Neap:

HSC Trial Examination 2007

Mathematics Extension 2

This paper must be kept under strict security and may only be used on or after the morning of Monday 13 August, 2007 as specified in the Neap Examination Timetable

General instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board-approved calculators may be used
- Write using blue or black pen
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

Attempt Questions 1–8

All questions are of equal value

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2007 HSC Mathematics Extension 2 Examination.

Total marks 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{x^3}{4+x^8}\,dx$, using the substitution $u = x^4$. **3**

(b) Find $\int e^{\sqrt{x}}\,dx$, using the substitution $u = \sqrt{x}$ and integration by parts. **3**

(c) (i) Show that $\sqrt{\frac{8-x}{x}} = \frac{4-x}{\sqrt{8x-x^2}} + \frac{4}{\sqrt{8x-x^2}}$. **2**

(ii) Hence evaluate $\int_0^2 \sqrt{\frac{8-x}{x}}\,dx$. **2**

(d) (i) Express $\frac{1}{2x^2-5x+2}$ in the form $\frac{A}{2x-1} + \frac{B}{x-2}$. **2**

(ii) Use the substitution $t = \tan\frac{x}{2}$ and your answer from part (i) to find $\int \frac{dx}{4-5\sin x}$. **3**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) A function $f(x)$ is defined by $f(x) = \log_e\left(\frac{2}{x-1}\right)$.

(i) State the domain of $f(x)$. **1**

(ii) Write down the equation of any asymptote to the graph of $y = f(x)$. **1**

(iii) Find the x-intercepts of $y = f(x)$. **1**

(iv) Is the function increasing or decreasing? Justify your answer **2**

(v) Use your answers to the above questions to sketch graphs of the following.

(α) $y = f(x)$ **1**

(β) $y = |f(x)|$ **1**

(γ) $y = \log_e\left|\frac{2}{x-1}\right|$ **2**

(b) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the x-axis at the points M and N.

A tangent drawn to the ellipse at a point $P\ (a\cos t, b\sin t)$ meets the tangents at M and N at the points Q and R respectively.

(i) Prove that the equation of the tangent at P is $\frac{x\cos t}{a} + \frac{y\sin t}{b} = 1$. **1**

(ii) Hence prove that, for all positions of P, $MQ \times NR$ is a constant. **2**

(c) (i) Show that 2 is a root of the equation $x^3 + 3x - 14 = 0$. **1**

Given that the other roots of the equation are α and β, show that $\alpha + \beta = -2$ and find the value of $\alpha\beta$.

(ii) Find the numerical coefficients of the equation whose roots are 5, $\alpha + 3$ and $\beta + 3$. **2**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Let $\alpha = 5 - 3i$ and $\beta = 2 + i$.

(i) Find $\alpha + \beta$. **1**

(ii) Find $\frac{\alpha}{\beta}$ in the form $x + iy$. **1**

(iii) If $z = x + iy$, sketch the region defined by $\text{Im}(z\alpha) < 3$. **2**

(b) The complex number $z = 1 + 2i$ is a root of the equation $z^2 - aiz + b = 0$, where a and b are real numbers.

(i) Find the values of a and b. **2**

(ii) Find the other root of the equation. **1**

(c) Sketch the region defined by $1 < |z - (1 + i\sqrt{3})| < 2$ and $0 \le \arg z \le \frac{\pi}{3}$. **3**

(d) Let $z = 2\sqrt{3} + 2i$.

(i) Express z in modulus–argument form. **2**

(ii) Hence solve the equation $\alpha^4 = 2\sqrt{3} + 2i$, giving your answer in modulus–argument form. **2**

(iii) Find the side length of the square formed by plotting the solutions to part (ii) on an Argand diagram and joining them together. **1**

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Show that the equation of the normal to the curve $xy = c^2$ at the point $P\left(cp, \frac{c}{p}\right)$ is given by $p^3x - py = c(p^4 - 1)$. 3

(b) The position of a particle moving in the Cartesian plane at a time t is given by the parametric equations

$$x = 5\cos t$$
$$y = 12\sin t$$

(i) Eliminate t from the two equations above. 1

(ii) Sketch the path of the particle in the x–y plane. 1

(iii) Without using the area formula for an ellipse, show by integration that the area of this ellipse is 60π square units. 4

(c) (i) If α, β and γ are the roots of $x^3 - mx^2 + nx - p = 0$, find the following in terms of m, n and p

(α) $\alpha + \beta + \gamma$ 1

(β) $\alpha^2 + \beta^2 + \gamma^2$ 1

(γ) $\alpha^3 + \beta^3 + \gamma^3$ 1

(ii) Using the above results, find a solution of the simultaneous equations 3

$$u + v + w = 5$$
$$u^2 + v^2 + w^2 = 13$$
$$u^3 + v^3 + w^3 = 35$$

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The binomial theorem states that the expansion of an expression of the form $(x + y)^n$, where n is an arbitrary integer, is given by the equation

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

(i) For the case $n = 0$, we see 4

$$(x + y)^0 = \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k = \frac{0!}{0! \times 0!} x^0 y^0 = 1$$

Proceeding from the case $n = 0$, apply the method of mathematical induction to prove that the binomial theorem holds for arbitrary positive integers n.

(ii) A game of chance consists of n independent trials, each of which may be won or lost. The possible number of wins (W) and losses (L) in each game ranges between 0 and n. The binomial distribution can be used to characterise the expected frequency of wins and losses in a sequence of such games. 1

The expected frequencies of wins ($\mathrm{E}(W)$) and losses ($\mathrm{E}(L)$) are the same. What conditions (if any) must be satisfied by the probability of a win ($\mathrm{P}(W)$) or a loss ($\mathrm{P}(L)$) in each trial for this to be the case?

(iii) Another game of chance also consists of n independent trials, each of which may be won or lost. The outcome of a game depends not only on the number of wins and losses that occur, but also on the order in which they occur. 1

What is the number of distinct possible outcomes of this game?

(iv) When a binomial expression is raised to a positive integer power, an expression arises of the form 1

$$(x + y)^n = a_0x^n + a_1x^{n-1}y^1 + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n$$

where $a_k = \binom{n}{k}$ for $k = 0$ to $k = n$.

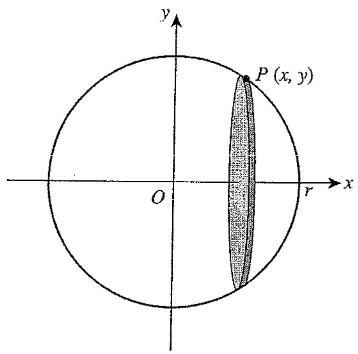
Consider now the relationship between the binomial expansion and Pascal's triangle. The first row of Pascal's triangle has a single entry (1). The second row of Pascal's triangle contains two entries (1, 1), corresponding to the case where $n = 1$ in the binomial expansion, and the third row contains three entries (1, 2, 1), corresponding to the case where $n = 2$ in the binomial expansion, and so on.

Show that the sum of the entries in the $(n + 1)$th row of Pascal's triangle corresponds to the number of distinct possible outcomes of the game described in part (iii) above.

Question 5 continues on page 7

Question 5 (continued) **Marks**

(b) A circle centred at the origin with equation $x^2 + y^2 = r^2$ is shown below, along with the variable point $P(x, y)$ on the circle. **3**



Use the method of slicing, and the diagram above, to show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

(c) A projectile is launched from a point 100 metres above ground level at an angle of elevation of 60° to the horizontal and with a speed of 20 metres per second. Assume that the acceleration due to gravity is $g = 9.8 \text{ m s}^{-2}$ downwards and that friction (or air resistance) is negligible.

Draw a rough sketch representing this situation.

(i) Determine the time of flight of the projectile. **1**

(ii) Determine the impact speed of the projectile at ground level. **2**

(iii) On solving a quadratic equation to determine the time of flight of the projectile, both a positive and a negative value for the time of flight may be obtained. What is the significance of the negative value, and how can it be used to verify that our previous answers are admissible? **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let ω be a complex cube root of unity, where $\omega \neq 1$.

(i) Explain why $\omega^2 + \omega + 1 = 0$. **1**

(ii) If $\alpha = (1 + 2\omega)(1 + \omega^2)$ and $\beta = (1 + \omega)(1 + 2\omega^2)$, show that $\alpha\beta = \alpha + \beta$. **3**

(b) (i) Given that $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n}$ is real for all positive integers n. **1**

(ii) Hence or otherwise, solve the equation $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$. **4**

Give your answers in the form $z = x + iy$.

(c) Let $I_n = \int_0^1 x(1 - x^5)^n dx$, where $n \geq 0$ is an integer.

(i) Show that $I_n = \frac{5n}{5n+2} I_{n-1}$, for $n \geq 1$. **3**

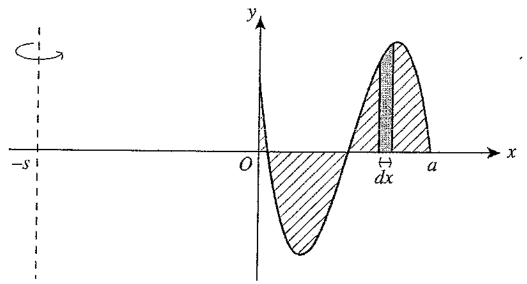
(ii) Show that $I_n = \frac{5^n n!}{2 \times 7 \times 12 \times \ldots \times (5n+2)}$, for $n \geq 1$. **2**

(iii) Hence evaluate I_4. **1**

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) Let $f(x)$ be a smooth, continuous function. The shaded region in the diagram below shows the area between the graph of $y = f(x)$ and the x-axis for x values between 0 and a.



Rotating this cross-sectional area about the line $x = -s$ $(s > 0)$ generates a solid of volume V.

(i) Express the differential volume of the shell (dV) arising from rotating an arbitrary strip of width dx, such as that indicated on the diagram, about the origin of rotation $x = -s$, in terms of π, $f(x)$, s, x and dx. **1**

(ii) Formulate an integral which expresses the total volume V as the sum of all such differential volumes. **1**

(iii) Suppose that the function $f(x)$ is symmetric about the vertical line $x = 0$ between the ordinate values $x = -a$ and $x = +a$. Also suppose that the region enclosed by $f(x)$ and the x-axis between the ordinate values $x = a$ and $x = -a$ has a total area A. As in parts (i) and (ii), a solid of revolution can be generated by rotating this region about the origin of rotation $x = -s$. **2**

Show that there exist at least some values of a for which the total volume of this solid of revolution is given by $V_{\text{total}} = 2\pi sA$.

(iv) What condition must be imposed on a in order that $V_{\text{total}} = 2\pi sA$? **1**

(v) If a satisfies the condition in (iv), what other constraint could be imposed on $f(x)$ to obtain an identical total volume $V_{\text{total}} = 2\pi sA$? **1**

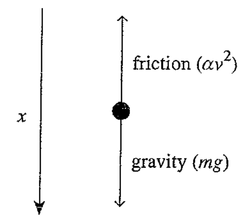
Question 7 continues on page 10

Question 7 (continued)

Marks

(b) In the case of turbulent flow, the resistive frictional force on an object travelling through a fluid is often modelled as being proportional to the square of the travelling velocity.

In the case where a particle is falling vertically through the atmosphere, gravity is providing downward acceleration and the resistive frictional force is effectively acting upward, since friction always opposes motion. The situation is represented diagramatically below.



Expressing the object's net acceleration in terms of its velocity and a positive coefficient α yields

$$ma_{\text{net}} = mg - \alpha v^2$$

According to this representation, the net acceleration of the object is therefore given by

$$a_{\text{net}} = g - \beta v^2, \text{ where } \beta = \frac{\alpha}{m}$$

The coefficient β depends on the surface profile and mass of the object.

(i) Show that setting the initial height of the object as $x = 0$ and assuming that the initial velocity is also zero yields the following equation of motion. **3**

$$\ln(g - \beta v^2) = -2\beta x + \ln g$$

(ii) Show that this allows the velocity to be written as a function of position in the following way. **2**

$$v(x) = \sqrt{\frac{g}{\beta}(1 - e^{-2\beta x})}$$

(iii) By using the result from (ii), state the object's terminal velocity (i.e. its maximum possible velocity). **1**

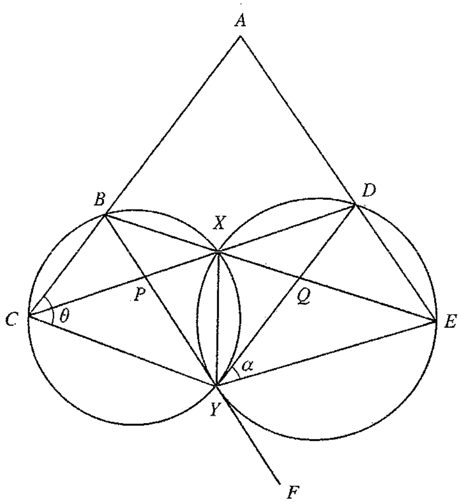
(iv) Calculate the terminal velocity of a table tennis ball in free fall through the atmosphere. Take $g = 9.8 \text{ m s}^{-2}$ and $\beta = 0.20 \text{ m}^{-1}$, and give your answer in kilometres per hour. **1**

(v) How far will the table tennis ball fall before it reaches 80% of its terminal velocity? **2**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



In the diagram above, two circles intersect at X and Y. The tangent to the larger circle at Y meets the smaller circle again at B, and BX is produced to meet the larger circle again at E.

A point C is chosen on the smaller circle to make a cyclic quadrilateral $CBXY$, and the straight lines CBA, $CPXD$, CY, EDA, EY and YQD are joined as shown.

Trace or copy the diagram into your answer booklet.

(i) Let $\angle BCY = \theta$ and $\angle EYD = \alpha$. Show that $\angle EYF = \theta$ and $\angle BYC = \alpha$. **3**

(ii) Show that $YD \parallel CA$. **2**

(iii) Show that Y, D, A and C are concyclic. **1**

(iv) Find a fourth set of four concyclic including Y in the diagram. **1**

(v) If $BP = XP$, show that $BCYQ$ is a parallelogram. **2**

(b) (i) Prove that $1 - r + r^2 - r^3 + \dots + r^{2n} = \frac{r^{2n+1}}{r+1} + \frac{1}{r+1}$, where n is a positive integer and $r \neq -1$. **1**

(ii) For $x > -1$, deduce that $\int_0^x \frac{r^{2n+1}}{r+1}\,dr + \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{x^{2n+1}}{2n+1}$, where n is a positive integer. **2**

(iii) For $0 \le x \le 1$, find $\lim_{n \to \infty} \int_0^x \frac{r^{2n+1}}{r+1}\,dr$, giving reasons for your answer. **2**

(iv) Hence find an infinite series with a limiting sum of $\ln 2$. **1**

End of paper

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x, \quad x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax}, \quad a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax, \quad a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax, \quad a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax, \quad a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax, \quad a \neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln(x + \sqrt{x^2-a^2}), \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln(x + \sqrt{x^2+a^2})$

Note: $\ln x = \log_e x, \quad x > 0$

Neap:

HSC Trial Examination 2007

Mathematics Extension 2

Solutions and marking guidelines

Question 1

Sample answer	Syllabus outcomes and marking guide
(a) $\int \frac{x^3}{4+x^8}dx$ Let $u = x^4$. Then $du = 4x^3 dx$. $\int \frac{x^3}{4+x^8}dx = \frac{1}{4}\int \frac{4x^3}{4+x^8}dx$ $= \frac{1}{4}\int \frac{du}{4+u^2}$ $= \frac{1}{4} \times \frac{1}{2}\tan^{-1}\frac{u}{2} + c$ $= \frac{1}{8}\tan^{-1}\frac{x^4}{2} + c$	E8 • Correct solution . . . . 3 • Correct solution without c . . . . 2 • Correct substitution to give $\frac{1}{4}\int \frac{du}{4+u^2}$ . . . 1
(b) $\int e^{\sqrt{x}}dx$ Let $u = \sqrt{x}$ Then $x = u^2$, and $dx = 2u\,du$. $\int e^{\sqrt{x}}dx = \int e^u \times 2u\,du$ $= e^u 2u - \int 2e^u du$ $= e^u 2u - 2e^u + c$ $= 2e^{\sqrt{x}}(\sqrt{x} - 1) + c$	E8 • Correct solution (disregard omission of c) 3 • Correct use of integration by parts, leaving answer in terms of u. . . . . 2 • Correct substitution of u 1
(c) (i) $\sqrt{\frac{8-x}{x}} = \frac{\sqrt{8-x}}{\sqrt{x}} \times \frac{\sqrt{8-x}}{\sqrt{8-x}}$ $= \frac{8-x}{\sqrt{8x-x^2}}$ $= \frac{4-x}{\sqrt{8x-x^2}} + \frac{4}{\sqrt{8x-x^2}}$	E4 • Completely correct process . . . . 2 • Multiplies by $\frac{\sqrt{8-x}}{\sqrt{8-x}}$ 1
(ii) $\int_0^2 \sqrt{\frac{8-x}{x}}dx = \int_0^2\left(\frac{4-x}{\sqrt{8x-x^2}} + \frac{4}{\sqrt{8x-x^2}}\right)dx$ $= \left[\sqrt{8x-x^2}\right]_0^2 + \int_0^2 \frac{4}{\sqrt{16-(x-4)^2}}dx$ $= 2\sqrt{3} + \left[4\sin^{-1}\left(\frac{x-4}{4}\right)\right]_0^2$ $= 2\sqrt{3} + 4\left(-\frac{\pi}{6} + \frac{\pi}{2}\right)$ $= 2\sqrt{3} + \frac{4\pi}{3}$	E8 • Correct solution . . . . 2 • Correct integration and evaluation of one of the fractions OR • Correct integration of both fractions with wrong or no evaluation 1

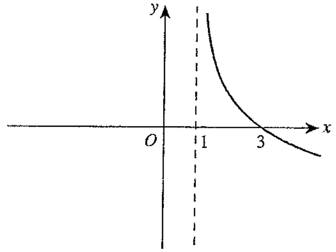
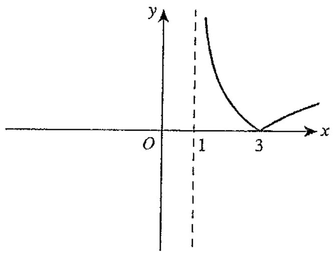
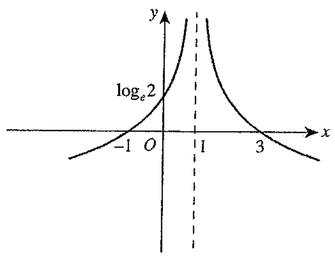
Question 1 (Continued)

Sample answer	Syllabus outcomes and marking guide
(d) (i) $\dfrac{1}{2x^2-5x+2}=\dfrac{1}{(2x-1)(x-2)}$ Let $\dfrac{1}{2x^2-5x+2}=\dfrac{A}{2x-1}+\dfrac{B}{x-2}$ Then $A(x-2)+B(2x-1)=1$ Put $x=2$: $3B=1$, so $B=\dfrac{1}{3}$ Put $x=\dfrac{1}{2}$: $-\dfrac{3A}{2}=1$, so $A=-\dfrac{2}{3}$ Hence $\dfrac{1}{2x^2-5x+2}=\dfrac{1}{3(x-2)}-\dfrac{2}{3(2x-1)}$ OR $Ax-2A+2Bx-B=1$ Equating coefficients, $A+2B=0$, $A=-2B$ $-2A-B=1$ Substituting in $A=-2B$ gives $3B=1$. Hence $B=\dfrac{1}{3}$ and $A=-\dfrac{2}{3}$	E8 • Correct values for A and B, using either method . . . . 2 • Either A or B correct OR • Correct process with one error. 1
(ii) Let $t=\tan\dfrac{x}{2}$ Then $\dfrac{x}{2}=\tan^{-1}t$ $x=2\tan^{-1}t$ $dx=\dfrac{2dt}{1+t^2}$ $4-5\sin x=\dfrac{4(1+t^2)-10t}{1+t^2}$ $\displaystyle\int\frac{dx}{4-5\sin x}=\int\frac{1+t^2}{4t^2-10t+4}\times\frac{2dt}{1+t^2}$ $\displaystyle=\int\frac{dt}{2t^2-5t+2}$ $\displaystyle=\frac{1}{3}\int\left(\frac{1}{t-2}-\frac{2}{2t-1}\right)dt$ $=\dfrac{1}{3}(\ln\lvert t-2\rvert-\ln\lvert 2t-1\rvert)+c$ $=\dfrac{1}{3}\left(\ln\left\lvert\tan\dfrac{x}{2}-2\right\rvert-\ln\left\lvert 2\tan\dfrac{x}{2}-1\right\rvert\right)+c$ $=\dfrac{1}{3}\left(\ln\left\lvert\dfrac{\tan\frac{x}{2}-1}{2\tan\frac{x}{2}-1}\right\rvert\right)+c$	E8 • Correct solution in either form (ignore omission of c). . . . . 3 • Finds correct integral in terms of t (following from previous error). . . . . 2 • Finds $dx=\dfrac{2dt}{1+t^2}$ and the correct equivalent expression for $4-5\sin x$ 1

Question 2

Sample answer	Syllabus outcomes and marking guide
(a) (i) $f(x)=\log_e\left(\dfrac{2}{x-1}\right)$ $f(x)$ exists when $\dfrac{2}{x-1}$ is positive. $\dfrac{2}{x-1}>0$ when $x-1>0$ That is, $x>1$ Hence the domain is given by $x>1$.	E2, E6, E9 • Correct domain 1
(ii) As $x\to1^+$, $\dfrac{2}{x-1}\to+\infty$, so $\log_e\left(\dfrac{2}{x-1}\right)\to\infty$. So $x=1$ is a vertical asymptote to $y=f(x)$.	E2, E6, E9 • Correct answer 1
(iii) For an x-intercept, $\log_e\left(\dfrac{2}{x-1}\right)=0$. $\dfrac{2}{x-1}=1$ $x-1=2$ $x=3$ There is only one x-intercept, (3, 0)	E2, E6, E9 • Correct x-intercept. 1
(iv) $y=f(x)=\log_e\left(\dfrac{2}{x-1}\right)$ $y=\log_e 2-\log_e(x-1)$ $y'=-\dfrac{1}{x-1}$ Since $x>1$, $y'<0$. Hence $y=f(x)$ is always decreasing.	E2, E6, E9 • Correct derivative AND • Correct justification . . . . 2 • Correct derivative OR • Correct justification 1

Question 2 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(a) (v) (α) $y=f(x)$ 	E2, E6, E9 • Correct graph 1
(β) $y=\lvert f(x)\rvert$ 	E2, E6, E9 • Correct graph 1
(γ) $y=\log_e\left\lvert\dfrac{2}{x-1}\right\rvert$ 	E2, E6, E9 • Correct graph 2

Question 2 **(Continued)**

Sample answer

(b) (i) Let $P\ (x_1, y_1)$ be a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Differentiating, $\dfrac{2x}{a^2}+\dfrac{2y}{b^2}y'=0$

Hence at $P\ (x_1, y_1)$, $y'=-\dfrac{b^2x_1}{a^2y_1}$

So the equation of the tangent at P is

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$$

$$\frac{b^2x_1x}{a^2b^2}+\frac{a^2y_1y}{a^2b^2}=\frac{b^2x_1^2}{a^2b^2}+\frac{a^2y_1^2}{a^2b^2}\ \text{(dividing by } a^2b^2\text{)}$$

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \text{since } P \text{ is on the ellipse}$$

So the equation of the tangent is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

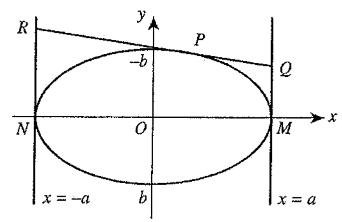
Substituting in $x_1=a\cos t,\ y_1=b\sin t$ gives

$$\frac{x\cos t}{a}+\frac{y\sin t}{b}=1$$

Syllabus outcomes and marking guide

E2, E4
- Correct solution 1

(ii)



Let point P have coordinates $(a\cos t, b\sin t)$.

Equation of the tangent at P is $\dfrac{x\cos t}{a}+\dfrac{y\sin t}{b}=1$ (1)

The point Q is obtained by substituting $x=a$ in (1).

$$\cos t+\frac{y\sin t}{b}=1$$

$$y=\frac{b}{\sin t}(1-\cos t)$$

Hence $MQ=\left\lvert\dfrac{b}{\sin t}(1-\cos t)\right\rvert$.

In a similar way, $NR=\left\lvert\dfrac{b}{\sin t}(1+\cos t)\right\rvert$.

$$MQ\times NR=\left\lvert\frac{b}{\sin t}(1-\cos t)\times\frac{b}{\sin t}(1+\cos t)\right\rvert$$

$$=\left\lvert\frac{b^2}{\sin^2 t}(1-\cos^2 t)\right\rvert$$

$$=b^2,\ \text{since } 1-\cos^2 t=\sin^2 t$$

Hence $MQ\times NR$ is a constant.

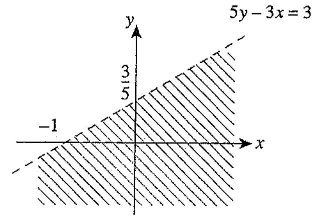
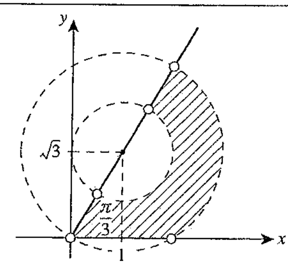
Syllabus outcomes and marking guide

E2, E4
- Correct solution 2
- Makes the correct substitution with no more than one error 1

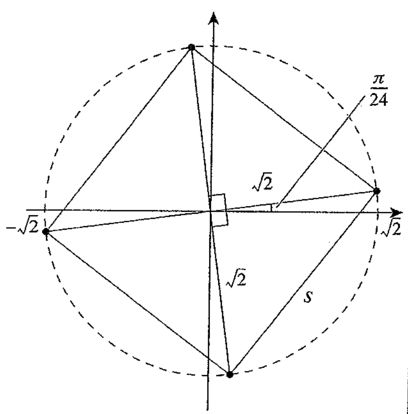
Question 2 (Continued)

Sample answer	Syllabus outcomes and marking guide
(c) (i) $x^3+3x-14=0$ When $x=2$, $\text{LHS}=8+6-14$ $=0$ $=\text{RHS}$ So $x-2$ is a root. Let α and β be the other two roots. Then $2+\alpha+\beta=-\frac{b}{a}$ $=0$ Hence $\alpha+\beta=-2$ Also, the sum of the products of the roots taken two at a time equals $\frac{c}{a}$, so $2\alpha+2\beta+\alpha\beta=3$ $2(\alpha+\beta)+\alpha\beta=3$ $2\times(-2)+\alpha\beta=3$ $\alpha\beta=7$	E1, E2, E4 • Correct solution 1
(ii) To find the coefficients we substitute $y-3$ for x. $(y-3)^3+3(y-3)-14=0$ $y^3-9y^2+27y-27+3y-9-14=0$ $y^3-9y^2+30y-50=0$	E1, E2, E4 • Correct equation........................ 2 • Correct substitution for x with no more that one error........................ 1

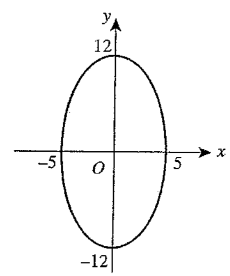
Question 3

Sample answer	Syllabus outcomes and marking guide
(a) (i) $\alpha=5-3i$ and $\beta=2+i$ $\alpha+\beta=7-2i$	E3 • Correct answer........................ 1
(ii) $\frac{\alpha}{\beta}=\frac{5-3i}{2+1}\times\frac{2-i}{2-i}$ $=\frac{7}{5}-\frac{11i}{5}$	E3 • Correct answer........................ 1
(iii) $z=x+iy$ $z\alpha=(x+iy)(5-3i)$ $=5x+3y-3ix+5iy$ $\text{Im}(z\alpha)=5y-3x$ Hence $\text{Im}(z\alpha)<3$ is given by $5y-3x<3$. 	E3 • Correct region, showing intercepts and dotted line........................ 2 • Correct expression for $\text{Im}(z\alpha)$ OR • Correct region following from incorrect $\text{Im}(z\alpha)$........................ 1
(b) (i) $z=1+2i$ is a root of the equation $z^2-aiz+b=0$. $(1+2i)^2-ai(1+2i)+b=0$ $-3+4i-ai+2a+b=0$ Equating real and imaginary parts, as a and b are real: $2a+b-3=0$ and $4-a=0$ Therefore $a=4$ and $b=-5$ So $z^2-4iz-5=0$.	E3 • Correct values for a and b........................ 2 • Correct substitution and expansion in original equation OR • One correct value only........................ 1
(ii) The equation is $z^2-4iz-5=0$. Let the other root be α. Then $\alpha+(1+2i)=4i$ $\alpha=-1+2i$ So the other root is $-1+2i$.	E3 • Correct answer........................ 1
(c) 	E3 • Completely correct region, including showing that the point $(1, \sqrt{3})$ lies on the diameter........................ 3 • Correct regions, but circles not dotted OR • Correct regions, but points of intersection not marked with open circles........................ 2 • Correct region between dotted circles OR • Correct regions between the lines $y=0$ and $y=\sqrt{3}x$........................ 1

Question 3 (Continued)

Sample answer	Syllabus outcomes and marking guide
(d) (i) $z = 2\sqrt{3} + 2i$ $\lvert z\rvert = 4$, $\operatorname{Arg} z = \frac{\pi}{6}$ $\therefore z = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$	E3 • Correct answer . . . . 2 • Correct modulus OR • Correct argument 1
(ii) As $\alpha^4 = 2\sqrt{3} + 2i$, $\alpha = z^{\frac{1}{4}}$. So one solution is given by $\alpha = \sqrt{2}\left(\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right)$ The spacing of the solutions around a circle on an Argand diagram is $\frac{2\pi}{4} = \frac{\pi}{2}$. Hence all solutions for α are given by $\alpha = \sqrt{2}\left(\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right)$, $\sqrt{2}\left(\cos\frac{13\pi}{24} + i\sin\frac{13\pi}{24}\right)$, $\sqrt{2}\left(\cos\frac{-11\pi}{24} + i\sin\frac{-11\pi}{24}\right)$, $\sqrt{2}\left(\cos\frac{-23\pi}{24} + i\sin\frac{-23\pi}{24}\right)$	E3 • Correct solutions following from previous answer . . . . 2 • Finds one answer OR • Determines spacing of solutions 1
(iii) $\lvert\alpha\rvert = \sqrt{2}$ and the spacing of the solutions is $\frac{\pi}{2}$.  By Pythagoras' theorem, $s^2 = (\sqrt{2})^2 + (\sqrt{2})^2$. Hence $s = \sqrt{2+2}$ $= 2$ units	E3 • Correct side length following from previous answer 1

Question 4

Sample answer	Syllabus outcomes and marking guide
(a) $xy = c^2$ $\frac{d}{dx}(xy) = 0$ $x\frac{dy}{dx} + y = 0$ $\frac{dy}{dx} = -\frac{y}{x}$ At $P\left(cp, \frac{c}{p}\right)$, $\frac{dy}{dx} = \frac{-\frac{c}{p}}{cp} = -\frac{1}{p^2}$ So the gradient of the normal is p^2. The equation of the normal is $y - \frac{c}{p} = p^2(x - cp)$ $py - c = p^3x - cp^4$ $p^3x - py = cp^4 - c$ $p^3x - py = c(p^4 - 1)$, as required.	E2, E3, E4 • Correct solution . . . . 3 • Correctly finds the derivative and the gradient of the normal . . . . 2 • Correctly finds the derivative. 1
(b) (i) $x = 5\cos t$ $\cos t = \frac{x}{5}$ $y = 12\sin t$ $\sin t = \frac{y}{12}$ Since $\sin^2 t + \cos^2 t = 1$, $\frac{x^2}{25} + \frac{y^2}{144} = 1$, which is the equation of an ellipse.	E2, E3, E4 • Finds the correct Cartesian equation of the ellipse. 1
(ii) The Cartesian equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{144} = 1$. Hence $a = 5$, $b = 12$. Hence the graph is as below  	E2, E3, E4 • Correct graph 1

Question 4 (Continued)

Sample answer	Syllabus outcomes and marking guide
(b) (iii) Area $= 4\int_0^5 y\,dx$	E2, E3, E4 • Correct solution 4
$= 4\int_0^5 \frac{12}{5}\sqrt{25-x^2}\,dx$	• Correctly expresses the area and makes the right substitution with no more than one error 3
$= \frac{12}{5} \times 4\int_0^5 \sqrt{25-x^2}\,dx$	• Correctly represents the area 2
$= \frac{12}{5} \times 25\pi$ since $4\int_0^5 \sqrt{25-x^2}\,dx$ = area of a circle of radius 5 $= 25\pi$ $= 60\pi$	• Correctly expresses y in terms of x 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

Question 4 (Continued)

Sample answer	Syllabus outcomes and marking guide
(c) (i) (α) $x^3 - mx^2 + nx - p = 0$ $\alpha + \beta + \gamma = m$	E2, E4 • Correct answer 1
(β) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$ $= m^2 - 2n$	E2, E4 • Correct answer 1
(γ) $x^3 = mx^2 - nx + p$ Substituting each of the roots into this equation gives $\alpha^3 = m\alpha^2 - n\alpha + p$ (1) $\beta^3 = m\beta^2 - n\beta + p$ (2) $\gamma^3 = m\gamma^2 - n\gamma + p$ (3) Finding (1) + (2) + (3) gives $\alpha^3 + \beta^3 + \gamma^3 = m(\alpha^2 + \beta^2 + \gamma^2) - n(\alpha + \beta + \gamma) + 3p$ $= m(m^2 - 2n) - n \times m + 3p$ $= m^3 - 2mn - mn + 3p$ $= m^3 - 3mn + 3p$	E2, E4 • Correct answer 1
(ii) Let us consider the two sets of equations $u + v + w = 5$ $\alpha + \beta + \gamma = m$ $u^2 + v^2 + w^2 = 13$ and $\alpha^2 + \beta^2 + \gamma^2 = m^2 - 2n$ $u^3 + v^3 + w^3 = 35$ $\alpha^3 + \beta^3 + \gamma^3 = m^3 - 3mn + 3p$ By comparing the two, we see that u, v and w are roots of $x^3 - mx^2 + nx - p = 0$, where $m = 5$ $n = 6$ (from $m^2 - 2n = 13$) $p = 0$ (from $m^3 - 3mn + 3p = 35$) So u, v and w are roots of $x^3 - 5x^2 + 6x = 0$. $x(x^2 - 5x + 6) = 0$ $x(x-2)(x-3) = 0$ $x = 0, 2, 3$ So a solution is $u = 0$, $v = 2$ and $w = 3$.	E2, E4 • Correct solution . . . . 3 • Correctly identifies the similarity between the two sets if equations and finds the correct values of m, n and p . . . . 2 • Recognises the comparison between the two sets of equations and attempts to find m, n and p. 1

Question 5

Sample answer

(a) (i) Assume that the binomial relationship holds for an arbitrary positive integer $n = m$,

$$(x+y)^m = \sum_{k=0}^{m}\binom{m}{k}x^{m-k}y^k$$

If it is true for $n = m$, we now aim to prove it is true for

$n = m+1$, that is $(x+y)^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k}x^{m+1-k}y^k$

Proof:

$$(x+y)^{m+1} = (x+y)(x+y)^m$$

$$= x(x+y)^m + y(x+y)^m$$

$$= x\sum_{k=0}^{m}\binom{m}{k}x^{m-k}y^k + y\sum_{j=0}^{m}\binom{m}{j}x^{m-j}y^j$$

$$= \sum_{k=0}^{m}\binom{m}{k}x^{m+1-k}y^k + \sum_{j=0}^{m}\binom{m}{j}x^{m-j}y^{j+1}$$

Now let $k = j+1$ in the second summation

$$= x^{m+1} + \sum_{k=1}^{m}\binom{m}{k}x^{m+1-k}y^k + \sum_{k=1}^{m+1}\binom{m}{k-1}x^{m+1-k}y^k$$

$$= x^{m+1} + \sum_{k=1}^{m}\binom{m}{k}x^{m+1-k}y^k + \sum_{k=1}^{m}\binom{m}{k-1}x^{m+1-k}y^k + y^{m+1}$$

$$= x^{m+1} + y^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k}x^{m+1-k}y^k + \binom{m}{k-1}x^{m+1-k}y^k\right]$$

$$= x^{m+1} + y^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right]x^{m+1-k}y^k$$

$$= x^{m+1} + y^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k}x^{m+1-k}y^k$$

$$= \sum_{k=0}^{m+1}\binom{m+1}{k}x^{m+1-k}y^k$$

Question 5 (a) (i) sample answer continues on page 14

Syllabus outcomes and marking guide

E4, E8, E9

- Correctly demonstrates the result $(x+y)^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k}x^{m+1-k}y^k$ and proceeds with a statement of inductive proof. 4
- Correctly demonstrates the result $(x+y)^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k}x^{m+1-k}y^k$ . . 3
- Proceeds with a correct $(m+1)$ substitution to obtain an expression of the form $x^{m+1} + y^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k}x^{m+1-k}y^k + \binom{m}{k-1}x^{m+1-k}y^k\right]$ 2
- Substitutes the case $(m+1)$ correctly, but does not proceed correctly any further. . . 1

Question 5 (Continued)

Sample answer	Syllabus outcomes and marking guide
(a) (i) (continued) So if $(x+y)^m = \sum_{k=0}^{m}\binom{m}{k}x^{m-k}y^k$ then $(x+y)^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k}x^{m+1-k}y^k$ as required. We have now proven that if there exists an n for which the binomial relationship holds, then it also holds for $n+1$ and all subsequent integers. Since we have shown that $(x+y)^0 = \sum_{k=0}^{0}\binom{0}{k}x^{0-k}y^k$, we have now proven that the binomial relationship holds for all n.	
(ii) $P(A) = P(B) = \frac{1}{2}$	E2, E4, E8, E9 • Correct answer. 1
(iii) Since there are n trials and each has a distinct pair of possible outcomes, the number of distinct possible outcomes as a whole is 2^n.	E2, E8, E9 • Correct answer. 1
(iv) The binomial expansion $(x+y)^n$ yields the $(n+1)$th row of Pascal's triangle when we set $x = y = 1$. On setting $x = y = 1$, we obtain $(1+1)^n = 2^n$, which is the number of distinct possible outcomes in part (iii).	E2, E8, E9 • Correct answer. 1

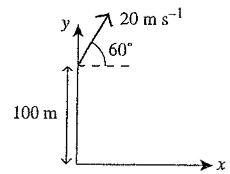
Question 5 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(b) $V_{\text{disk}} = A_{\text{disk}}dx$ where V_{disk} is the surface area of the disk and dx is the disk's thickness.	E7 • Obtains a correct answer via a correct method 3
$V_{\text{disk}} = \pi r_{\text{disk}}^2 dx$ $= \pi(\sqrt{r^2 - x^2})^2 dx$ $= \pi(r^2 - x^2)dx$	• Determines the integrand as $\pi(r^2 - x^2)$ and sets it in an appropriate integral with correct limits 2
$V_{\text{sphere}} = \int_{-r}^{r} \pi(r^2 - x^2)dx$ $= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r}$ $= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$ $= \pi\left(2r^3 - \frac{2r^3}{3}\right)$ $= \pi\left(\frac{6r^3}{3} - \frac{2r^3}{3}\right)$ $= \pi\left(\frac{4r^3}{3}\right)$ $= \frac{4}{3}\pi r^3$	• Writes the volume of the disk as $V_{\text{disk}} = \pi r_{\text{disk}}^2 dx$ 1

Question 5 **(Continued)**

Sample answer / **Syllabus outcomes and marking guide**

(c) (i)



Let us define $s_x, v_x, a_x, s_y, v_y, a_y$ as displacement, velocity and acceleration in the x and y directions respectively.

The time of flight, t_f, is then a value for which $s_y = 0$.

$$s_y = u_y t_f + \frac{a_y t_f^2}{2} + 100$$

$$0 = \frac{a_y}{2} t_f^2 + u_y t_f + 100$$

$$t_f = \frac{-u_y \pm \sqrt{u_y^2 - 4 \times \frac{a_y}{2} \times 100}}{a_y}$$

Now $u_y = \sin 60° \times 20 \text{ m s}^{-1}$ and $a_y = -9.8 \text{ m s}^{-2}$

$= 17.321 \text{ m s}^{-1}$

$$t_f = \frac{-17.321 \pm \sqrt{(17.321)^2 + 200 \times 9.8}}{-9.8}$$

$= 6.6183, -3.08$

Taking the positive value gives

$t_f = 6.6183 \text{ s} \approx 6.6 \text{ s}$

E4, E6, E8
- Obtains a correct solution via a correct method 1

(ii) $v_x = 20\cos 60°$

$= 10$ metres per second

$v_y = u_y + a_y t_f$

$= 20\sin 60° - 9.8 \times 6.6183$

$= 17.3205 - 64.85934$

$= -47.53884$ metres per second

$\text{speed} = \sqrt{v_y^2 + v_x^2}$

$= \sqrt{(-47.53884)^2 + 10^2}$

$= 48.58$ metres per second

E4, E6, E8
- Correctly performs a vectorial addition to determine impact speed 2
- Correctly determines the horizontal and vertical components of velocity 1

(iii) The negative value for the time of flight corresponds to the time at which the projectile would have had to have been launched from ground level in order to trace out the same path.

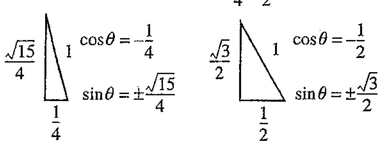
To verify our present calculations, we can check that the launch and impact speeds at ground level would have values of the same magnitude as one another.

E4, E6, E8
- States the significance of the negative time of flight value

AND

- Observes that these values should give identical speeds at ground level 2
- States the significance of the negative time of flight value 1

Question 6

	Sample answer	Syllabus outcomes and marking guide
(a) (i)	$x^3-1=(x-1)(x^2+x+1)$ So the cube roots of unity come from either $x-1=0$ or $x^2+x+1=0$ As $\omega\neq 1$, $\omega^2+\omega+1=0$.	E3 • Correct explanation 1
(ii)	$\alpha=(1+2\omega)(1+\omega^2)$ $=1+\omega^2+2\omega+2\omega^3$ $=2\omega^2+2\omega+2-(1+\omega^2)+2\omega^3$ $=2(\omega^2+\omega+1)+2\omega^3-(1+\omega)$ $=1-\omega^2$, since $\omega^3=1$ and $\omega^2+\omega+1=0$. $\beta=(1+\omega)(1+2\omega^2)$ $=1+2\omega^2+\omega+2\omega^3$ $=2\omega^2+2\omega+2-(1+\omega)+2\omega^3$ $=2(\omega^2+\omega+1)-(1+\omega)+2\omega^3$ $=1-\omega$ as $\omega^3=1$ and $\omega^2+\omega+1=0$ $\alpha+\beta=2-\omega-\omega^2$ $=3-(1+\omega+\omega^2)$ $=3$ as $1+\omega+\omega^2=0$ $\alpha\beta=(1-\omega^2)(1-\omega)$ $=1-\omega-\omega^2+\omega^3$ $=2-(1+\omega+\omega^2)+\omega^3$ $=3$ as $1+\omega+\omega^2=0$ and $\omega^3=0$ So $\alpha+\beta=\alpha\beta$	E2, E3 • Shows that $\alpha+\beta=\alpha\beta$ with full reasoning throughout .... 3 • Finds expressions for α and β with full reasoning .... 2 • Finds expressions for α and β with no reasoning 1
(b) (i)	$z=\cos\theta+i\sin\theta$ $z^n=\cos n\theta+i\sin n\theta$ by de Moivre's theorem $z^{-n}=\cos(-n\theta)+i\sin(-n\theta)$ by de Moivre's theorem $=\cos n\theta-i\sin n\theta$ $z^n+z^{-n}=2\cos n\theta$, which is real for all positive integers n.	E3 • Correct proof 1
(ii)	$2z^4+3z^3+5z^2+3z+2=0$ $2z^2+3z+5+3z^{-1}+2z^{-2}=0$ (dividing through by z^2) $2(z^2+z^{-2})+3(z+z^{-1})+5=0$ $2(2\cos 2\theta)+3(2\cos\theta)+5=0$ $4\cos 2\theta+6\cos\theta+5=0$ $4(2\cos^2\theta-1)+6\cos\theta+5=0$ $8\cos^2\theta+6\cos\theta+5=0$ $(4\cos\theta+1)(2\cos\theta+1)=0$ $\cos\theta=-\frac{1}{4},-\frac{1}{2}$  Hence $z=-\frac{1}{4}\pm\frac{\sqrt{15}}{4}i,\ -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$	E3, E2 • Correct solution .... 4 • Correct values for $\cos\theta$ .... 3 • Arrives at $4\cos 2\theta+6\cos\theta+5=0$ .... 2 • Divides through by z^2 correctly 1

HSC Mathematics Extension 2 Trial Examination Solutions and marking guidelines

Question 6 (Continued)

	Sample answer	Syllabus outcomes and marking guide
(c) (i)	$I_n=\int_0^1 x(1-x^5)^n dx$ $=\left[\frac{x^2}{2}(1-x^5)^n\right]_0^1-\frac{1}{2}\int_0^1 -5x^4nx^2(1-x^5)^{n-1}dx$ $=0-\frac{5n}{2}\int_0^1(-x^5)\times x(1-x^5)^{n-1}dx$ $=\frac{5n}{2}\int_0^1[(1-x^5)-1](1-x^5)^{n-1}x\,dx$ $=-\frac{5n}{2}\int_0^1[x(1-x^5)^n-x(1-x^5)^{n-1}]dx$ $I_n=-\frac{5n}{2}I_n+\frac{5n}{2}I_{n-1}$ $5nI_n+2I_n=5nI_{n-1}$ $I_n(5n+2)=5nI_{n-1}$ $I_n=\frac{5n}{5n+2}I_{n-1}$	E2, E8 • Correct solution .... 3 • Correctly determines the constant outside the remaining integral AND • Uses the step $[(1-x^5)-1]$ .... 2 • Correct first step of integration including evaluation of the first part of the integral 1
(ii)	$I_0=\int_0^1 x\,dx$ $=\left[\frac{x^2}{2}\right]_0^1$ $=\frac{1}{2}$ $I_1=\frac{5(1)}{5(1)+2}\times\frac{1}{2}$ $I_2=\frac{5(2)}{5(2)+2}\times\frac{5(1)}{5(1)+2}\times\frac{1}{2}$ $I_3=\frac{5(3)}{5(3)+2}\times\frac{5(2)}{5(2)+2}\times\frac{5(1)}{5(1)+2}\times\frac{1}{2}$ So $I_n=\frac{1\times 5(1)\times 5(2)\times 5(3)\times\ldots\times 5n}{2[5(1)+2][5(2)+2][5(3)+2]\times\ldots\times[5(n)+2]}$ $=\frac{5^n(1\times 2\times 3\times\ldots\times n)}{2\times 7\times 12\times\ldots\times[5(n)+2]}$ $=\frac{5^n n!}{2\times 7\times 12\times\ldots\times[5(n)+2]}$	E2, E8 • Correct solution .... 2 • Correct evaluation of I_0 1
(iii)	$I_4=\frac{5^4\times 4!}{2\times 7\times 12\times 17\times 22}$ $=\frac{625}{2618}$	E2 • Correct answer 1

Question 7

	Sample answer	Syllabus outcomes and marking guide
(a)	(i) $dV = \lvert f(x)\rvert 2\pi(s+x)dx$	E6, E7, E9 • Correct answer 1
	(ii) $V = \int_0^a \lvert f(x)\rvert 2\pi(s+x)dx$ $= 2\pi\int_0^a \lvert f(x)\rvert(s+x)dx$	E6, E7, E9 • Correct answer 1
	(iii) $V_{\text{total}} = 2\pi\int_{-a}^{a} \lvert f(x)\rvert(s+x)dx$ $= 2\pi\int_{-a}^{a} \lvert f(x)\rvert s\,dx + 2\pi\int_{-a}^{a}\lvert f(x)\rvert x\,dx$ $= 2\pi s\int_{-a}^{a}\lvert f(x)\rvert dx + 0$ $= 2\pi sA$	E6, E7, E9 • Demonstrates a fully correct answer, having shown that $\int_{-a}^{a}\lvert f(x)\rvert x\,dx$ vanishes ...... 2 • Correctly writes the volume as the sum of two integrals and recognises that $\int_{-a}^{a}\lvert f(x)\rvert x\,dx$ vanishes 1
	(iv) $s \ge a$, or equivalently, $a \le s$.	E6, E7, E9 • Correct answer 1
	(v) $f(x)$ could be an odd function (i.e. $f(x) = -f(-x)$) over the region $x = -a$ to $x = a$.	E6, E7, E9 • Correct answer 1

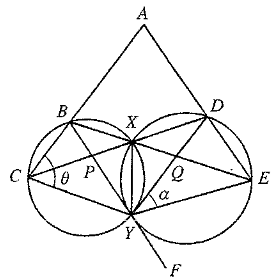
Question 7 (Continued)

	Sample answer	Syllabus outcomes and marking guide
(b)	(i) $a_{\text{net}} = g - \beta v^2$ $v\frac{dv}{dx} = g - \beta v^2$ $\frac{v}{g-\beta v^2}dv = dx$ $\int\frac{v}{g-\beta v^2}dv = \int dx$ $\int\frac{-2\beta v}{g-\beta v^2}dv = -2\beta\int dx$ $\ln(g-\beta v^2) = -2\beta x + C$ C is a constant of integration whose value can be determined from the initial conditions $x = 0$ and $v = 0$. $\ln(g-0) = -2\beta\times 0 + C$ $C = \ln g$ So $\ln(g-\beta v^2) = -2\beta x + \ln g$.	E5, E9 • Reaches the correct answer by a logically and algebraically correct method .... 3 • Correctly evaluates an appropriate integral to obtain $\ln(g-\beta v^2) = -2\beta x + C$ .... 2 • Recognises that acceleration must be written in the form $v\frac{dv}{dx}$ 1
	(ii) $\ln(g-\beta v^2) = -2\beta x + \ln g$ $\ln(g-\beta v^2) - \ln g = -2\beta x$ $\ln\left(\frac{g-\beta v^2}{g}\right) = -2\beta x$ $\ln\left(1-\frac{\beta v^2}{g}\right) = -2\beta x$ $e^{-2\beta x} = 1 - \frac{\beta v^2}{g}$ $\frac{\beta v^2}{g} = 1 - e^{-2\beta x}$ $v^2 = \frac{g}{\beta}(1-e^{-2\beta x})$ $v = \sqrt{\frac{g}{\beta}(1-e^{-2\beta x})}$	E5, E9 • Reaches the correct answer via a correct method .... 2 • Correctly converts the equation into an exponential form 1
	(iii) $v_t = \lim_{x\to\infty}\sqrt{\frac{g}{\beta}(1-e^{-2\beta x})}$ $= \sqrt{\frac{g}{\beta}}\lim_{x\to\infty}\sqrt{1-e^{-2\beta x}}$ $= \sqrt{\frac{g}{\beta}}$	E5, E9 • Correct answer 1
	(iv) $v_t = \sqrt{\frac{9.8}{0.2}}$ $= 7$ metres per second $= 3.6\times 7$ kilometres per hour $= 25.2$ kilometres per hour	E5, E9 • Correct answer 1
	(v) $2\beta x = \ln g - \ln(g-\beta v^2)$ $x = \frac{1}{2\beta}\ln\left(\frac{g}{g-\beta v^2}\right)$ $= \frac{1}{2\times 0.2}\ln\left(\frac{9.8}{9.8-0.2\times(0.8v_t)^2}\right)$ $= 2.5\ln\left(\frac{9.8}{9.8-0.2\times 0.64\times 7^2}\right)$ $= 2.55$ metres	E5, E9 • Substitutes appropriately for v and obtains a correct answer via a correct method .... 2 • Uses the correct equation to solve the problem, and adapts it to a correct form 1

Question 8

Sample answer | **Syllabus outcomes and marking guide**

(a) (i)



Let $\angle BCY = \theta$ and $\angle EYD = \alpha$

$\angle EXY = \angle BCY = \theta$ (exterior angle of cyclic quadrilateral $BCYX$ equals opposite interior angle)

$\angle EYF = \angle EXY = \theta$ (angle between tangent and chord equals angle in alternate segment)

$\angle EXD = \angle EYD = \alpha$ (angles in the same segment are equal)

$\angle BXC = \angle EXD = \alpha$ (vertically opposite)

$\angle BYC = \angle BXC = \alpha$ (angles in the same segment)

So $\angle BYC = \alpha$ and $\angle EYF = \theta$.

E2, E9
- Correctly proves that the angles are equal using the facts that angles in the same segment are equal, that the exterior angle of a cyclic quadrilateral equals the opposite interior angle, that the angle between a tangent and a chord equals the angle in the alternate segment, and that vertically opposite angles are equal 3
- Correctly uses any three of the properties listed above 2
- Correctly uses any two of the properties listed above 1

(ii) Let $\angle PYD = \beta$.

Then $\alpha + \beta + \theta = 180°$ (straight angle PYF)

So $\angle ACY + \angle CYD = 180°$

So $CA \parallel YD$ (co-interior angles are supplementary)

E2, E9
- Correctly shows that $\alpha + \beta + \theta = 180°$ (or equivalent), then uses the fact that co-interior angles are supplementary. . . . 2
- Makes significant progress towards proof 1

(iii) $\angle EDY = \angle EXY = \theta$ (angles in the same segment)

Hence Y, D, A and C are concyclic, as exterior $\angle EDY$ equals opposite interior $\angle BCY$.

E2, E9
- Correct statement. 1

(iv) A, E, Y and C are concyclic, as exterior $\angle EYF$ equals opposite interior $\angle CAD$.

E2, E9
- Gives any correct set of concyclic points including Y. 1

(v) $\angle XBP = \angle PXB = \alpha$ (base angles of isosceles ΔBXP are equal)

$\therefore \angle XBP = \alpha$

So $BQ \parallel CY$ and $BCYQ$ is a parallelogram, as both pairs of opposite sides are parallel.

E2, E9
- Complete correct proof 2
- Shows that $\angle XBP = \angle BYC$ 1

Question 8 **(Continued)**

Sample answer | **Syllabus outcomes and marking guide**

(b) (i) To prove that $1 - r + r^2 - r^3 + \ldots + r^{2n} = \frac{r^{2n+1}}{r+1} + \frac{1}{r+1}$

The left-hand side is the sum of a geometric progression with first term 1, common ratio $-r$ and $2n + 1$ terms

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\text{LHS} = \frac{1[(-r)^{2n+1} - 1]}{-r - 1}$$

$$= \frac{-1(r^{2n+1} + 1)}{-1(r+1)}$$

$$= \frac{r^{2n+1}}{r+1} + \frac{1}{r+1} = \text{RHS}$$

Alternative solution:

$$\text{LHS} \times (1 + r) = 1 - r + r^2 - \ldots + r^{2n} + r - r^2 + \ldots + r^{2n-1}$$

$$= 1 + r^{2n+1}$$

$$= \text{RHS} \times (1 + r)$$

E2, E9
- Correct proof 1

(ii) To show that

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{r^{2n+1}}{r+1}dr$$

From (i), $1 - r + r^2 - r^3 + \ldots + r^{2n} = \frac{r^{2n+1}}{r+1} + \frac{1}{r+1}$.

$$\frac{1}{r+1} = 1 - r + r^2 - r^3 + \ldots + r^{2n} - \frac{r^{2n+1}}{r+1}$$

$$\int_0^x \frac{dr}{r+1} = \int_0^x \left(1 - r + r^2 - r^3 + \ldots + r^{2n} - \frac{r^{2n+1}}{r+1}\right)dr$$

$$\Big[\ln(r+1)\Big]_0^x = \left[r - \frac{r^2}{2} + \frac{r^3}{3} - \frac{r^4}{4} + \ldots + \frac{r^{2n+1}}{2n+1}\right]_0^x - \int_0^x \frac{r^{2n+1}}{r+1}dr$$

$$\ln(x + 1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{r^{2n+1}}{r+1}dr$$

E2, E8, E9
- Correct proof 2
- Takes integrals of both sides

OR
- Rearranges 1

(iii) $\int_0^x \frac{r^{2n+1}}{r+1}dr < \int_0^x \frac{r^{2n+1}}{r}dr$ for $x > 0$

$\frac{r^{2n+1}}{r+1} < r^{2n+1}$, since $0 < r < 1$.

Hence $\int_0^x \frac{r^{2n+1}}{r+1}dr < \int_0^x r^{2n+1}dr$

$$< \frac{x^{2n+2}}{2n+2}$$

$\frac{x^{2n+2}}{2n+2} \to 0$ as $n \to \infty$ since $0 \le x \le 1$.

Hence $\int_0^x \frac{r^{2n+1}}{r+1}dr \to 0$ as $n \to \infty$.

E2, E9
- Correct result with reasoning (it is essential to include $0 \le x \le 1$) 2
- States initial inequality and establishes equal limits 1

Question 8 (Continued)

	Sample answer	Syllabus outcomes and marking guide
(b) (iv)	$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{r^{2n+1}}{r+1}dr$ Let $x = 1$: $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n+1} - \int_0^1 \frac{r^{2n+1}}{r+1}dr$ Taking the limiting sum as $n \to \infty$, $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots,$ since the integral approaches 0 as $n \to \infty$.	E2 • Correct answer 1