric progression. **4**

(c) (i) Let k be a zero of a polynomial $F(x)$ and also of its derivative $F'(x)$. Prove that k is a zero of $F(x)$ of multiplicity at least 2. **2**

(ii) Show that $y = 1$ is a double root of the equation **2**

$$y^{2t} - ty^{t+1} = 1 - ty^{t-1}$$

where t is a positive integer.

(d) Consider the polynomial $k(t) = t^4 + at^3 + bt^2 + at + 1$, where a and b are real numbers.

(i) Show that if θ is a zero of $k(t)$, then $\frac{1}{\theta}$ is also a zero of $k(t)$. **2**

(ii) Hence, or otherwise, write down all four zeros of $k(t)$, given that $(1 + i)$ is a zero of $k(t)$. (There is no need to calculate a or b.) **2**

 TENME2_QA_06.FM Copyright © 2006 Neap

HSC Mathematics Extension 2 Trial Examination

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int e^x(1 + e^x)^5\,dx$. **2**

(b) Find $\int \frac{dt}{\sqrt{7 + 6t - t^2}}$. **2**

(c) Using the substitution $t = \tan\frac{\theta}{2}$, find **3**

$$\int \frac{2}{4 + 3\sin\theta}\,d\theta.$$

(d) Find $\int \frac{y^5 - 7y^2 + 8}{y^3 - 8}\,dy$. **4**

(e) If $U_m = \int_0^k (k^2 - x^2)^m\,dx$ for $m \geq 1$, **4**

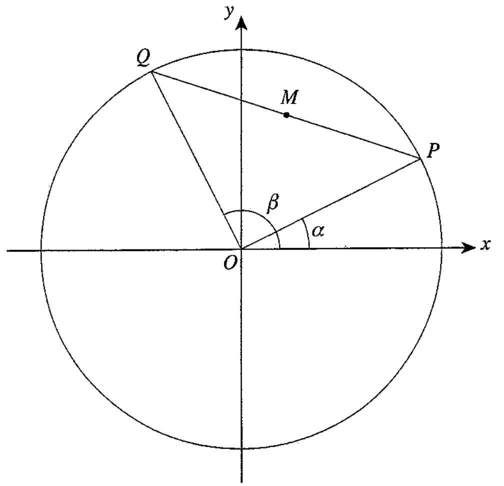
show that $U_m = \frac{2k^2 m}{2m + 1} U_{m-1}$.

Copyright © 2006 Neap TENME2_QA_06.FM

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows a chord PQ of the circle $x^2 + y^2 = a^2$. The coordinates of P and Q are $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ respectively, where α and β are parameters. O is the centre of the circle and M is the midpoint of PQ.



(i) Write down the coordinates of M. **1**

(ii) Write down a relationship between α and β, given that PQ subtends a right angle at the origin. **1**

(iii) Express the coordinates of M in terms of α. **1**

(iv) Find the equation of the locus of M. **3**

(b) The ellipse E has the equation

$$4x^2 + 9y^2 = 36.$$

(i) Write down:

(α) its eccentricity **1**

(β) the coordinates of its foci S and S' **1**

(γ) the equation of each directrix **1**

(δ) the length of the major axis. **1**

(ii) Sketch the ellipse E. Show the x and y intercepts as well as the features found in parts (β) and (γ) of part (i) above. **1**

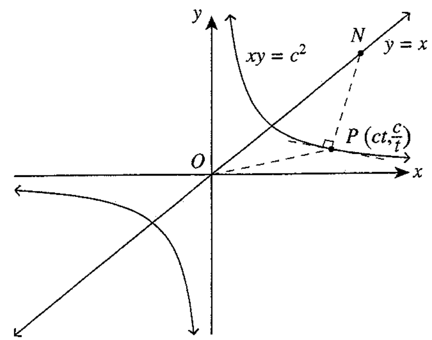
Question 5 continues on page 7

 TENME2_QA_06.FM Copyright © 2006 Neap

Marks

Question 5 (continued)

(c) The diagram below shows the hyperbola $xy = c^2$. The point $P\left(ct, \frac{c}{t}\right)$ lies on the curve, where $t \neq 0$. The normal at P intersects the straight line $y = x$ at N. O is the origin.



(i) Prove that the equation of the normal at P is **1**

$$y = t^2x + \frac{c}{t} - c^3.$$

(ii) Find the coordinates of N. **1**

(iii) Show that triangle OPN is isosceles. **2**

End of Question 5

Copyright © 2006 Neap TENME2_QA_06.FM

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The region enclosed by the parabola $y=\frac{1}{4}(2x-5)^2$ and the straight line $y=6\frac{1}{4}$ is rotated about the y-axis. **4**

Use the method of cylindrical shells to find the exact volume of the solid formed.

(b) The region bounded by the parabolas $y=6-x^2$ and $y=\frac{1}{2}x^2$ forms the base of a solid.

Cross-sections by planes perpendicular to the y-axis are semicircles, with their diameters in the base of the solid.

(i) Find the points of intersection of the two parabolas. **1**

(ii) Find the volume of the solid. **4**

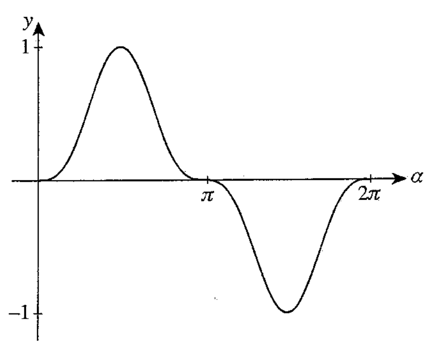
(c) Consider the complex number $z=\cos\alpha+i\sin\alpha$.

(i) Prove that $z^n-\frac{1}{z^n}=2i\sin n\alpha$. **2**

(ii) Expand $\left(z-\frac{1}{z}\right)^3$, and use the result from part (i) to show that **2**

$$\sin^3\alpha=\frac{3}{4}\sin\alpha-\frac{1}{4}\sin3\alpha.$$

(iii) Below is a sketch of $y=\sin^3\alpha$, for $0\le\alpha\le2\pi$. **2**

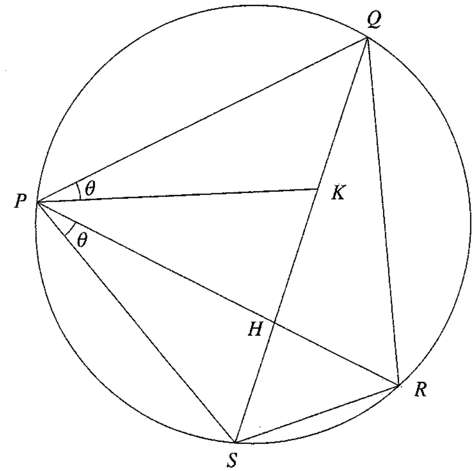


Find the area of the region between the curve and the α-axis, for $0\le\alpha\le2\pi$.

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows a cyclic quadrilateral $PQRS$ circumscribed by a circle. The diagonals of this quadrilateral meet at H.
K is a point on QS such that angle SPR equals angle QPK.



(i) Show that the triangles PQK and PRS are similar. **2**

(ii) Show that the triangles PQR and PKS are similar. **2**

(iii) Hence prove Ptolemy's theorem: **3**

"In any cyclic quadrilateral, the sum of the products of the lengths of opposite sides is equal to the product of the lengths of the diagonals."

That is, prove that in this cyclic quadrilateral $PQRS$,

$$PQ\times RS+PS\times QR=PR\times QS.$$

(b) A body is projected vertically upwards from the ground with initial velocity v_0 in a medium that produces a resistance force per unit mass of kv^2, where v is the velocity and k is a positive constant.

Take acceleration due to gravity as g m s^{-2}.

(i) Prove that the maximum height H of the body above the ground is **4**

$$H=\frac{1}{2k}\log_e\left(1+\frac{kv_0^2}{g}\right).$$

(ii) Show that in order to double the maximum height reached, the initial velocity must be increased by a factor of **4**

$$(e^{2kH}+1)^{\frac{1}{2}}.$$

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) On a racetrack for go-karts, a circular bend of radius 10 metres is banked at 45° to the horizontal. Given that the maximum frictional force R (up or down the bank) is at most $\frac{1}{9}$ of the normal reaction N, find the range of speeds (in exact form) at which a go-kart of mass m kilograms can safely negotiate the bend. **5**

Take the acceleration due to gravity to be 10 m s^{-2}.

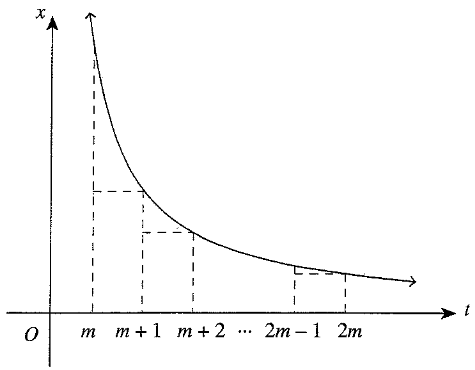
(b) (i) Prove that $\frac{1}{2p+1} + \frac{1}{2p+2} > \frac{1}{p+1}$, for all $p > 0$. **2**

(ii) Consider the statement **4**

$$\chi(m): \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m} \geq \frac{37}{60}.$$

Show by mathematical induction that $\chi(m)$ is true for all integers $m \geq 3$.

(iii) The diagram below shows the graph of $x = \frac{1}{t}$, for $t > 0$.



(α) By comparing areas, show that **1**

$$\int_m^{m+1} \frac{1}{t}\,dt > \frac{1}{m+1}.$$

(β) Hence, without using a calculator, show that **3**

$$\log_e 2 > \frac{37}{60}.$$

End of paper

Copyright © 2006 Neap

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln(x + \sqrt{x^2-a^2}), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x + \sqrt{x^2+a^2})$

Note: $\ln x = \log_e x, \quad x > 0$

Copyright © 2006 Neap

Neap.
advancing education

HSC Trial Examination 2006

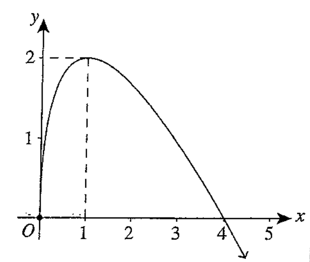
Mathematics Extension 2

Solutions and marking guidelines

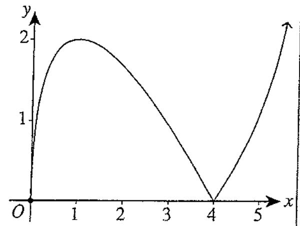
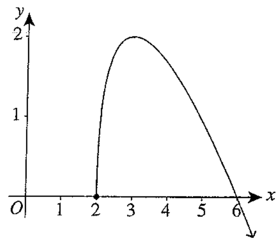
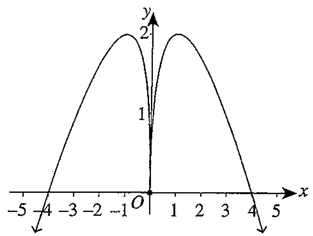
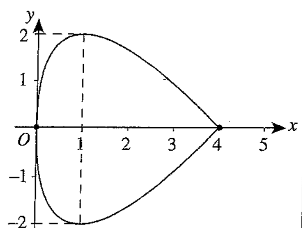
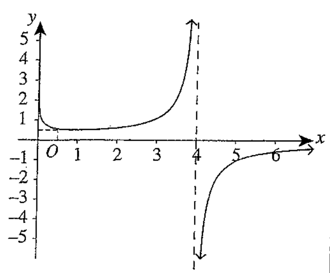
Neap Trial Exams are licensed to be photocopied or placed on the school intranet and used only within the confines of the school purchasing them, for the purpose of examining that school's students only. They may not be otherwise reproduced or distributed. The copyright of Neap Trial Exams remains with Neap No Neap Trial Exam or any part thereof is to be issued or passed on by any person to any party inclusive of other schools, non-practising teachers, coaching colleges, tutors, parents, students, publishing agencies or websites without the express written consent of Neap

Copyright © 2006 Neap ABN 49 910 906 643 PO Box 214 St Leonards NSW 2065 Tel: (02) 9438 1386 Fax: (02) 9438 1385 TENME2_SS_06.FM

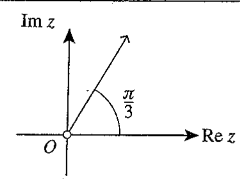
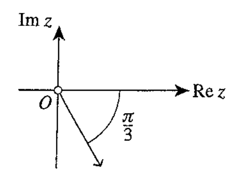
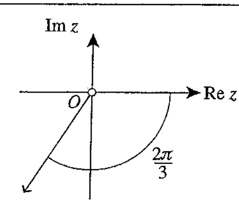
Question 1

	Sample answer	Syllabus outcomes and marking guide
(a)	$g(x)=4\sqrt{x}-2x$ The domain of $g(x)$ is $x \geq 0$.	E6 • Correct domain 1
(b)	Let $g(x)=0$ for x intercepts $4\sqrt{x}-2x=0$ $x=2\sqrt{x}$ $x^2=4x$ $x^2-4x=0$ $x(x-4)=0$ $x=0$ or 4	E6 • Correct intercepts........ 1
(c)	$g(x)=4x^{\frac{1}{2}}-2x$ $g'(x)=2x^{-\frac{1}{2}}-2$ $g''(x)=2\times-\frac{1}{2}x^{-\frac{3}{2}}$ $g''(x)=-\frac{1}{\sqrt{x^3}}$ Now $\frac{1}{\sqrt{x^3}}>0$ for $x>0$ Hence $g''(x)<0$ for $x>0$ $\therefore y=g(x)$ is concave downwards for all $x>0$.	E6 • Correct working........ 1
(d)	For stationary points $g'(x)=0$ $\frac{2}{\sqrt{x}}-2=0$ $\sqrt{x}=1$ $x=1$ and $g(1)=2$ $(1, 2)$ is a maximum because the curve is concave downwards.	E6 • Correct solution........ 1
(e)	$y=g(x)$ 	E6 • Correct graph........ 1

Question 1 (Continued)

Sample answer	Syllabus outcomes and marking guide
(f) (i) $y = \lvert g(x)\rvert$ 	E6 • Correct graph .... 2 • Substantially correct but missing one of intercepts, turning points or correct behaviour at end points 1
(ii) $y = g(x-2)$ 	E6 • Correct graph .... 2 • Substantially correct but missing one of intercepts, turning points or correct behaviour at end points 1
(iii) $y = g(\lvert x\rvert)$ 	E6 • Correct graph .... 2 • Substantially correct but missing one of intercepts, turning points or correct behaviour at end points 1
(iv) $\lvert y\rvert = g(x)$ 	E6 • Correct graph .... 2 • Substantially correct but missing one of intercepts, turning points or correct behaviour at end points 1
(v) $y = \dfrac{1}{g(x)}$ 	E6 • Correct graph .... 2 • Substantially correct but missing one of intercepts, turning points, asymptotes or correct behaviour at end points 1

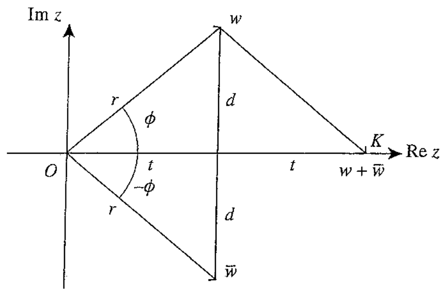
Question 2

Sample answer	Syllabus outcomes and marking guide
(a) $\dfrac{\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)}{\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$ $= \cos\left(\frac{5\pi}{12} + \frac{3\pi}{4} - \frac{2\pi}{3}\right) + i\sin\left(\frac{5\pi}{12} + \frac{3\pi}{4} - \frac{2\pi}{3}\right)$ $= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ $= 0 + i$	E1 • Correct answer .... 2 • Substantially correct 1
(b) $(p+qi)(1-2i) = (1-p) - qi$ $p - 2pi + qi - 2qi^2 = (1-p) - qi$ $(p+2q) + (-2p+q)i = (1-p) - qi$ Equating real and imaginary parts $p + 2q = 1 - p \Rightarrow 2p + 2q = 1$ ...(1) $-2p + q = -q \Rightarrow 2p - 2q = 0$ ...(2) $4p = 1$ Solving (1) and (2) simultaneously, $p = \frac{1}{4}$ $\therefore q = \frac{1}{4}$	E1 • Correct solution .... 3 • Correct method with no more than one mistake in the working .... 2 • Obtains one of the two simultaneous equations. OR • Makes a reasonable attempt 1
(c) (i) $\arg z = \frac{\pi}{3}$ 	E3 • Correct diagram 1
(ii) If $\arg \bar{z} = \frac{\pi}{3}$, then $\arg z = -\frac{\pi}{3}$. 	E3 • Correct diagram 1
(iii) If $\arg(-z) = \frac{\pi}{3}$, then $\arg(-1) + \arg z = \frac{\pi}{3}$. $\arg z = \frac{\pi}{3} - \pi$ $= -\frac{2\pi}{3}$ 	E3 • Correct diagram 1

Question 2 (Continued)

Sample answer | **Syllabus outcomes and marking guide**

(d)



(i) $w = r(\cos\phi + i\sin\phi)$

The distance between w and $\overline{w} = 2 \times d$

$= 2 \times r\sin\phi$

$= 2r\sin\phi$

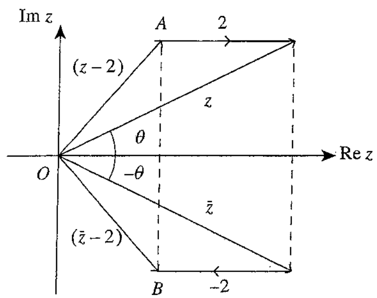
E3
- Correct solution 2
- Correct diagram but no attempt at showing the result.
  OR
- An incorrect diagram with a reasonable attempt at showing the result 1

(ii) $w + \overline{w}$ is represented by $\overrightarrow{OK}$

$w + \overline{w} = 2t$

E1
- Correct answer 1

(e)

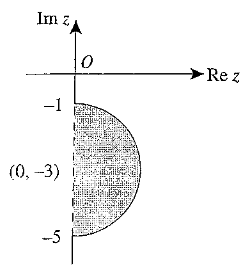


$AO = OB$ (by symmetry) $\Rightarrow |z-2| = |\bar{z}-2|$

E3
- Correct solution 2
- Correct diagram but no attempt at showing the result.
  OR
- An incorrect diagram with a reasonable attempt at showing the result 1

(f) $|z + 3i| \le 2$

Part of the boundary is a circle: centre $(0, -3)$ and radius $r = 2$.



E3
- Correct diagram with shaded region 2
- Substantially correct diagram 1

Question 3

Sample answer | **Syllabus outcomes and marking guide**

(a) $x^3 + 4x^2 - 3x + 1 = 0$ has roots p, q, r.

To find the equation whose roots are $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$.

Let $y = \frac{1}{x}$. Since $x = p, q, r$, $y = \frac{1}{p}, \frac{1}{q}, \frac{1}{r}$

Put $x = \frac{1}{y}$ into the above equation.

$\frac{1}{y^3} + \frac{4}{y^2} - \frac{3}{y} + 1 = 0$

$1 + 4y - 3y^2 + y^3 = 0$

Hence the required equation in x is: $x^3 - 3x^2 + 4x + 1 = 0$

E4
- Correct solution 3
- Correct approach with no more than one algebraic error 2
- Reasonable attempt at solution 1

(b) $3x^3 - 26x^2 + 52x - 24 = 0$

Let the roots be $\frac{\alpha}{r}$, α, αr, where α is a root and r is the common ratio of the geometric progression.

Sum of roots: $\frac{\alpha}{r} + \alpha + \alpha r = \frac{26}{3}$

$\alpha\left(\frac{1}{r} + 1 + r\right) = \frac{26}{3}$ ... (1)

Product of roots: $\frac{\alpha}{r} \times \alpha \times \alpha r = 8$

$\alpha = 2$ ... (2)

Substitute into (1): $\frac{1}{r} + 1 + r = \frac{13}{3}$

$3r^2 - 10r + 3 = 0$

$(3r-1)(r-3) = 0$

$r = 3$ or $\frac{1}{3}$

When $r = 3$ the roots are $\frac{2}{3}$, 2, and 6.

This will reverse for $r = \frac{1}{3}$.

E3
- Correct roots 4
- Correct approach with no more than one algebraic error 3
- Correctly forms the pair of simultaneous equations 2
- Forms an equation in α and r 1

Question 3 (Continued)

Sample answer	Syllabus outcomes and marking guide
(c) (i) By the factor theorem, since k is a zero of $F(x)$, $F(x)=(x-k)Q(x)$ for some polynomial $Q(x)$. Differentiating, $F'(x)=Q(x)+(x-k)Q'(x)$ so $F'(k)=Q(k)+0$ and since k is a zero of $F'(x)$, $Q(k)=0$ Again by the factor theorem, $(x-k)$ is a factor of $Q(x)$, and so $(x-k)^2$ is a factor of $F(x)$.	E4 • Correct reasoning . . . . 2 • Correct approach with poor reasoning . . . 1
(ii) $y^{2t}-ty^{t+1}=1-ty^{t-1}$ $y^{2t}-ty^{t+1}+ty^{t-1}-1=0$ Let $P(y)=y^{2t}-ty^{t+1}+ty^{t-1}-1$ $P(1)=1-t+t-1=0$ $P'(y)=2ty^{2t-1}-t(t+1)y^{t}+t(t-1)y^{t-2}$ $P'(1)=2t-t^2-t+t^2-t=0$ So $y=1$ is a zero of $P(y)$ and $P'(y)$. Hence it must be a double root of $P(y)=0$.	E4 • Correct reasoning . . . . 2 • Correct approach with no more than one algebraic error 1
(d) (i) $k(t)=t^4+at^3+bt^2+at+1$ Since θ is a zero, $k(\theta)=0$. That is, $\theta^4+a\theta^3+b\theta^2+a\theta+1=0$ ... (1) Hence $k\left(\frac{1}{\theta}\right)=\frac{1}{\theta^4}+\frac{a}{\theta^3}+\frac{b}{\theta^2}+\frac{a}{\theta}+1$ $=\frac{1+a\theta+b\theta^2+a\theta^3+\theta^4}{\theta^4}$ $=0$ (by (1))	E3, E4 • Correct solution . . . . 2 • Substantially correct 1
(ii) Since $(1+i)$ is a zero, then using part (i) $\frac{1}{1+i}$ is also a zero. Realising the denominator, $\frac{1-i}{2}$ is also a zero, since all the coefficients are real numbers. Thus by the conjugate root theorem the roots are $1+i,\ 1-i,\ \frac{1}{2}+\frac{i}{2}$ and $\frac{1}{2}-\frac{i}{2}$.	E3, E4 • Correct solution . . . . 2 • Substantially correct 1

Question 4

Sample answer	Syllabus outcomes and marking guide
(a) $I=\int e^x(1+e^x)^5dx$ Let $u=1+e^x$ $\frac{du}{dx}=e^x$ $du=e^xdx$ $I=\int u^5du$ $=\frac{u^6}{6}+C$ $I=\frac{1}{6}(1+e^x)^6+C$	E8 • Correct solution . . . . 2 • Appropriate substitution done correctly OR • Correct modified primitive. 1
(b) $I=\int\frac{dt}{\sqrt{7+6t-t^2}}$ $I=\int\frac{dt}{\sqrt{16-(t^2-6t+9)}}$ $I=\int\frac{dt}{\sqrt{4^2-(t-3)^2}}$ $I=\sin^{-1}\left(\frac{t-3}{4}\right)+C$	E8 • Correct solution . . . . 2 • Reasonable attempt to use the method of completing the square 1
(c) $I=\int\frac{2}{4+3\sin\theta}d\theta$ Let $t=\tan\frac{\theta}{2}$ $\frac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{\theta}{2}$ $2dt=\left(1+\tan^2\frac{\theta}{2}\right)d\theta$ $=(1+t^2)d\theta$ $\frac{2dt}{1+t^2}=d\theta$ Now $\frac{2}{4+3\sin\theta}=\frac{2}{4+3\left(\frac{2t}{1+t^2}\right)}$ $=\frac{2(1+t^2)}{4(1+t^2)+6t}$ $=\frac{2(1+t^2)}{2(2t^2+3t+2)}$ $=\frac{1+t^2}{2t^2+3t+2}$ Hence $I=\int\frac{2dt}{2t^2+3t+2}$ Now $2t^2+3t+2=2\left(t^2+\frac{3}{2}t+\frac{9}{16}\right)+2-2\left(\frac{9}{16}\right)$ $=2\left(t+\frac{3}{4}\right)^2+2\left(1-\frac{9}{16}\right)$ $=2\left[\left(t+\frac{3}{4}\right)^2+\frac{7}{16}\right]$ Hence $I=\int\frac{dt}{\left(\frac{\sqrt{7}}{4}\right)^2+\left(t+\frac{3}{4}\right)^2}$ $I=\frac{1}{\frac{\sqrt{7}}{4}}\tan^{-1}\left(\frac{t+\frac{3}{4}}{\frac{\sqrt{7}}{4}}\right)+C$ $I=\frac{4}{\sqrt{7}}\tan^{-1}\left(\frac{4t+3}{\sqrt{7}}\right)+C$	E8 • Correct solution . . . . 3 • Correctly applies the t-formulae and attempts to complete the square with no more than one algebraic error . . . . 2 • Makes a reasonable attempt to apply the t-formulae. 1

Question 4 (Continued)

Sample answer

(d) $$I = \int \frac{y^5 - 7y^2 + 8}{y^3 - 8}\,dy$$

$$I = \int \frac{y^5 - 8y^2 + y^2 + 8}{(y-2)(y^2 + 2y + 4)}\,dy \qquad \#$$

$$I = \int \frac{y^2(y^3 - 8) + (y^2 + 8)}{(y-2)(y^2 + 2y + 4)}\,dy$$

$$I = \int \left[y^2 + \frac{y^2 + 8}{(y-2)(y^2 + 2y + 4)}\right]dy$$

Let $\dfrac{y^2 + 8}{(y-2)(y^2 + 2y + 4)} \equiv \dfrac{A}{y-2} + \dfrac{By + C}{y^2 + 2y + 4}$

This identity gives $A = 1$, $B = 0$ and $C = -2$

$$I = \int \left(y^2 + \frac{1}{y-2} - \frac{2}{y^2 + 2y + 4}\right)dy$$

$$I = \int \left[y^2 + \frac{1}{y-2} - \frac{2}{(\sqrt{3})^2 + (y+1)^2}\right]dy$$

$$I = \frac{y^3}{3} + \ln|y-2| - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{y+1}{\sqrt{3}}\right) + C$$

Note: sometimes it can be more judicious to perform long division first.

Syllabus outcomes and marking guide

E8

- Correct solution 4
- Correctly applies the method of partial fractions with no more than one algebraic error 3
- Correctly applies the method of partial fractions to find values for A, B and C . . . 2
- Makes a reasonable attempt to simplify the integral 1

Question 4 (Continued)

Sample answer

(e) $$U_m = \int_0^k (k^2 - x^2)^m dx,\ m \ge 0$$

$$U_m = \int_0^k (k^2 - x^2) \times 1\,dx$$

Let $u = (k^2 - x^2)^m \Rightarrow \dfrac{du}{dx} = m(k^2 - x^2)^{m-1} \times -2x$

$$= -2m(k^2 - x^2)^{m-1}x$$

and let $v' = 1 \Rightarrow v = x$

$$U_m = [x \times (k^2 - x^2)^m]_0^k - \int_0^k -2m(k^2 - x^2)^{m-1}x \times x\,dx$$

$$U_m = [k(k^2 - k^2)^m - 0(0^2 - 0^2)^m] + 2m\int_0^k x^2(k^2 - x^2)^{m-1}dx$$

$$U_m = 2m\int_0^k \frac{x^2(k^2 - x^2)^m}{k^2 - x^2}dx \qquad \frac{x^2}{k^2 - x^2} = \frac{k^2 - (k^2 - x^2)}{k^2 - x^2} = \frac{k^2}{k^2 - x^2} - 1$$

$$U_m = 2m\int_0^k (k^2 - x^2)^m\left(\frac{k^2}{k^2 - x^2} - 1\right)dx$$

$$U_m = 2m\int_0^k \frac{k^2(k^2 - x^2)^m}{k^2 - x^2}dx - 2m\int_0^k (k^2 - x^2)^m dx$$

$$U_m = 2mk^2\int_0^k (k^2 - x^2)^{m-1}dx - 2m\int_0^k (k^2 - x^2)^m dx$$

$$U_m = 2mk^2U_{m-1} - 2mU_m$$

$$U_m + 2mU_m = 2mk^2U_{m-1}$$

$$U_m(1 + 2m) = 2mk^2U_{m-1}$$

$$U_m = \frac{2mk^2}{2m + 1}U_{m-1}$$

Syllabus outcomes and marking guide

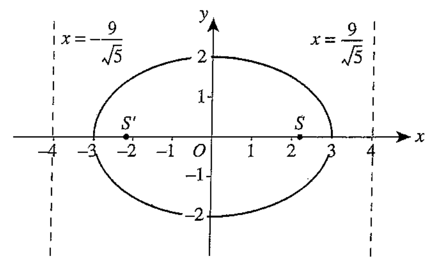
E8

- Correct solution 4
- Substantially correct solution using the method of integration by parts with no more than one algebraic error 3
- Correctly forms the following integral but is unable to simplify further 2
  $$2m\int_0^k x^2(k^2 - x^2)^{m-1}dx$$
- Makes a reasonable attempt to apply the method of integration by parts. 1

Question 5

Sample answer	Syllabus outcomes and marking guide
(a) (i) $M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ $x_M=\frac{a\cos\alpha+a\cos\beta}{2}$ $y_M=\frac{a\sin\alpha+a\sin\beta}{2}$	E9 • Correct coordinates 1
(ii) $\beta-\alpha=90°$	E9 • Correct answer 1
(iii) $\beta=90°+\alpha$ $x_M=\frac{a\cos\alpha+a\cos(90°+\alpha)}{2}$ $=\frac{a\cos\alpha-a\sin\alpha}{2}$ $y_M=\frac{a\sin\alpha+a\sin(90°+\alpha)}{2}$ $=\frac{a\sin\alpha+a\cos\alpha}{2}$	E9 • Correct solution 1
(iv) $2x=a\cos\alpha-a\sin\alpha$ .. (1) $2y=a\sin\alpha+a\cos\alpha$ ... (2) Squaring both sides of (1) and (2) and adding, $4x^2+4y^2=a^2(\cos^2\alpha+\sin^2\alpha-2\sin\alpha\cos\alpha)+a^2(\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha)$ $4x^2+4y^2=a^2(1-2\sin\alpha\cos\alpha+1+2\sin\alpha\cos\alpha)$ $4x^2+4y^2=2a^2$ $x^2+y^2=\frac{a^2}{2}$ Note: the required locus is a circle with centre (0, 0) and radius $\frac{a\sqrt{2}}{2}$.	E9 • Correct solution . . . . 3 • Correct approach with no more than one algebraic error . . . . 2 • Makes a reasonable attempt at solving the simultaneous equations 1

Question 5 (Continued)

Sample answer	Syllabus outcomes and marking guide
(b) (i)	
(α) $4x^2+9y^2=36$ $\frac{x^2}{9}+\frac{y^2}{4}=1$ $a=3,\ b=2$ $b^2=a^2(1-e^2)$ $4=9(1-e^2)$ $\frac{4}{9}=1-e^2$ $e^2=\frac{5}{9}$ $e=\frac{\sqrt{5}}{3}$, since $e>0$	E4 • Correct eccentricity 1
(β) Foci $(\pm ae,0)\Rightarrow\pm(\sqrt{5},0)$	E4 • Correct foci (using answer from part (i)) . 1
(γ) Directrices $x=\pm\frac{a}{e}\Rightarrow x=\pm\frac{9}{\sqrt{5}}$	E4 • Correct directrices 1
(δ) The major axis is 6 units in length.	E4 • Correct answer 1
(ii) 	E4 • Correct diagram showing all necessary information. 1

Question 5 (Continued)

Sample answer	Syllabus outcomes and marking guide
(c) (i) $xy = c^2$ and $P\left(ct, \frac{c}{t}\right)$ $y = c^2x^{-1}$ $\frac{dy}{dx} = -c^2x^{-2}$ # $= -\frac{c^2}{x^2}$ At P, gradient of tangent, m_1: $m_1 = -\frac{c^2}{c^2t^2} \Rightarrow m_1 = -\frac{1}{t^2}$ The gradient of the normal at P is t^2. Hence the equation of the normal is $y - y_1 = m(x - x_1)$ $y - \frac{c}{t} = t^2(x - ct)$ $y = t^2x + \frac{c}{t} - ct^3$ .. (1) # Note: parametric differentiation is an alternative here.	E4 • Correct solution 1
(ii) Put $y = x$ ... (2) Solving (1) and (2) $x = t^2x + \frac{c}{t} - ct^3$ $t^2x - x = \frac{ct^4 - c}{t}$ $x(t^2 - 1) = \frac{c(t^2 - 1)(t^2 + 1)}{t}$ $x = \frac{c}{t}(t^2 + 1)$ provided $t \neq \pm 1$ Hence N is $\left(\frac{c}{t}(t^2 + 1), \frac{c}{t}(t^2 + 1)\right)$	E4 • Correct coordinates of N 1

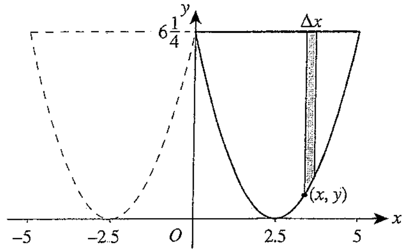
Question 5 (Continued)

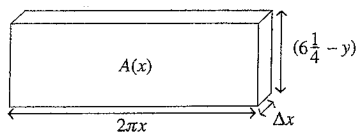
Sample answer	Syllabus outcomes and marking guide
(iii) Show that $OP = NP$ for triangle OPN to be isosceles. Using $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ $OP^2 = (ct)^2 + \left(\frac{c}{t}\right)^2$ $OP^2 = c^2t^2 + \frac{c^2}{t^2}$ $OP^2 = c^2\left(t^2 + \frac{1}{t^2}\right)$ $OP^2 = c^2\left(\frac{t^4 + 1}{t^2}\right)$ $OP = \frac{c}{t}\sqrt{t^4 + 1}$ $NP^2 = \left[\frac{c}{t}(t^2+1) - ct\right]^2 + \left[\frac{c}{t}(t^2+1) - \frac{c}{t}\right]^2$ $NP^2 = c^2\left(\frac{t^2 + 1}{t} - t\right)^2 + \frac{c^2}{t^2}(t^2+1-1)^2$ $NP^2 = \frac{c^2}{t^2} + \frac{c^2}{t^2}(t^4)$ $NP = \frac{c}{t}\sqrt{1 + t^4}$ Hence $NP = OP$ and so triangle OPN is isosceles.	E4 • Correct solution .... 2 • Substantially correct with no more than one mistake 1

Question 6

Sample answer

(a) $y=\frac{1}{4}(2x-5)^2$ and $y=6\frac{1}{4}$





$$A(x)=2\pi x\left(6\frac{1}{4}-y\right)$$

$$\Delta V\approx 2\pi x\left(6\frac{1}{4}-y\right)\Delta x$$

Now, $\left(6\frac{1}{4}-y\right)=\frac{25}{4}-\frac{1}{4}(2x-5)^2$

$$=\frac{1}{4}[25-(4x^2-20x+25)]$$

$$=\frac{1}{4}(25-4x^2+20x-25)$$

$$=\frac{1}{4}(20x-4x^2)$$

$$=5x-x^2$$

$$V=\lim_{\Delta x\to 0}\sum_{x=0}^{5}2\pi x(5x-x^2)\Delta x$$

Hence, $V=2\pi\int_0^5(5x^2-x^3)dx$

$$V=2\pi\left[\frac{5x^3}{3}-\frac{x^4}{4}\right]_0^5$$

$$V=2\pi\left[\frac{625}{3}-\frac{625}{4}\right]$$

$$V=\frac{625\pi}{6}\text{ cubic units}$$

Syllabus outcomes and marking guide

E7

- Correct solution 4
- Correctly uses the method of cylindrical shells with no more than one mistake ... 3
- Writes the expression for ΔV in terms of x 2
- Writes the expression for $A(x)$ 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

Question 6 **(Continued)**

Sample answer

(b) (i) $y=6-x^2$ .. (1)

$y=\frac{1}{2}x^2$ ... (2)

Solving simultaneously,

$$\frac{1}{2}x^2=6-x^2$$

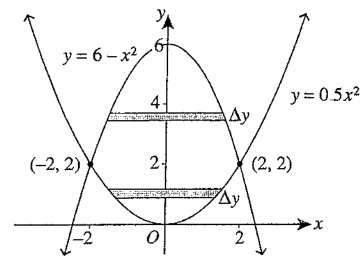
$$x^2=12-2x^2$$

$$3x^2=12$$

$$x^2=4$$

$$x=\pm 2$$

At $x=\pm 2$, $y=2$ in both cases.



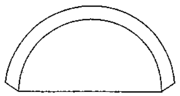
Syllabus outcomes and marking guide

E7

- Correct coordinates of the two points of intersection. 1

Question 6 (Continued)

Sample answer	Syllabus outcomes and marking guide
(ii) Area of a semicircular disc:	E7



$y = 6 - x^2$ $\qquad$ $y = \frac{1}{2}x^2$

$A = \pi x^2 \times \frac{1}{2}$ $\qquad$ $A = \pi x^2 \times \frac{1}{2}$

Volume of discs:

$\Delta V \approx \frac{\pi x^2}{2}\Delta y$ $\qquad$ $\Delta V \approx \frac{\pi x^2}{2}\Delta y$

$$V = \lim_{\Delta y \to 0}\sum_{y=2}^{6}\pi\left(\frac{6-y}{2}\right)\Delta y + \lim_{\Delta y \to 0}\sum_{y=0}^{2}\pi\left(\frac{2y}{2}\right)\Delta y$$

$$V = \frac{\pi}{2}\int_2^6 (6-y)dy + \pi\int_0^2 ydy$$

$$V = \frac{\pi}{2}\left[6y - \frac{y^2}{2}\right]_2^6 + \pi\left[\frac{y^2}{2}\right]_0^2$$

$$V = \frac{\pi}{2}[(36-18)-(12-2)] + \pi\left[\frac{4}{2} - 0\right]$$

$$V = \frac{\pi}{2}(18-10) + 2\pi$$

$V = 6\pi$ cubic units

Marking guide (E7):
- Correct solution 4
- Correctly uses the method of cross-sections with no more than one mistake 3
- Writes the following definite integral to determine the volume of the solid 2
  $$V = \frac{\pi}{6}\int_2^6 (6-y)dy + \pi\int_0^2 ydy$$
- Makes a reasonable attempt at using the method of cross-sections 1

(c) (i) $z^n - \frac{1}{z^n} = (\cos\alpha + i\sin\alpha)^n - (\cos\alpha + i\sin\alpha)^{-n}$

$= \cos n\alpha + i\sin n\alpha - (\cos n\alpha - i\sin n\alpha)$

$= \cos n\alpha + i\sin n\alpha - \cos n\alpha + i\sin n\alpha$

$= 2i\sin n\alpha$

Marking guide (E3):
- Correct solution 2
- Substantially correct 1

Question 6 (Continued)

Sample answer	Syllabus outcomes and marking guide

(ii) Now $z - \frac{1}{z} = 2i\sin\theta \quad (n = 1)$

so $\left(z - \frac{1}{z}\right)^3 = 8i^3\sin^3\theta$

$-8i\sin^3\theta = z^3 - 3z^2\left(-\frac{1}{z}\right) + 3z\left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3$

$(-8\sin^3\theta)i = z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$

$= \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$

$= 2i\sin 3\theta - 3(2i\sin\theta)$, by part (i)

$= (2\sin 3\theta - 6\sin\theta)i$

Hence $-8\sin^3\theta = -6\sin\theta + 2\sin 3\theta$

$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

Marking guide (E3, E4, E9):
- Correct solution 2
- Makes a reasonable attempt to show the result 1

(iii) area = 2 × (area between 0 and π)

$= 2\int_0^{\pi} \sin^3\alpha d\alpha$

$= 2\int_0^{\pi}\left(\frac{3}{4}\sin\alpha - \frac{1}{4}\sin 3\alpha\right)d\alpha$

$= 2\left[\frac{3}{4}(-\cos\alpha) - \frac{1}{4}\left(\frac{1}{3}\right) - \cos 3\alpha\right]_0^{\pi}$

$= 2\left[-\frac{3}{4}\cos\alpha + \frac{1}{12}\cos 3\alpha\right]_0^{\pi}$

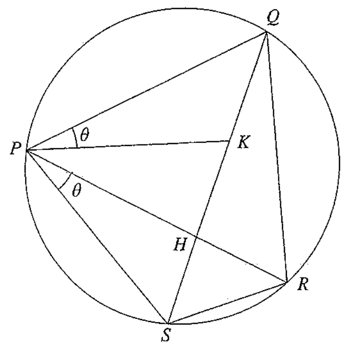
$= 2\left[\left(-\frac{3}{4}\right)(-1) + \frac{1}{12}(-1) - \left(-\frac{3}{4} + \frac{1}{12}\right)\right]$

$= 2\left(\frac{3}{4} - \frac{1}{12} + \frac{3}{4} - \frac{1}{12}\right)$

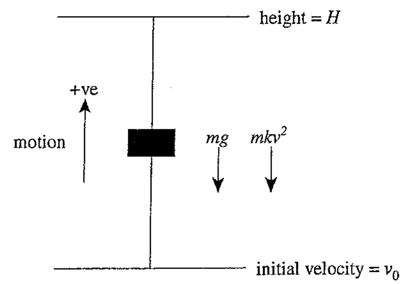
$= 2\frac{2}{3}$ square units

Marking guide (E9):
- Correct solution 2
- Makes a reasonable attempt to determine the area. 1

Question 7

Sample answer	Syllabus outcomes and marking guide
(a) (i) 	E9 • Correct solution . . . . . 2 • Correct approach with poor or incomplete reasoning 1
Let $\angle SPR = \angle QPK = \theta$ In the triangles PQK and PRS $\angle QPK = \angle RPS = \theta$ (given) $\angle PQK = \angle PRS$ (angles in the same arc PS) Hence triangles PQK and PRS are equiangular and hence similar	
(ii) In the triangles PQR and PKS $\angle PRQ = \angle PSK$ (angles in the same arc PQ) $\angle QPR = \angle KPS = \theta + \angle KPH$ Hence triangles PQR and PKS are equiangular and hence similar.	E9 • Correct solution . . . . . 2 • Correct approach with poor or incomplete reasoning 1
(iii) From part (i) $\dfrac{PQ}{PR} = \dfrac{PK}{PS} = \dfrac{QK}{RS}$ So $PQ \times RS = PR \times QK$ From part (ii) $\dfrac{PQ}{QK} = \dfrac{PR}{PS} = \dfrac{QR}{KS}$ So $PR \times KS = PS \times QR$ Hence $PQ \times RS + PS \times QR = PR \times QK + PR \times KS$ $PQ \times RS + PS \times QR = PR(QK + KS)$ $PQ \times RS + PS \times QR = PR \times QS$	E9 • Correct solution . . . . . 3 • Correctly shows results (i) and (ii) . . . . . 2 • Makes a reasonable attempt to prove the result 1

Question 7 (Continued)

Sample answer	Syllabus outcomes and marking guide
(b) (i) 	E5 • Correct solution . . . . . 4 • Correct approach with no more than one algebraic error . . . . . 3 • Writes the following equation . . . . . 2 $\dfrac{dx}{dv} = -\dfrac{v}{10 + kv^2}$ • Draws a correct diagram and finds the equation of motion 1
Using $F = ma$, the equation of motion is: $ma = -mg - mkv^2$ Now, with $m = 1$ and $a = v\dfrac{dv}{dx}$ and $g = 10$ $v\dfrac{dv}{dx} = -(10 + kv^2)$ $\dfrac{dv}{dx} = -\dfrac{10 + kv^2}{v}$ $\dfrac{dx}{dv} = -\dfrac{v}{10 + kv^2}$ $\displaystyle\int_0^H \frac{dx}{dv}dv = -\int_{v_0}^0 \frac{v}{10 + kv^2}dv$ $\displaystyle[x]_0^H = -\frac{1}{2k}\int_{v_0}^0 \frac{2kv}{10 + kv^2}dv$ $H = -\dfrac{1}{2k}[\ln(10 + kv^2)]_{v_0}^0$ $H = -\dfrac{1}{2k}[\ln 10 - \ln(10 + kv_0^2)]$ $H = \dfrac{1}{2k}\ln\left(\dfrac{10 + kv_0^2}{10}\right)$	

Question 7 **(Continued)**

Sample answer

(b) (ii) $2kH = \ln\left(\frac{10 + kv_0^2}{10}\right)$

$$\frac{10 + kv_0^2}{10} = e^{2kH}$$

$$10 + kv_0^2 = 10e^{2kH}$$

$$kv_0^2 = 10e^{2kH} - 10$$

$$v_0^2 = \frac{10}{k}(e^{2kH} - 1) \quad \text{... (1)}$$

Now, if an initial velocity of u generates the maximum height of $2H$,

$$u^2 = \frac{10}{k}(e^{2k(2H)} - 1) \quad \text{... (2)}$$

$(2) \div (1)$

$$\frac{u^2}{v_0^2} = \frac{e^{4kH} - 1}{e^{2kH} - 1}$$

$$\frac{u^2}{v_0^2} = \frac{(e^{2kH} + 1)(e^{2kH} - 1)}{e^{2kH} - 1}$$

$$\frac{u^2}{v_0^2} = e^{2kH} + 1$$

$$u = v_0\sqrt{e^{2kH} + 1}$$

Hence the need to increase the initial velocity by a factor of $(e^{2kH} + 1)^{\frac{1}{2}}$.

Syllabus outcomes and marking guide

E5

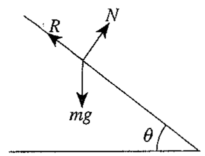
- Correct solution 4
- Correct approach with a reasonable attempt to solve the simultaneous equations 3
- Writes the correct pair of simultaneous equations 2
- Correctly forms one of the equations 1

Question 8

Sample answer

(a) $R = \frac{N}{9}$, $\theta = 45°$, $r = 10$ metres, mass $= m$ kg.

For downward movement:



Resolving vertically:

$$mg = N\cos\theta + R\sin\theta$$

$$m \times 10 = 9R\cos\theta + R\sin\theta$$

(but $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$ since $\theta = 45°$)

$$10m = \frac{10}{\sqrt{2}}R$$

$$\frac{R}{m} = \sqrt{2} \quad \text{...(1)}$$

Resolving horizontally:

$$N\sin\theta - R\cos\theta = \frac{mv^2}{r}$$

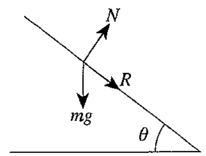
$$9R\sin\theta - R\cos\theta = \frac{mv^2}{r}$$

$$\frac{8}{\sqrt{2}}R = \frac{mv^2}{10}$$

$$\frac{R}{m} = \frac{\sqrt{2}v^2}{80} \quad \text{...(2)}$$

Solving (1) and (2): $\frac{\sqrt{2}v^2}{80} = \sqrt{2} \Rightarrow v = 4\sqrt{5}$ m s^{-1}

For upward movement:



Resolving vertically:

$$mg = N\cos\theta - R\sin\theta$$

$$10m = 9R\cos\theta - R\sin\theta$$

$$10m = \frac{8}{\sqrt{2}}R$$

$$\frac{R}{m} = \frac{10\sqrt{2}}{8} \quad \text{...(3)}$$

Resolving horizontally:

$$N\sin\theta + R\cos\theta = \frac{mv^2}{r}$$

$$9R\sin\theta + R\cos\theta = \frac{mv^2}{10}$$

$$\frac{10}{\sqrt{2}}R = \frac{mv^2}{10}$$

$$\frac{R}{m} = \frac{\sqrt{2}v^2}{100} \quad \text{...(4)}$$

Equating (3) and (4): $\frac{\sqrt{2}v^2}{100} = \frac{10\sqrt{2}}{8}$

$$v^2 = \frac{1000}{8}$$

$$v = \frac{10\sqrt{10}}{2\sqrt{2}}$$

$$v = 5\sqrt{5} \text{ m s}^{-1}$$

Hence $4\sqrt{4} \le v \le 5\sqrt{5}$

Syllabus outcomes and marking guide

E5

- Correct solution 5
- Substantially correct with no more than one mistake 4
- Correctly solves one pair of simultaneous equations 3
- Correctly forms one pair of simultaneous equations 2
- Makes a reasonable attempt to form one of the equations of motion 1

Question 8 (Continued)

Sample answer | **Syllabus outcomes and marking guide**

(b) (i) For $p > 0$, $2p + 1 < 2p + 2$

$$\therefore \frac{1}{2p+1} > \frac{1}{2p+2}$$

$$\therefore \frac{1}{2p+1} + \frac{1}{2p+2} > \frac{1}{2p+2} + \frac{1}{2p+2}$$

$$\frac{1}{2p+1} + \frac{1}{2p+2} > \frac{2}{2p+2}$$

$$\frac{1}{2p+1} + \frac{1}{2p+2} > \frac{1}{p+1}$$

E9
- Correct solution 2
- Substantially correct solution.......... 1

(ii) $\chi(m)$ is $\frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m} \geq \frac{37}{60}$

for all positive integers $m \geq 3$

$\chi(3)$: $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60} \geq \frac{37}{60}$ i.e. true for $m = 3$.

If $\chi(k)$ is true, then

$$\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{2k} \geq \frac{37}{60} \qquad \ldots\text{(A)}$$

now on considering $m = k + 1$:

$$\chi(k+1): \frac{1}{k+2} + \frac{1}{k+3} + \ldots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}$$

$$= \left(\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{2k}\right) + \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$$

$\geq \frac{37}{60}$ (from (A) above) + 0 from part (i), as

$$\frac{1}{2k+1} + \frac{1}{2k+2} > \frac{1}{k+1} \Rightarrow \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1} > 0$$

Thus $\chi(k+1)$ is true whenever $\chi(k)$ is true, and $\chi(3)$ is true.

Therefore $\chi(m)$ is true for all integers $m \geq 3$.

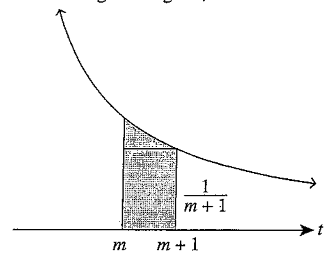
E9
- Correct solution 4
- Substantially correct with no more than one algebraic error 3
- Forms the statement for $m = k + 1$ 2
- Shows the result true for $m = 3$ 1

Question 8 (Continued)

Sample answer | **Syllabus outcomes and marking guide**

(b) (iii)

(α) Considering the diagram,



shaded area > area of rectangle

$$\int_m^{m+1} \frac{1}{t}\,dt > 1 \times \frac{1}{m+1}$$

$$\int_m^{m+1} \frac{1}{t}\,dt > \frac{1}{m+1}$$

E9
- Correct solution 1

(β) As in part (α),

$$\int_{m+1}^{m+2} \frac{1}{t}\,dt > \frac{1}{m+2}$$

$$\int_{m+2}^{m+3} \frac{1}{t}\,dt > \frac{1}{m+3}$$

$$\int_{2m-1}^{2m} \frac{1}{t}\,dt > \frac{1}{2m}$$

Adding these inequalities together

$$\int_m^{m+1} \frac{1}{t}\,dt + \int_{m+1}^{m+2} \frac{1}{t}\,dt + \ldots + \int_{2m-1}^{2m} \frac{1}{t}\,dt > \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m}$$

So $\int_m^{2m} \frac{1}{t}\,dt > \frac{37}{60}$ from part (ii)

$$[\log_e|t|]_m^{2m} > \frac{37}{60}$$

$$\log_e 2m - \log_e m > \frac{37}{60}$$

$$\log_e\left(\frac{2m}{m}\right) > \frac{37}{60}$$

$$\log_e 2 > \frac{37}{60} \text{ as } m > 0$$

E9
- Correct solution 3
- Correct approach with poor or incomplete reasoning 2
- Makes some attempt at the summation of the integrals 1