Neap:

HSC Trial Examination 2013

Mathematics Extension 1

This paper must be kept under strict security and may only be used on or after the morning of Friday 9 August, 2013 as specified in the Neap Examination Timetable.

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Section I – 10 marks
10 multiple-choice questions

Section II – 60 marks
4 short-answer questions

Total marks – 70
Attempt Questions 1–14

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2013 HSC Mathematics Extension 1 Examination.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I – 10 marks
Attempt Questions 1–10
All questions are of equal value

Use the multiple-choice answer sheet for Questions 1–10.

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1. The inequality $\frac{x}{|x-1|} > 0$ has the solution

(A) $x < 0, x \neq 1$
(B) $x > 1, x \neq 0$
(C) $x < 1, x \neq 0$
(D) $x > 0, x \neq 1$

2. The angle θ is made between the tangents at the point of intersection of the curves; $y = x^2$ and $y = \frac{1}{2}x^2 + 2$.

Which of the following is tan θ?

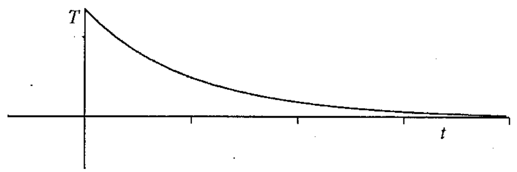
(A) $\frac{2}{1}$
(B) $\frac{1}{2}$
(C) $\frac{2}{9}$
(D) $-\frac{6}{7}$

3. Consider a projectile launched with an initial velocity v at an angle θ to the horizontal. Assume that $g = 9.8$ m/s^2 and that air resistance is negligible.

Which of the following statements is correct?

(A) The acceleration of the projectile decreases during its upward flight.
(B) The acceleration of the projectile is greatest during its upward flight.
(C) The acceleration of the projectile increases during its downward flight.
(D) The acceleration of the projectile remains constant during its entire flight.

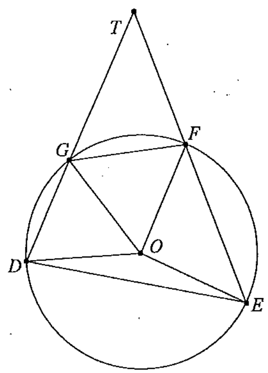
4. Part of a temperature graph is shown below:



Which of the following statements are true?

(A) $\frac{dT}{dt} > 0, \frac{d^2T}{dt^2} < 0$
(B) $\frac{dT}{dt} > 0, \frac{d^2T}{dt^2} > 0$
(C) $\frac{dT}{dt} < 0, \frac{d^2T}{dt^2} < 0$
(D) $\frac{dT}{dt} < 0, \frac{d^2T}{dt^2} > 0$

5. Which of the following statements is always true?



(A) $\angle GDE = \angle TFG$
(B) $\angle GFO = \angle GDO$
(C) $\angle FED = 180 - \angle GDE$
(D) $\angle TFG = \angle TGF$

6. The area enclosed between the curve $y = x^3 - 1$, the y-axis and the lines $y = 1$ and $y = 2$ is given by

(A) $\int_1^2 (x^3 - 1)\,dy$

(B) $\int_1^2 (\sqrt[3]{y+1})\,dy$

(C) $\int_1^2 (\sqrt[3]{y} + 1)\,dy$

(D) $\int_1^2 (y+1)\,dy$

7. Which equation shows a particle **not** moving in simple harmonic motion?

(A) $x = a\sin(nt + \alpha)$

(B) $x = a\tan(nt + \alpha)$

(C) $x = a\cos(nt + \alpha) - a\sin(nt + \alpha)$

(D) $x = a\cos(nt + \alpha)$

8. The domain and range for $y = \sin^{-1} x$ are:

(A) $-1 \le x \le 1; 0 \le y \le \pi$

(B) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}; -1 \le y \le 1$

(C) $-1 \le x \le 1; -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(D) $0 \le x \le \pi; -1 \le y \le 1$

9. A polynomial equation has $f(-1) < 0$ and $f(-2) > 0$. By halving the interval, it is found that $f(-1\frac{1}{2}) < 0$.

A root of the equation lies between:

(A) $x = -2$, and $x = -1\frac{1}{2}$

(B) $x = -1\frac{1}{2}$, and $x = -1$

(C) $x = -1\frac{1}{2}$, and $x = -1\frac{1}{4}$

(D) $x = -1\frac{3}{4}$, and $x = -1\frac{1}{2}$

10. Find $\int \frac{dx}{\sqrt{4 - x^2}}$.

(A) $\frac{1}{4}\sin^{-1}\left(\frac{x}{4}\right) + C$

(B) $\sin^{-1}\left(\frac{x}{4}\right) + C$

(C) $\frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right) + C$

(D) $\sin^{-1}\left(\frac{x}{2}\right) + C$

Section II – 60 marks
Attempt Questions 11–14
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove the identity $\frac{\sin(A+C) + \sin A + \sin(A-C)}{\sin(B+C) + \sin B + \sin(B-C)} = \frac{\sin A}{\sin B}$ **2**

(ii) Given that $\cot A = \frac{1}{4}$ and $\cot(A+B) = 2$, find $\cot B$. **2**

(b) Water is poured into a container at a rate of 8 cm³/s. If the volume of the water in the container is given by $V = \frac{3}{2}(h^2 + 8h)$ cm³ where h cm is the depth of the water, find the rate of change of the depth of the water when $V = 72$ cm³. **3**

(c) Find the coefficient of a^5b^3 in the expansion of $\left(a - \frac{b}{2}\right)^8$. **2**

(d) A particle moves in a straight line so that its acceleration is given by $\frac{dv}{dt} = x - 1$, where v is its velocity and x is its displacement from the origin. Initially the particle is at the origin and has $v = 1$.

(i) Show that $v^2 = (x-1)^2$. **2**

(ii) Find x as a function of t, that is $x(t)$. **4**

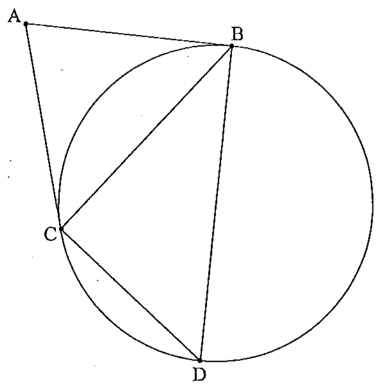
Marks

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) AB and AC are tangents to a circle. D is a point on the circle such that $\angle BDC = \angle BAC$ and $2\angle DBC = \angle BAC$.

(i) Show that DB is a diameter. **4**

(ii) Show that $BC = AB$. **1**



(b) Consider the function $f(x) = 4x - x^3$.

(i) Sketch $y = f(x)$, showing the x and y intercepts and the coordinates of the stationary points. **3**

(ii) Find the largest domain containing the origin for which $f(x)$ has an inverse function, $f^{-1}(x)$. **1**

(iii) State the domain of $f^{-1}(x)$. **1**

(iv) Find the gradient of the inverse function at the origin. **1**

(c) Let $S_n = 1\times 2 + 2\times 3 + 3\times 4 + \cdots + (n-1)\times n$.

Prove, using Mathematical Induction, that $S_n = \dfrac{(n-1)n(n+1)}{3}$, for all integers n, $n \geq 2$. **4**

Marks

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Solve the inequality $2x - 1 \leq x^2 - 4 < 12$. **3**

(b) The polynomial $x^3 + px^2 + qx + 5$, where p and q are constants, leaves remainders of 7 and 17 when divided by $x-2$ and $x+3$ respectively.

(i) Find the value of p and q. **3**

(ii) Find the remainder when $P(x)$ is divided by $x-4$. **1**

(c) Find the coordinates of the point, P that divides the interval MN with $M(1,4)$ and $N(5,2)$ in the ratio $-1:3$. **2**

(d) Using t-results solve the equation $\sin 2x = \tan x$ for $0 \leq x \leq \pi$. **3**

(e) Using the substitution $u = 1+2x$, find $\displaystyle\int \frac{6dx}{\sqrt{(1+2x)^3}}$. **3**

Marks

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Seven people are sitting around a table.

(i) How many seating arrangements are possible? **1**

(ii) Two people, Kevin and Julia, do not sit next to each other. **2**

How many seating arrangements are now possible?

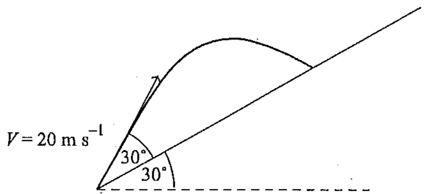
(b) A heated metal ball is dropped into a liquid. As the ball cools, its temperature, T °C, t minutes after it enters the liquid, is given by $T = 200e^{-0.1t} + 25, t > 0$.

(i) Find the value of t for which $T = 50$. **2**

(ii) Find the rate at which the temperature of the ball is decreasing at the instant when $t = 20$. **2**

(iii) From the equation of T given above, explain why the temperature of the ball can never fall to 20°C. **1**

(c) A ball is projected with an initial velocity of 20 m/s at an angle of 30° from a road inclined at 30° to the horizontal. Let $g = -10$ m/s^2. Ignore air resistance.



(i) How far is the ball thrown up the road? Assume the ball hits the road at an angle of 30°. **5**

(ii) Find the speed with which the ball hits the road. **2**

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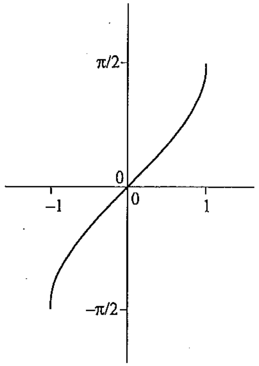
Mathematics Extension 1

Solutions and marking guidelines

Section I

Sample answer	Syllabus outcomes and marking guide
Question 1 D Since $\|x-1\|>0, x\neq 1$ it is reasonable to say that $x>0$ will provide solutions to the inequality as long as $x\neq 1$.	HE7 Band 4-5
Question 2 C Finding the gradient of each curve at the points of intersection $(\pm 2,4)$ we get $m_1=4$ and $m_2=2$, hence $\tan\theta=\frac{\|4-2\|}{1+4\times 2}=\frac{2}{9}$	PE3 Band 3-4
Question 3 D The fact that gravity, g is constant indicates that the acceleration remains constant throughout flight.	HE3 Band 2-3
Question 4 D The graph of T(t) shows that the function is decreasing $\left(\frac{dT}{dt}<0\right)$ and that the curve is concave up $\left(\frac{d^2T}{dt^2}>0\right)$.	HE5 Band 2-3
Question 5 A The external angle of a cyclic quadrilateral is always equal to the interior opposite angle.	HE2 Band 3-4
Question 6 B As it is an area to the y-axis we need $\int_1^2 f(y).dy$, so given $y=x^3-1$ we get $f(y)=\sqrt[3]{y+1}$. The area is $\int_1^2 \sqrt[3]{y+1}.dy$.	HE7 Band 4-5
Question 7 B The functions $a\sin(nt+\alpha)$, $a\cos(nt+\alpha)$, $a\sin(nt+\alpha)-a\cos(nt+\alpha)$ are all functions that represent SHM. Therefore, $a\tan(nt+\alpha)$ does not represent SHM.	HE3 Band 4-5

Sample answer	Syllabus outcomes and marking guide
Question 8 C The curve $y = \sin^{-1} x$ has domain and range as in the diagram:	HE4 Band 3-4



Sample answer	Syllabus outcomes and marking guide
Question 9 A Since change in sign exists between -2 and $-1\frac{1}{2}$ then a root of the equation lies between these two values of x.	HE7 Band 3-4
Question 10 D $\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{dx}{\sqrt{2^2-x^2}}$. Hence using standard integrals $\int \frac{dx}{\sqrt{2^2-x^2}} = \sin^{-1}\left(\frac{x}{2}\right) + C$	HE4 Band 3-4

Section II

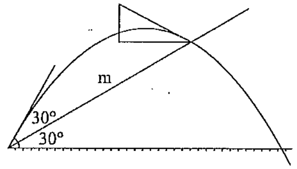
Sample answer	Syllabus outcomes and marking guide
Question 11	
(a) (i) $\frac{\sin(A+C)+\sin A+\sin(A-C)}{\sin(B+C)+\sin B+\sin(A-C)}$ $= \frac{\sin A\cos C+\cos A\sin C+\sin A+\sin A\cos C-\sin C\cos A}{\sin B\cos C+\sin C\cos B+\sin B+\sin A\cos C-\sin C\cos A}$ $= \frac{\sin A(\cos C+1+\cos C)}{\sin B(\cos C+1+\cos C)}$ $= \frac{\sin A}{\sin B}$	HE7 Band 3-4 • Correctly factorises and shows the correct result . . . . 2 • Simplifies the given expression 1
(ii) $\cot A = \frac{1}{4}$ and $\cot(A+B) = \frac{1}{\tan(A+B)}$ $= \frac{1-\tan A\tan B}{\tan A+\tan B}$ $= \frac{1-4\tan B}{4+\tan B}$ $\therefore \frac{1-4\tan B}{4+\tan B} = 2$ $1-4\tan B = 8+2\tan B$ $6\tan B = -7$ $\tan B = -\frac{7}{6}$ Hence, $\cot B = -\frac{6}{7}$	HE7 Band 4-5 • Correctly solves to find $\tan B$ . . . . 2 • Correctly converts $\cot(A+B)$ 1
(b) $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$ $\frac{dh}{dt} = 8 \div \frac{3}{2}(2h+8) = \frac{8}{3(h+4)}$ What is h when $V = 72$: $\frac{3}{2}(h^2+8h) = 72$ $h^2+8h-48 = 0$ $(h-4)(h+12) = 0$ $h = 4$, since $h > 0$ Thus, the rate of change of the depth of water is $\frac{1}{3} cms^{-1}$.	HE5 Band 5-6 • Correctly finds the rate of change. . . . . 3 • Correctly finds $h = 4$ . . . . 2 • Correctly finds $\frac{dh}{dt}$ 1

Sample answer	Syllabus outcomes and marking guide
(c) $\left(a-\frac{b}{2}\right)^8 = {}^8C_0(a)^{8-0}\left(\frac{b}{2}\right)^0 + {}^8C_1(a)^{8-1}\left(\frac{b}{2}\right)^1 + \cdots + {}^8C_n(a)^{8-n}\left(\frac{b}{2}\right)^n$ $T_{r+1} = {}^8C_r(a)^{8-r}\left(\frac{-b}{2}\right)^r$ Coefficient of a^5b^3 is when $r=3$: Hence, coefficient is ${}^8C_3(1)^5\left(\frac{-1}{2}\right)^3 = -7$	HE3 Band 4-5 • Correctly finds the answer .... 2 • Correctly determines that $r=3$....... 1
(d) (i) $a=(x-1)$ $\frac{d\left(\frac{1}{2}v^2\right)}{dx}=(x-1)$ $\frac{1}{2}v^2=\frac{1}{2}(x-1)^2+C_1$ When $x=0, v=1$ $C_1=0$ and $\frac{1}{2}v^2=\frac{1}{2}(x-1)^2$ $v^2=(x-1)^2$	HE5 Band 5-6 • Correctly finds $C=0$ and shows the final result .... 2 • Uses $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.................. 1
(ii) $\left(\frac{dx}{dt}\right)^2=(x-1)^2$ $\frac{dx}{dt}=\pm(x-1)$ Since, $v=1$ when $x=0$ $\frac{dx}{dt}=-(x-1)$ $\int\frac{dx}{(1-x)}=\int dt$ $-\ln(1-x)=t+C_2$ Since, $t=0$, $x=0$ $C_2=0$, $1-x=e^{-t}$ $x=1-e^{-t}$	HE5 Band 4-5 • Correctly writes $x(t)$ .... 4 • Correctly integrates................. 3 • Correctly shows $\frac{dx}{dt}=-(x-1)$ ...... 2 • Uses $v=\frac{dx}{dt}$..................... 1

Sample answer	Syllabus outcomes and marking guide
(iii) Domain for $f^{-1}(x)$: $\lvert x\rvert \le \frac{16\sqrt{3}}{9}$	HE4 Band 4-5 • Gives the correct answer. 1
(iv) Gradient of $f^{-1}(x)$ at $(0,0)$ is $\frac{1}{4}$ since $f'(0)=4$	HE4 Band 5-6 • Gives the correct answer. 1
(c) Step 1: Show true for $n=2$ $S_n = 1\times2+2\times3+\cdots+(n-1)\times n$ $n=2$: $LHS = 1\times2=2$ $RHS=\frac{(2-1)\times2\times(2+1)}{3}=2$ Therefore, true for $n=2$ Step 2: Assume true for $n=k$ $1\times2+2\times3+\cdots+(k-1)\times k$ $=\frac{(k-1)k(k+1)}{3}$ Step 3: Show true for $n=k+1$ $S_{k+1}=S_k+T_{k+1}$ $S_{k+1}=\frac{(k-1)k(k+1)}{3}+k(k+1)$ $S_{k+1}=\frac{1}{3}k(k+1)[(k-1)+3]$ $S_{k+1}=\frac{k(k+1)(k+2)}{3}$ $=\frac{[(k+1)-1][k+1][(k+1)+1]}{3}$ If true for $n=k$, then true for $n=k+1$ Step 4: Induction process Since true for $n=2$, then true for $n=2+1=3$, and then true for $n=3+1=4$, and so on for all integers $n>1$.	HE2 Band 4-5 • Correctly writes final conclusion to induction. . . . . 4 • Correctly shows true for $n=k+1$ . . . . 3 • Correctly substitutes into S_{k+1} . . . . 2 • Shows true for $n=2$ 1
Question 13	
(a) $2x-1\le x^2-4<12$ $\therefore x^2-2x-3\ge0$ and $x^2-16<0$ $(x+1)(x-3)\ge0$ $(x+4)(x-4)<0$ $x\le-1, x\ge3$ $-4<x<4$ hence, $-4<x\le-1$ and $3\le x<4$	PE3 Band 3-4 • Gives the correct answer. . . . . 3 • Correctly solves inequalities. . . . . 2 • Writes two quadratic inequalities 1

Sample answer	Syllabus outcomes and marking guide
(b) (i) $P(x)=x^3+px^2+qx+5$ $P(2)=7, P(-3)=17$ $\therefore 8+4p+2q+5=7$ $\therefore -27+9p-3q+5=17$ $4p+2q=-6$ and $9p-3q=39$ $2p+q=-3$ $3p-q=13$ Adding these equations; $5p=10$ $p=2, q=-7$	PE3 Band 3-4 • Correctly solves equations for p and q. . . . . 3 • Determines simultaneous equations . . . 2 • Uses remainder theorem 1
(ii) $\therefore P(x)=x^3+2x^2-7x+5$ $P(4)=64+32-28+5$ $P(4)=73$	PE3 Band 3-4 • Gives the correct answer. 1
(c) $P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ $P\left(\frac{-1(5)+3(1)}{2}, \frac{-1(2)+3(4)}{2}\right)$ $P(-1,5)$	PE3 Band 3-4 • Gives the correct answer. . . . . 2 • Uses correct formula. 1
(d) $\frac{2t}{1+t^2}=t$ $2t=t+t^3$ $t^3-t=0$ $t(t^2-1)=0$ $t=0,\pm1$ $\tan x=0,\pm1$ $x=0,\frac{\pi}{4},\frac{3\pi}{4},\pi\ (0\le x\ge\pi)$	HE7 Band 4-5 • Gives the correct answers . . . . 3 • Writes correct values for t. . . . . 2 • Substitutes using t-results 1
(e) Putting $u=1+2x$ $\therefore du=2dx$ $\int\frac{6}{\sqrt{(1+2x)^3}}dx$ $=3\int\frac{2}{\sqrt{(1+2x)^3}}dx$ $=3\int\frac{du}{u^{\frac{3}{2}}}$ $=3\int u^{-\frac{3}{2}}du$ $=3\left[-2u^{-\frac{1}{2}}\right]+C$ $=-\frac{6}{\sqrt{(1+2x)}}+C$	HE7 Band 4-5 • Gives the correct answer. . . . . 3 • Correctly integrates. . . . . 2 • Correctly changes variable 1

Sample answer	Syllabus outcomes and marking guide
Question 14	
(a) (i) $(7-1)! = 720$	PE3 Band 3-4 • Gives the correct answer. 1
(ii) Kevin and Julia sit together in $2 \times 5! = 240$ ways Hence, Kevin and Julia do not sit together in $720 - 240 = 480$ ways.	PE3 Band 4-5 • Gives the correct answer. . . . . 2 • Finds ways of sitting together. 1
(b) (i) $T = 50$ $50 = 200e^{-0.1t} + 25$ $e^{-0.1t} = \frac{1}{8}$ $-0.1t = \ln\left(\frac{1}{8}\right)$ $t = 30\ln 2$ seconds	HE3 Band 3-4 • Gives the correct answer. . . . . 2 • Writes an equation to solve. 1
(ii) $\frac{dT}{dt} = -0.1 \times 200e^{-0.1t}$ $t = 20$ $\frac{dT}{dt} = -20e^{-0.1\times 20}$ $\frac{dT}{dt} = -\frac{20}{e^2}$ C/minute	HE3 Band 4-5 • Gives the correct answer. . . . . 2 • Finds $\frac{dh}{dt}$ 1
(iii) As $t \to \infty$ $T = \lim\limits_{t\to\infty}\left(25 + \frac{200}{e^{0.1t}}\right) \to 25$	HE3 Band 4-5 • Establishes that $T \to 25$ 1

Sample answer	Syllabus outcomes and marking guide
(c) (i) Vertically: $\ddot{y} = -10$ $\dot{y} = -10t + 20\sin 60$ $\dot{y} = -10t + 10\sqrt{3}$ $y = -5t^2 + 10\sqrt{3}t$ Horizontally: $\ddot{x} = 0$ $\dot{x} = 20\cos 60$ $\dot{x} = 10$ $x = 10t$  $\tan 30 = \frac{-5t^2 + 10\sqrt{3}t}{10t}$ $\frac{1}{\sqrt{3}} = \frac{-5t^2 + 10\sqrt{3}t}{10t}$ $5\sqrt{3}t^2 - 20t = 0$ $5t\left(\sqrt{3}t - 4\right) = 0$ $t = 0, \frac{4}{\sqrt{3}}$ The time of flight to where projectile meets the inclined plane is $\frac{4\sqrt{3}}{3}$ seconds. Now, $\cos 30 = \frac{10t}{m}$ $m = \frac{10t}{\frac{\sqrt{3}}{2}}$ $m = \frac{20t}{\sqrt{3}}$ $m = \frac{20}{\sqrt{3}} \times \frac{4\sqrt{3}}{3}$ Distance up the incline is $\frac{80}{3}$ metres.	HE3 Band 5-6 • Correctly writes the distance up the incline to point of impact . . . . 5 • Correctly writes the time of flight to point of impact . . . . 4 • Uses tan 30 to find time of flight . . . . . 3 • Makes progress towards finding time of flight . . . . 2 • Correctly defines x and y components. 1
(ii) $V^2 = \dot{y}^2 + \dot{x}^2$ $V = \sqrt{\dot{y}^2 + \dot{x}^2}$ $V = \sqrt{\left(-10 \times \frac{4\sqrt{3}}{3} + 10\sqrt{3}\right)^2 + (10)^2}$ $V = \_\ 10\sqrt{3}$	