BOARD OF STUDIES
NEW SOUTH WALES

2007

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\sqrt{\pi^2 + 5}$ correct to two decimal places. **2**

(b) Solve $2x - 5 > -3$ and graph the solution on a number line. **2**

(c) Rationalise the denominator of $\dfrac{1}{\sqrt{3} - 1}$. **2**

(d) Find the limiting sum of the geometric series **2**

$$\frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \cdots .$$

(e) Factorise $2x^2 + 5x - 12$. **2**

(f) Find the equation of the line that passes through the point $(-1, 3)$ and is perpendicular to $2x + y + 4 = 0$. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate with respect to x:

(i) $\dfrac{2x}{e^x+1}$ **2**

(ii) $(1+\tan x)^{10}$. **2**

(b) (i) Find $\int (1+\cos 3x)\,dx$. **2**

(ii) Evaluate $\int_1^4 \dfrac{8}{x^2}\,dx$. **3**

(c) The point $P(\pi, 0)$ lies on the curve $y = x\sin x$. Find the equation of the tangent to the curve at P. **3**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

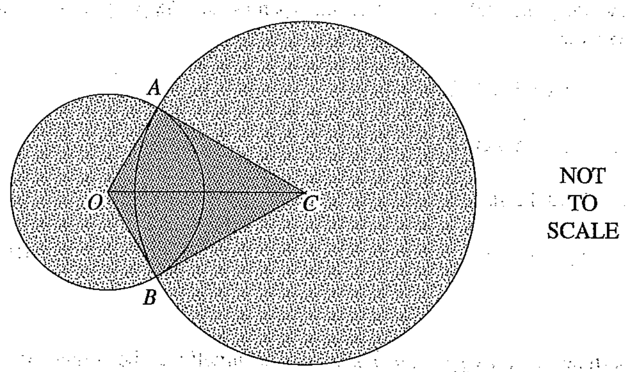
(a) Solve $\sqrt{2}\sin x = 1$ for $0 \le x \le 2\pi$. **2**

(b) Two ordinary dice are rolled. The score is the sum of the numbers on the top faces.

(i) What is the probability that the score is 10? **2**

(ii) What is the probability that the score is not 10? **1**

(c)



An advertising logo is formed from two circles, which intersect as shown in the diagram.

The circles intersect at A and B and have centres at O and C.

The radius of the circle centred at O is 1 metre and the radius of the circle centred at C is $\sqrt{3}$ metres. The length of OC is 2 metres.

(i) Use Pythagoras' theorem to show that $\angle OAC = \dfrac{\pi}{2}$. **1**

(ii) Find $\angle ACO$ and $\angle AOC$. **2**

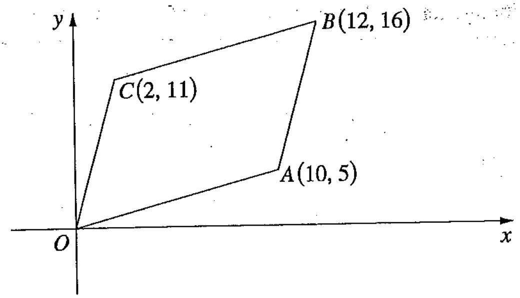
(iii) Find the area of the quadrilateral $AOBC$. **1**

(iv) Find the area of the major sector ACB. **1**

(v) Find the total area of the logo (the sum of all the shaded areas). **2**

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a)



In the diagram, A, B and C are the points $(10, 5)$, $(12, 16)$ and $(2, 11)$ respectively.

Copy or trace this diagram into your writing booklet.

(i) Find the distance AC. **1**

(ii) Find the midpoint of AC. **1**

(iii) Show that $OB \perp AC$. **2**

(iv) Find the midpoint of OB and hence explain why $OABC$ is a rhombus. **2**

(v) Hence, or otherwise, find the area of $OABC$. **1**

(b) Heather decides to swim every day to improve her fitness level.

On the first day she swims 750 metres, and on each day after that she swims 100 metres more than the previous day. That is, she swims 850 metres on the second day, 950 metres on the third day and so on.

(i) Write down a formula for the distance she swims on the nth day. **1**

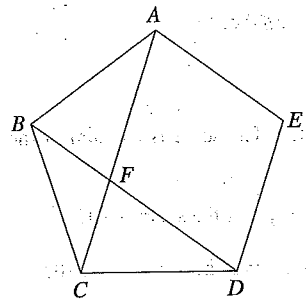
(ii) How far does she swim on the 10th day? **1**

(iii) What is the total distance she swims in the first 10 days? **1**

(iv) After how many days does the total distance she has swum equal the width of the English Channel, a distance of 34 kilometres? **2**

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a)



In the diagram, $ABCDE$ is a regular pentagon. The diagonals AC and BD intersect at F.

Copy or trace this diagram into your writing booklet.

(i) Show that the size of $\angle ABC$ is $108°$. **1**

(ii) Find the size of $\angle BAC$. Give reasons for your answer. **2**

(iii) By considering the sizes of angles, show that ΔABF is isosceles. **2**

(b) A particle is moving on the x-axis and is initially at the origin. Its velocity, v metres per second, at time t seconds is given by

$$v = \frac{2t}{16 + t^2}.$$

(i) What is the initial velocity of the particle? **1**

(ii) Find an expression for the acceleration of the particle. **2**

(iii) Find the time when the acceleration of the particle is zero. **1**

(iv) Find the position of the particle when $t = 4$. **3**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Solve the following equation for x: **2**

$$2e^{2x} - e^{x} = 0.$$

(b) Let $f(x) = x^4 - 4x^3$.

(i) Find the coordinates of the points where the curve crosses the axes. **2**

(ii) Find the coordinates of the stationary points and determine their nature. **4**

(iii) Find the coordinates of the points of inflexion. **1**

(iv) Sketch the graph of $y = f(x)$, indicating clearly the intercepts, stationary points and points of inflexion. **3**

End of Question 6

Please turn over

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Find the coordinates of the focus, S, of the parabola $y = x^2 + 4$. **2**

(ii) The graphs of $y = x^2 + 4$ and the line $y = x + k$ have only one point of intersection, P. Show that the x-coordinate of P satisfies **1**

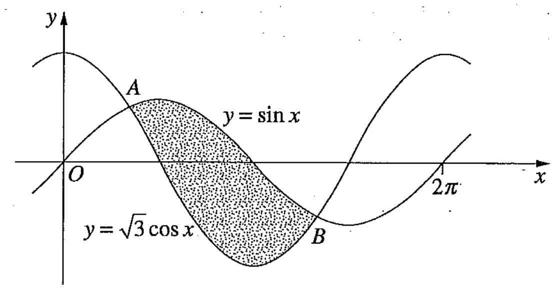
$$x^2 - x + 4 - k = 0.$$

(iii) Using the discriminant, or otherwise, find the value of k. **1**

(iv) Find the coordinates of P. **2**

(v) Show that SP is parallel to the directrix of the parabola. **1**

(b)



The diagram shows the graphs of $y = \sqrt{3}\cos x$ and $y = \sin x$. The first two points of intersection to the right of the y-axis are labelled A and B.

(i) Solve the equation $\sqrt{3}\cos x = \sin x$ to find the x-coordinates of A and B. **2**

(ii) Find the area of the shaded region in the diagram. **3**

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) One model for the number of mobile phones in use worldwide is the exponential growth model,

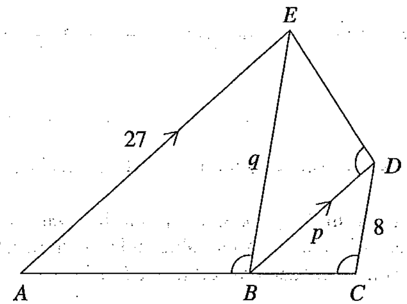
$$N = Ae^{kt},$$

where N is the estimate for the number of mobile phones in use (in millions), and t is the time in years after 1 January 2008.

(i) It is estimated that at the start of 2009, when $t = 1$, there will be 1600 million mobile phones in use, while at the start of 2010, when $t = 2$, there will be 2600 million. Find A and k. **3**

(ii) According to the model, during which month and year will the number of mobile phones in use first exceed 4000 million? **2**

(b)



In the diagram, AE is parallel to BD, $AE = 27$, $CD = 8$, $BD = p$, $BE = q$ and $\angle ABE$, $\angle BCD$ and $\angle BDE$ are equal.

Copy or trace this diagram into your writing booklet.

(i) Prove that $\Delta ABE \;|||\; \Delta BCD$. **2**

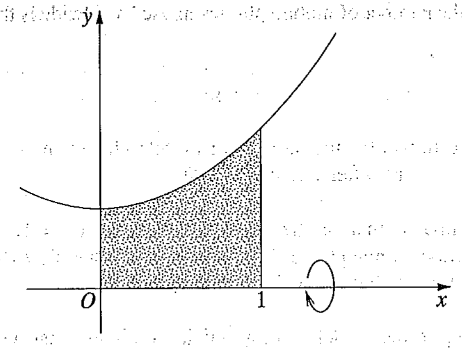
(ii) Prove that $\Delta EDB \;|||\; \Delta BCD$. **2**

(iii) Show that 8, p, q, 27 are the first four terms of a geometric series. **1**

(iv) Hence find the values of p and q. **2**

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) **3**



The shaded region in the diagram is bounded by the curve $y = x^2 + 1$, the x-axis, and the lines $x = 0$ and $x = 1$.

Find the volume of the solid of revolution formed when the shaded region is rotated about the x-axis.

(b) A pack of 52 cards consists of four suits with 13 cards in each suit.

(i) One card is drawn from the pack and kept on the table. A second card is drawn and placed beside it on the table. What is the probability that the second card is from a different suit to the first? **1**

(ii) The two cards are replaced and the pack shuffled. Four cards are chosen from the pack and placed side by side on the table. What is the probability that these four cards are all from different suits? **2**

Question 9 continues on page 11

Question 9 (continued) **Marks**

(c) Mr and Mrs Caine each decide to invest some money each year to help pay for their son's university education. The parents choose different investment strategies.

(i) Mr Caine makes 18 yearly contributions of \$1000 into an investment fund. He makes his first contribution on the day his son is born, and his final contribution on his son's seventeenth birthday. His investment earns 6% compound interest per annum. **3**

Find the total value of Mr Caine's investment on his son's eighteenth birthday.

(ii) Mrs Caine makes her contributions into another fund. She contributes \$1000 on the day of her son's birth, and increases her annual contribution by 6% each year. Her investment also earns 6% compound interest per annum. **2**

Find the total value of Mrs Caine's investment on her son's third birthday (just before she makes her fourth contribution).

(iii) Mrs Caine also makes her final contribution on her son's seventeenth birthday. **1**

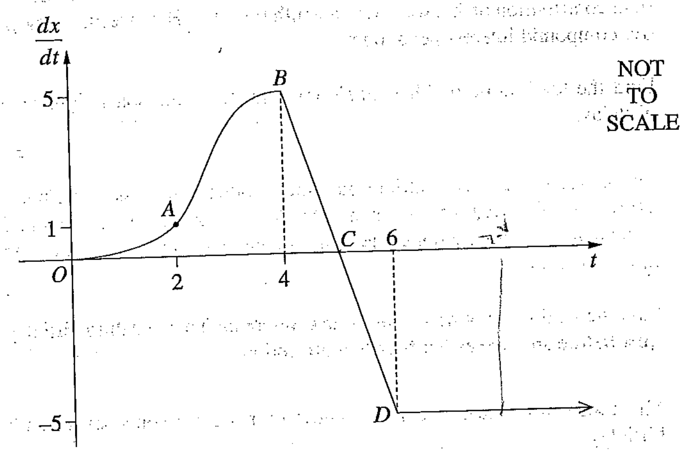
Find the total value of Mrs Caine's investment on her son's eighteenth birthday.

End of Question 9

Marks

Question 10 (12 marks) Use a SEPARATE writing booklet.

(a) An object is moving on the x-axis. The graph shows the velocity, $\frac{dx}{dt}$, of the object, as a function of time, t. The coordinates of the points shown on the graph are $A(2, 1)$, $B(4, 5)$, $C(5, 0)$ and $D(6, -5)$. The velocity is constant for $t \geq 6$.



(i) Using Simpson's rule, estimate the distance travelled between $t = 0$ and $t = 4$. **2**

(ii) The object is initially at the origin. During which time(s) is the displacement of the object decreasing? **1**

(iii) Estimate the time at which the object returns to the origin. Justify your answer. **2**

(iv) Sketch the displacement, x, as a function of time. **2**

Question 10 continues on page 13

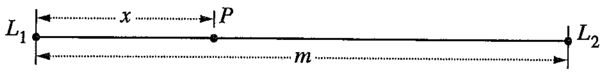
Question 10 (continued)

Marks

(b) The noise level, N, at a distance d metres from a single sound source of loudness L is given by the formula

$$N = \frac{L}{d^2}.$$

Two sound sources, of loudness L_1 and L_2 are placed m metres apart.



The point P lies on the line between the sound sources and is x metres from the sound source with loudness L_1.

(i) Write down a formula for the sum of the noise levels at P in terms of x. **1**

(ii) There is a point on the line between the sound sources where the sum of the noise levels is a minimum. **4**

Find an expression for x in terms of m, L_1 and L_2 if P is chosen to be this point.

End of paper

2007 Higher School Certificate Solutions

Mathematics

Question 1

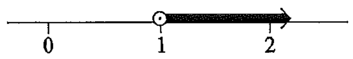
(a) $\sqrt{\pi^2+5}=3.8561\ldots$

$=3.86$ (to 2 d.p).

(b) $2x-5>-3$

$2x>2$

$x>1$



(c) $\dfrac{1}{\sqrt{3}-1}=\dfrac{1}{(\sqrt{3}-1)}\times\dfrac{(\sqrt{3}+1)}{(\sqrt{3}+1)}$

$=\dfrac{\sqrt{3}+1}{3-1}$

$=\dfrac{\sqrt{3}+1}{2}$ or $\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}$.

(d) $r=\dfrac{1}{4},\quad a=\dfrac{3}{4}$

$S^{\infty}=\dfrac{a}{1-r}\ ,\ |r|<1$

$=\dfrac{\frac{3}{4}}{1-\frac{1}{4}}$

$=1.$

(e) $2x^2+5x-12$

$=2x^2+8x-3x-12$

$=2x(x+4)-3(x+4)$

$=(2x-3)(x+4).$

$AB=2\times-12$

$A+B=5$

$A=8\ ,B=-3$

(f) $2x+y+4=0$

$y=-2x-4$

$\therefore$ Gradient of line, $m_L=-2$

$\therefore$ Gradient of perpendicular, $m_P=\dfrac{1}{2}$.

METHOD 1

$y=mx+b$

For $(-1,3)$ and $m_P=\dfrac{1}{2}$,

$3=\dfrac{1}{2}\times-1+b$

$3=\dfrac{-1}{2}+b$

$b=3\dfrac{1}{2}$ or $\dfrac{7}{2}$

$\therefore\ y=\dfrac{1}{2}x+\dfrac{7}{2}$

$x-2y+7=0$ (general form).

METHOD 2

$y-y_1=m(x-x_1)$

For $(-1,3)$ and $m_P=\dfrac{1}{2}$,

$y-3=\dfrac{1}{2}(x--1)$

$y-3=\dfrac{1}{2}(x+1)$

$2y-6=x+1$

$2y=x+7$

$\therefore y=\dfrac{1}{2}x+\dfrac{7}{2}$

or $x-2y+7=0$ (general form).

Question 2

(a) (i)

Quotient rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{vu'-uv'}{v^2}$

$\therefore\ \dfrac{d}{dx}\left(\dfrac{2x}{e^x+1}\right)=\dfrac{(e^x+1)\times2-2xe^x}{(e^x+1)^2}$

$=\dfrac{2e^x+2-2xe^x}{(e^x+1)^2}$

$=\dfrac{2(e^x+1-xe^x)}{(e^x+1)^2}.$

(ii)

$\dfrac{d}{dx}(1+\tan x)^{10}=10(1+\tan x)^9\times\sec^2 x$

$=10\sec^2 x(1+\tan x)^9.$

(b) (i) $\displaystyle\int(1+\cos 3x)\,dx=x+\frac{1}{3}\sin 3x+c.$

(ii) $\displaystyle\int_1^4\frac{8}{x^2}\,dx=\int_1^4 8x^{-2}\,dx$

$=\left[\dfrac{8x^{-1}}{-1}\right]_1^4$

$=\left[\dfrac{-8}{x}\right]_1^4$

$=\dfrac{-8}{4}-\dfrac{-8}{1}$

$=-2+8$

$=6.$

(c) Product rule: $\dfrac{d}{dx}(uv)=uv'+vu'$

Gradient, $m=\dfrac{dy}{dx}(x\sin x)$

$=x\times\cos x+1\times\sin x$

$=x\cos x+\sin x$

At $(\pi,0)$,

$m=\pi\times\cos\pi+\sin\pi$

$=\pi\times-1+0$

$=-\pi$

$\therefore$ Equation of tangent:

$y-y_1=m(x-x_1)$

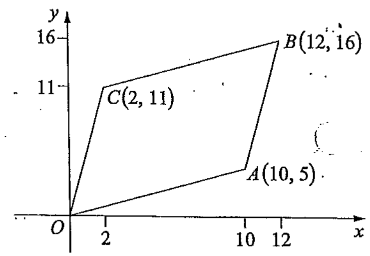
$y-0=-\pi(x-\pi)$

$y=-\pi x+\pi^2$

or $\pi x+y-\pi^2=0$ (general form).

Question 3

(a)



(i) Using the distance formula,

$d_{AC}=\sqrt{(2-10)^2+(11-5)^2}$

$=\sqrt{(-8)^2+6^2}$

$=\sqrt{100}$

$=10$ units.

(ii) Midpoint of AC

$=\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)$

$=\left(\dfrac{2+10}{2},\dfrac{11+5}{2}\right)$

$=(6,8).$

(iii)

$m_{OB}=\dfrac{16-0}{12-0}=\dfrac{16}{12}$ and $m_{AC}=\dfrac{5-11}{10-2}=$

$=\dfrac{4}{3}$ $\qquad=\dfrac{-3}{4}$

Now, $m_{OB}\times m_{AC}=\dfrac{4}{3}\times\dfrac{-3}{4}$

$=-1$

$\therefore\ OB\perp AC.$

(iv) Midpoint of $OB = \left(\frac{0+12}{2}, \frac{0+16}{2}\right)$

$= (6, 8)$

Since the diagonals of $OABC$ have the same midpoint they bisect each other, and do so at right angles (from part (iii)).

$\therefore OABC$ is a rhombus.

(v) $d_{OB} = \sqrt{12^2+16^2}$

$= \sqrt{400}$

$= 20$ units

From part (i), $d_{AC} = 10$ units.

$\therefore$ Area of rhombus

$= \frac{1}{2} \times$ product of diagonals

$= \frac{1}{2} \times 10 \times 20$

$= 100$ units^2.

(b) Heather swims 750, 850, 950, ... m. This is an arithmetic sequence with $a = 750$ and $d = 100$.

(i) $T_n = a + (n-1)\,d$

$\therefore\ T_n = 750 + (n-1)100$

$= 750 + 100n - 100$

$= 650 + 100n.$

(ii) $T_{10} = 650 + 100 \times 10$

$= 1650$ m

$\therefore$ She swims 1650 m on the 10th day.

(iii) $S_n = \frac{n}{2}[a + l]$

$\therefore S_{10} = \frac{10}{2}[750 + 1650]$

$= 5[2400]$

$= 12\ 000$ m

$\therefore$ She swims 12 000 m or 12 km in the first 10 days.

(iv) $S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore\ 34\ 000 = \frac{n}{2}[2 \times 750 + (n-1)100]$

$34\ 000 = \frac{n}{2}[1400 + 100n]$

$34\ 000 = 700n + 50n^2$

$680 = 14n + n^2$

$\therefore n^2 + 14n - 680 = 0$

$(n+34)(n-20) = 0$

$\therefore n = 20$ or -34

But $n > 0$.

$\therefore$ It takes her 20 days to swim a distance equal to the width of the English channel.

Question 4

(a) $\sqrt{2}\sin x = 1 \quad (0 \le x \le 2\pi)$

$\sin x = \frac{1}{\sqrt{2}}$

x lies in 1st, 2nd quadrants ($\sin x$ positive).

$\therefore x = \frac{\pi}{4}, \pi - \frac{\pi}{4}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}.$

(b) Sample space is shown in the table below.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$\therefore$ 36 possible outcomes.

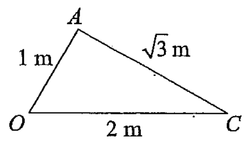
(i) $P(\text{score} = 10) = \frac{3}{36} = \frac{1}{12}.$

(ii) $P(\text{score} \ne 10) = 1 - P(\text{score} = 10)$

$= 1 - \frac{1}{12}$

$= \frac{11}{12}.$

(c) (i)



$OA^2 + AC^2 = 1^2 + (\sqrt{3})^2$

$= 4$

$= 2^2$

$= OC^2$

$\therefore$ Pythagoras' theorem holds.

$\therefore\quad \angle OAC = \frac{\pi}{2}.$

(ii) In ΔAOC,

$\tan \angle ACO = \frac{1}{\sqrt{3}}$

$\therefore\quad \angle ACO = \frac{\pi}{6}$

$\therefore\quad \angle AOC = \pi - \left(\frac{\pi}{2} + \frac{\pi}{6}\right)$

$= \frac{\pi}{3}.$

(iii) By symmetry,

Area $AOBC = 2 \times$ area ΔAOC

$= 2 \times \frac{1}{2} \times 1 \times \sqrt{3}$

$= \sqrt{3}$ m^2.

(iv) $\angle ACB = 2 \times \angle ACO$

$= 2 \times \frac{\pi}{6} = \frac{\pi}{3}$

$\therefore$Reflex $\angle ACB = 2\pi - \frac{\pi}{3}$

$= \frac{5\pi}{3}$

Area of sector $= \frac{1}{2}r^2\theta$ (θ in radi

$\therefore$ Area major sector ACB

$= \frac{1}{2} \times (\sqrt{3})^2 \times \frac{5\pi}{3}$

$= \frac{5\pi}{2}$ m^2.

(v) Similarly,

Reflex $\angle AOB = 2\pi - 2 \times \frac{\pi}{3}$

$= \frac{4\pi}{3}$

$\therefore$ Area major sector AOB

$= \frac{1}{2} \times 1^2 \times \frac{4\pi}{3}$

$= \frac{2\pi}{3}$ m^2

$\therefore$ Total area = area $AOBC$

+ area ACB (iv

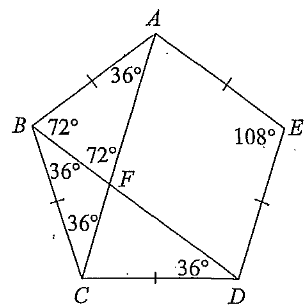
+ area AOB (v)

$= \sqrt{3} + \frac{5\pi}{2} + \frac{\pi}{3}$

$= \sqrt{3} + \frac{19\pi}{6}$ m^2.

Question 5

(a)



(i)

Angle sum of n-sided polygon $= (n-2) \times 180°$

Angle sum of pentagon $= (5-2) \times 180°$

$= 540°$

$\angle ABC = 540° \div 5$ (angles of regular pentagon are equal).

$= 108°$

(ii)

$AB = BC$ (regular pentagon)

$\angle BAC = \angle BCA$ (base $\angle$s of isosceles Δ)

$\therefore \angle BAC = \dfrac{180° - 108°}{2}$ (angle sum of Δ)

$= 36°$.

(iii)

$BC = CD$ (regular pentagon)

$\angle CBD = \angle CDB$ (base $\angle$s of isosceles Δ)

$\angle CBD = \dfrac{180° - 108°}{2}$ (angle sum of Δ)

$= 36°$

$\angle ABF = \angle ABC - \angle CBD$

$= 108° - 36°$

$= 72°$

$\angle AFB = 108° - (36° + 72°)$ (angle sum of Δ)

$= 72°$

$\therefore \angle ABF = \angle AFB$

$\therefore \Delta ABF$ is isosceles (base $\angle$s equal).

(b) (i) $v = \dfrac{2t}{16+t^2}$

Initially, $t = 0$.

$\therefore \quad v = \dfrac{2 \times 0}{16+0} = 0.$

$\therefore$ Initial velocity is zero.

(ii) $a = \dfrac{dv}{dt}\left(\dfrac{2t}{16+t^2}\right)$

$= \dfrac{2(16+t^2) - 2t(2t)}{(16+t^2)^2}$

$= \dfrac{32+2t^2-4t^2}{(16+t^2)^2}$

$= \dfrac{32-2t^2}{(16+t^2)^2}.$

(iii) When $a = 0$,

$\dfrac{32-2t^2}{(16+t^2)^2} = 0$

$32 - 2t^2 = 0$

$2(16 - t^2) = 0$

$2(4-t)(4+t) = 0$

$\therefore t = 4$ or -4

But $t \geq 0$.

$\therefore$ Acceleration is zero when $t = 4$ s.

(iv) $v = \dfrac{dx}{dt} = \dfrac{2t}{16+t^2}$

$\therefore x = \displaystyle\int \dfrac{2t}{16+t^2}\, dt$

$x = \ln(16+t^2) + c$

When $t = 0$, $x = 0$.

$\therefore 0 = \ln 16 + c$

$c = -\ln 16$

$\therefore x = \ln(16+t^2) - \ln 16$

$= \ln\left(\dfrac{16+t^2}{16}\right)$

When $t - 4$,

$x = \ln\left(\dfrac{16+16}{16}\right)$

$x = \ln 2$

$\therefore$ Particle's position is $\ln 2$ metres to the right of the origin when $t = 4$ s.

Question 6

(a) $2e^{2x} - e^x = 0$

$2(e^x)^2 - e^x = 0$

Let $a = e^x$.

$2a^2 - a = 0$

$a(2a-1) = 0$

$\therefore a = 0$ or $a = \dfrac{1}{2}$

$\therefore e^x = 0$ or $e^x = \dfrac{1}{2}$

But $e^x > 0$ for all x.

$\therefore$ Only solution:

$e^x = \dfrac{1}{2}$

$\log_e e^x = \log_e\left(\dfrac{1}{2}\right)$

$\therefore \ x = \log_e\left(\dfrac{1}{2}\right).$

(b) (i) Curve cuts x-axis when $f(x) = 0$

$f(x) = x^4 - 4x^3 = 0$

$x^3(x-4) = 0$

$\therefore \ x = 0, 4$

$\therefore$ Coordinates are $(0, 0)$ and $(4, 0)$.

(ii) Stationary points when $f'(x) = 0$.

$f'(x) = 4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$\therefore \ x = 0, 3$

At $x = 0$, $y = f(0) = 0 - 0 = 0$

At $x = 3$, $y = f(3) = 3^4 - 4(3^3)$

$= 81 - 108 = -27$

$\therefore$ Stationary points at $(0,0)$ and $(3,-27)$.

$f''(x) = 12x^2 - 24x$

At $x = 0$, $f''(0) = 0 - 0 = 0$

$\therefore$ Using 1st derivative test:

x	-1	0	1
$f'(x)$	-16	0	-8

$\therefore (0, 0)$ is a horizontal point of inflexion

At $x = 3$, $f''(3) = 12(3^2) - 24(3)$

$= 108 - 72$

$= 36$

$> 0 \Rightarrow$ concave up

$\therefore (3, -27)$ is a minimum turning point.

(iii) For points of inflexion, $f''(x) = 0$.

$f''(x) = 12x^2 - 24x = 0$

$12x(x-2) = 0$

$\therefore \ x = 0, 2$

At $x = 0$, $(0, 0)$ is a horizontal point of inflexion (from (ii)).

At $x = 2$, $y = f(2) = 2^4 - 4(2)^3$

$= 16 - 32$

$= -16$

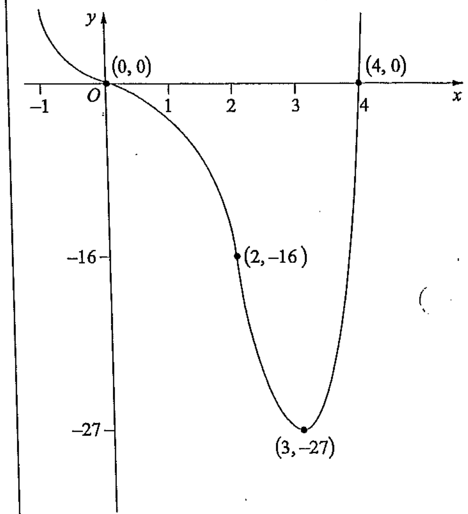
Considering concavity:

x	1	2	3
$f''(x)$	-12	0	36

$\therefore$ Concavity changes.

$\therefore$ Point of inflexion at $(2, -16)$.

(iv)



Question 7

(a) (i) $y = x^2 + 4$

i.e. $x^2 = y - 4$

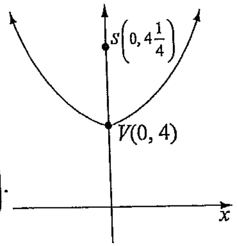
or $x^2 = 4a(y-4)$

Hence $4a = 1$

$a = \frac{1}{4}$

Vertex is $V(0, 4)$.

$\therefore$ Focus, S, is $\left(0, 4\frac{1}{4}\right)$.



(ii) At P, x satisfies both $y = x^2 + 4$ and $y = x + k$. To find the x value of P, solve the equations simultaneously.

$$x^2 + 4 = x + k$$

$$\therefore x^2 - x + 4 - k = 0.$$

(iii) Discriminant formula:

$$\Delta = b^2 - 4ac$$

For $x^2 - x + 4 - k = 0$,

$$\Delta = (-1)^2 - 4(1)(4-k)$$
$$= 1 - 4(4-k)$$
$$= 1 - 16 + 4k$$
$$= 4k - 15$$

For one point of intersection,

$$\Delta = 0 \text{ (equal roots)}$$

$$\therefore 4k - 15 = 0$$

$$k = \frac{15}{4}.$$

(iv) Substituting $k = \frac{15}{4}$ into

$$x^2 - x + 4 - k = 0$$

gives $x^2 - x + 4 - \frac{15}{4} = 0$

METHOD 1

$$x^2 - x + \frac{1}{4} = 0$$
$$\left(x - \frac{1}{2}\right)^2 = 0$$
$$\therefore x = \frac{1}{2}$$

METHOD 2

$$4x^2 - 4x + 1 = 0$$
$$(2x-1)^2 = 0$$
$$2x - 1 = 0$$
$$\therefore x = \frac{1}{2}$$

Substituting into

$$y = x^2 + 4$$

gives $y = \left(\frac{1}{2}\right)^2 + 4$

$$\therefore y = 4\frac{1}{4}$$

$$\therefore P \text{ is } \left(\frac{1}{2}, 4\frac{1}{4}\right).$$

(v) Gradient of interval SP is

$$m_{SP} = \frac{4\frac{1}{4} - 4\frac{1}{4}}{0 - \frac{1}{2}} = 0$$

Equation of directrix is $y = 3\frac{3}{4}$.

$\therefore$ Gradient of directrix $= 0$

$\therefore SP$ is parallel to the directrix.

(b) (i) $\sqrt{3}\cos x = \sin x$

$$\frac{\sqrt{3}\cos x}{\cos x} = \frac{\sin x}{\cos x}$$

$$\tan x = \sqrt{3}$$

x is in 1st, 3rd quadrants.

$$\therefore x = \frac{\pi}{3}, \pi + \frac{\pi}{3}$$
$$= \frac{\pi}{3}, \frac{4\pi}{3}$$

$\therefore x$-coordinate of $A = \frac{\pi}{3}$

x-coordinate of $B = \frac{4\pi}{3}$.

(ii) Area $= \int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \left(\sin x - \sqrt{3}\cos x\right) dx$

$$= \left[-\cos x - \sqrt{3}\sin x\right]_{\frac{\pi}{3}}^{\frac{4\pi}{3}}$$

$$= \left(-\cos\left(\frac{4\pi}{3}\right) - \sqrt{3}\sin\left(\frac{4\pi}{3}\right)\right) - \left(-\cos\left(\frac{\pi}{3}\right) - \sqrt{3}\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \left(-\left(-\frac{1}{2}\right) - \sqrt{3}\left(\frac{-\sqrt{3}}{2}\right)\right) - \left(-\left(\frac{1}{2}\right) - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{1}{2} + \frac{3}{2} + \frac{1}{2} + \frac{3}{2}$$

$$= 4 \text{ units}^2.$$

Question 8

(a) (i) $N = Ae^{kt}$

When $t = 1$, $N = 1600$ million

$$\therefore 1600 = Ae^{k\times 1} \quad —①$$

When $t = 2$, $N = 2600$ million

$$\therefore 2600 = Ae^{k\times 2} \quad —②$$

Solving simultaneously, ② ÷ ①:

$$\frac{2600}{1600} = \frac{Ae^{2k}}{Ae^{k}}$$

$$\frac{13}{8} = e^{2k-k}$$

$$\frac{13}{8} = e^{k}$$

$$\log_e\left(\frac{13}{8}\right) = \log_e e^k$$

$$\log_e\left(\frac{13}{8}\right) = k\log_e e$$

$$\therefore k = \log_e\left(\frac{13}{8}\right)$$

Substituting for k in ①,

$$1600 = Ae^{\log_e\left(\frac{13}{8}\right)}$$

$$1600 = A \times \frac{13}{8} \quad (Note: e^{\log_e a} = a)$$

$$A = 1600 \times \frac{8}{13}$$

$$\therefore A = \frac{12\,800}{13} \text{ million.}$$

(ii) $N = \frac{12\,800}{13}e^{kt}$

For $N > 4000$ million, $t = ?$

$$\frac{12\,800}{13}e^{kt} > 4000$$

Let $\frac{12\,800}{13}e^{kt} = 4000$

$$e^{kt} = 4000 \times \frac{13}{12\,8(}$$

$$e^{kt} = \frac{65}{16}$$

$$\log_e e^{kt} = \log_e\left(\frac{65}{16}\right)$$

$$kt = \log_e\left(\frac{6(}{16}\right)$$

$$\therefore t = \frac{\log_e\left(\frac{65}{16}\right)}{k}$$

$$t = \frac{\log_e\left(\frac{65}{16}\right)}{\log_e\left(\frac{13}{8}\right)}$$

$$= 2.8872\ldots$$

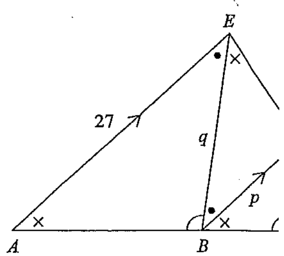
$\therefore t = 2$ years $+ (0.8872\ldots$ of

$= 2$ years $+ 10.6473\ldots$ mon

$\therefore$ Mobile phones first exceed

4000 million during No

(b)



(i) In ΔABE and ΔBCD,

$\angle ABE = \angle BCD$ (give

$\angle EAB = \angle DBC$ (corr

$\angle$s, A

$\therefore \Delta ABE \,|||\, \Delta BCD$ (equi nths)

10.

ding
D)
ar).

(ii) *METHOD 1*

$BE \parallel CD$ ($\angle ABE$ and $\angle BCD$ are equal corresponding $\angle$s)

In ΔEDB and ΔBCD,

$\angle BDE = \angle BCD$ (given)

$\angle EBD = \angle BDC$ (alternate $\angle$s, $BE \parallel CD$)

$\therefore \Delta EDB \parallel\!\mid \Delta BCD$ (equiangular).

METHOD 2

In ΔABE and ΔEDB,

$\angle ABE = \angle BDE$ (given)

$\angle AEB = \angle EBD$ (alternate $\angle$s, $AE \parallel BD$)

$\therefore \Delta ABE \parallel\!\mid \Delta EDB$ (equiangular)

$\therefore \Delta EDB \parallel\!\mid \Delta BCD$ (both similar to ΔABE).

(iii) For geometric series, need to show $\frac{p}{8} = \frac{q}{p} = \frac{27}{q}$.

Since $\Delta ABE \parallel\!\mid \Delta BCD$,

$$\therefore \frac{AB}{BC} = \frac{BE}{CD} = \frac{EA}{DB} \quad \text{(corresponding sides in proportion)}$$

$$\frac{AB}{BC} = \frac{q}{8} = \frac{27}{p}$$

$$\therefore pq = 27 \times 8$$

$$\therefore \frac{p}{8} = \frac{27}{q} \quad \text{—①}$$

Since $\Delta EDB \parallel\!\mid \Delta BCD$,

$$\therefore \frac{ED}{BC} = \frac{EB}{BD} = \frac{DB}{CD}$$

$$\frac{ED}{BC} = \frac{q}{p} = \frac{p}{8}$$

$$\therefore \frac{p}{8} = \frac{q}{p} \quad \text{—②}$$

From ① and ②, $\frac{p}{8} = \frac{q}{p} = \frac{27}{q}$

$\therefore 8, p, q, 27$ are the first four terms of a geometric series.

(iv) $T_n = ar^{n-1}$

$T_4 = 27, \quad a = 8, \quad r = ?$

$$8r^{4-1} = 27$$

$$r^3 = \frac{27}{8}$$

$$r = \frac{3}{2}$$

$$\therefore \quad p = T_2 = 8 \times \frac{3}{2} = 12$$

$$q = T_3 = 8 \times \left(\frac{3}{2}\right)^2 = 18.$$

Question 9

(a)
$$V_x = \pi\int_0^1 y^2 dx$$
$$= \pi\int_0^1 (x^2+1)^2\,dx$$
$$= \pi\int_0^1 (x^4+2x^2+1)\,dx$$
$$= \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^1$$
$$= \pi\left[\left(\frac{1}{5}+\frac{2}{3}+1\right) - 0\right]$$
$$= \pi\left[\frac{28}{15}\right]$$
$$= \frac{28\pi}{15} \text{ units}^3.$$

(b) (i) After first card is drawn, 51 card remain, of which 3×13 are of a different suit.

$\therefore P$(2nd card different suit)

$$= \frac{39}{51} \text{ or } \frac{13}{17}.$$

(ii) P(all different suits)

$= P(\text{any suit}) \times P(\text{any of other 3 suits}) \times P(\text{any of other 2 suits}) \times P(\text{last remaining suit})$

$$= \left(\frac{52}{52}\right) \times \left(\frac{39}{51}\right) \times \left(\frac{26}{50}\right) \times \left(\frac{13}{49}\right)$$

$$= \frac{2197}{20\,825} \text{ or } 0.1055 \text{ (to 4 d.p.).}$$

(c) (i)

1st \$1000 invested: $A_1 = \$1000 \times 1.06^{18}$

2nd \$1000 invested: $A_2 = \$1000 \times 1.06^{17}$

⋮

Last \$1000 invested: $A_{18} = \$1000 \times 1.06^1$

$\therefore$ Total contributions $= \$1000(1.06^{18} + 1.06^{17} + \ldots + 1.06^2 + 1.06^1)$

$= \$1000(1.06 + 1.06^2 + \ldots + 1.06^{18})$

which is geometric series with

$a = 1.06, \quad r = 1.06, \quad n = 18.$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$S_{18} = \frac{1.06(1.06^{18}-1)}{1.06-1} = \frac{1.06(1.06^{18}-1)}{0.06}$$

$\therefore$ Total value of investment

$= \$1,000 \times S_{18}$

$= \$32\,759.99.$

(ii) For first 3 years:

$A_1 = \$1000 \times 1.06^3$

$A_2 = (\$1000 \times 1.06) \times 1.06^2 = \$1000 \times 1.06^3$

$A_3 = (\$1000 \times 1.06 \times 1.06) \times 1.06^1 = \$1000 \times (1.06)^3$

$\therefore$ Total value after 3 years

$= A_1 + A_2 + A_3$

$= \$1000 \times 1.06^3 \times 3$

$= \$3573.05.$

(iii) From (ii), after 18 years:

$A_1 = \$1000 \times 1.06^{18}$

$A_2 = (\$1000 \times 1.06) \times 1.06$

$= \$1000 \times 1.06^{18}$ … e

$\therefore$ Total value of investme

$= A_1 + A_2 + \ldots + A_{18}$

$= \$1000 \times 1.06^{18} \times 18$

$= \$51\,378.10.$

Question 10

(a) (i) Distance travelled $= \int_0^4 v$

where $v = \frac{dx}{dt}$

METHOD 1 Function method, usin

$$\int_a^b f(t)\,dt \doteqdot \frac{b-a}{6}\left\{f(a) + 4f\left(\frac{a+b}{2}\right.\right.$$

t	0	2	4
$\frac{dx}{dt}$	0	1	5

$$\int_0^4 v\,dt \doteqdot \frac{4-0}{6}\{0 + 4\times 1 + 5\}$$
$$\doteqdot \frac{4}{6} \times 9$$
$$\doteqdot 6.$$

METHOD 2 Weights method, usin

$$\int_a^b f(x)\,dx \doteqdot \frac{\text{total width}}{\Sigma \text{ wts}} \times \sum\left(\frac{dx}{dt} \times\right.$$

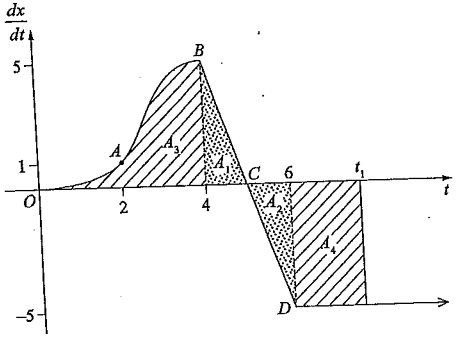
t	0	2	4
$\frac{dx}{dt}$	0	1	5
weights	1	4	1
$\frac{dx}{dt} \times wts$	0	4	5

$$\int_0^4 v\,dt \doteqdot \frac{4-0}{6} \times 9$$
$$\doteqdot 6$$

$\therefore$ By Simpson's rule, distance tra between $t = 0$ and $t = 4$ is approx

(ii) Displacement (x) is decreasing when $\frac{dx}{dt} < 0$. From graph, when $t > 5$.

(iii) Object will return to origin when the area above the axis is equal to the area below the axis.



$$A_1 = A_2 = \frac{1}{2} \times 1 \times 5$$
$$= 2.5 \text{ units}^2$$

$\therefore$ Need to find t_1 when

$$A_3 = A_4$$

From (i), $6 \doteqdot 5 \times (t_1 - 6)$

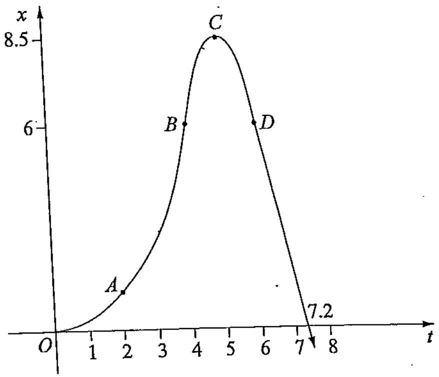
$$\frac{6}{5} \doteqdot t_1 - 6$$
$$t_1 \doteqdot 7.2 \text{ s.}$$

(iv) $\frac{dx}{dt}$ above t-axis $= x$ has positive gradient

$\frac{dx}{dt}$ on t-axis $= x$ has turning point

$\frac{dx}{dt}$ under t-axis $= x$ has negative gradient

$\therefore$ Displacement curve starts at origin, has a max turning point at C, and cuts t-axis at approx 7.2.



Note: Displacement values on the vertical axis are derived from calculating the areas under the given velocity curve. For example, the area under the velocity curve from A to C is $6 + 2.5 = 8.5$. These values are approximate only unless the velocity curve from O to B is cubic, in which case the use of Simpson's rule would give the exact values.

(b) (i) $N = \frac{L}{d^2}$

$$N = \frac{L_1}{x^2} + \frac{L_2}{(m-x)^2} \quad \text{(Since } PL_2 = m - x\text{)}$$

(ii) $N = L_1 x^{-2} + L_2 (m-x)^{-2}$

$$\frac{dN}{dx} = -2L_1 x^{-3} - 2L_2 (m-x)^{-3} \times -$$
$$= \frac{-2L_1}{x^3} + \frac{2L_2}{(m-x)^3}$$

Stationary points when $\frac{dN}{dx} = 0$.

$$\frac{-2L_1}{x^3} + \frac{2L_2}{(m-x)^3} = 0$$
$$\frac{-2L_1}{x^3} = \frac{2L_2}{(m-x)^3}$$
$$(m-x)^3 \times 2L_1 = x^3 \times 2L_2$$
$$\frac{(m-x)^3}{x^3} = \frac{2L_2}{2L_1}$$
$$\frac{m-x}{x} = \sqrt[3]{\frac{L_2}{L_1}}$$
$$\frac{m}{x} - 1 = \sqrt[3]{\frac{L_2}{L_1}}$$
$$\therefore \frac{m}{x} = 1 + \sqrt[3]{\frac{L_2}{L_1}}$$
$$m = x\left(1 + \sqrt[3]{\frac{L_2}{L_1}}\right)$$
$$x = \frac{m}{1 + \sqrt[3]{\frac{L_2}{L_1}}}$$

Test:

$$\frac{d^2N}{dx^2} = 6L_1 x^{-4} - 6L_2 (m-x)^{-4} \times -1$$
$$= \frac{6L_1}{x^4} + \frac{6L_2}{(m-x)^4}$$

> 0 (since $L_1, L_2, x^4, (m-x)^4 > ($

$\Rightarrow$ concave up

$\therefore$ Minimum noise level when $x = \frac{m}{1 + \sqrt[3]{\frac{L_2}{L_1}}}$.

End of Mathematics solutions