Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2010
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Monday, 9 August 2010

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided as a separate page
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

6400 - 1

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}; \quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

Note: $\ln x = \log_e x, \quad x > 0$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find

(i) $\displaystyle\int \frac{e^x - e^{-x}}{\left(e^x + e^{-x}\right)^2}\,dx.$ **2**

(ii) $\displaystyle\int \frac{dx}{\sqrt{6x - x^2}}.$ **2**

(b) Use integration by parts to find $\displaystyle\int \ln\left(x^2+1\right)dx.$ **3**

(c) By means of the substitution $x = 2 + \sin^2\theta$, for $0 \le \theta \le \dfrac{\pi}{2}$, evaluate **4**

$$\int_{2\frac{1}{4}}^{2\frac{1}{2}} \frac{dx}{\sqrt{(3-x)(x-2)}}.$$

(d) (i) Use the result $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, to show that **2**

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx.$$

(ii) Hence, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx.$ **2**

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = x + iy$, where x and y are real numbers. If $\overline{z}$ is its complex conjugate and $2z - \overline{z} = 1 + 6i$ find z. **2**

(b) (i) Express $z = 1 + \sqrt{3}\,i$ in modulus-argument form. **1**

(ii) Hence, or otherwise, show that $z^7 - 64z = 0$. **2**

(c) For the complex number $z = x + iy$, where x and y are real numbers, find, and clearly sketch, the curve in the Argand diagram for which $\operatorname{Re}(z^2 - 3) = 0$. **3**

(d) Let $OABC$ represent a square on an Argand diagram, where O is the origin and A is represented by the complex number z. **3**

Find the complex numbers represented by B.

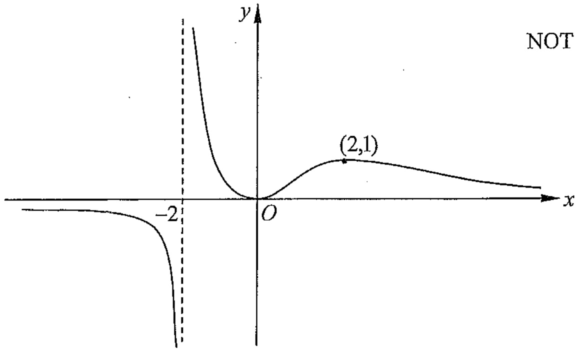
(e) (i) Solve $z^4 = 1$ for all z. **1**

(ii) Hence, or otherwise, solve $z^4 = (z-1)^4$. **3**

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The function defined by $y = f(x)$ is drawn below.

NOT TO SCALE



Draw separate, one-third page sketches, of the following:

(i) $y = f(-x)$. **2**

(ii) $y = \dfrac{1}{f(x)}$. **2**

(iii) $y = \left[f(x)\right]^2$. **2**

(b) The equation of a curve is $x^2 + 3xy + 4y^2 = 58$. **3**

Find the equation of the normal to the curve at the point $(2, 3)$.

Question 3 continues on page 5

(c) The polynomial $P(x) = x^3 + 3x^2 - 24x + 28$ has a double root. **3**

Find all solutions to $P(x) = 0$.

(d) Using the method of cylindrical shells, find the volume of the solid formed by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$. **3**

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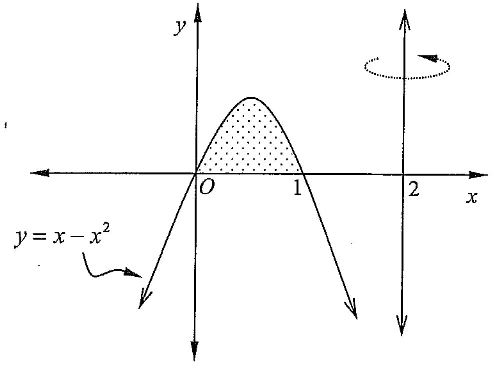


End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) (i) In the complex plane show that the locus of z satisfying $|z+3|+|z-3|=10$ is an ellipse and find its equation. **3**

(ii) Draw a neat sketch of the ellipse, clearly indicating its foci and the equations of its directrices. **1**

(b) Consider $P(x)=x^4-2Ax^3+B$, where A and B are constants, $A\neq 0$. **2**

The roots of $P(x)=0$ are α, β, γ and $\alpha+\beta+\gamma$.

Show that $B=A^4$.

(c) A mass of 5 kilograms attached to the end of a light fishing line describes a circular path with radius 60 centimetres about a point P on a smooth table. It completes 2 revolutions per second.

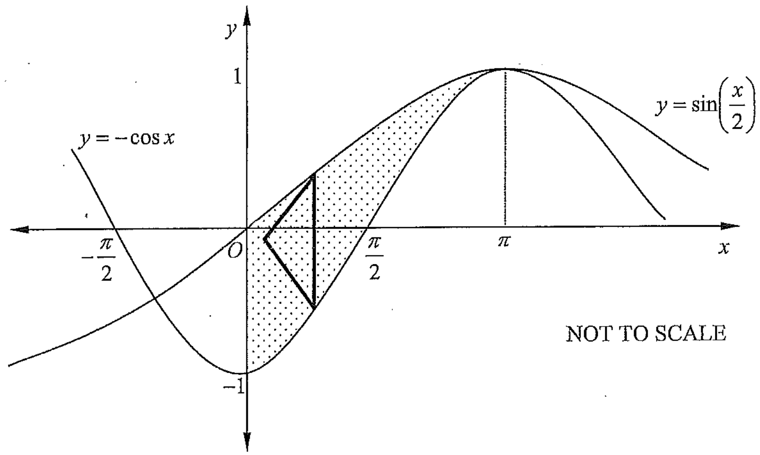
(i) Find the tension in the fishing line. **2**

(ii) The line breaks under a tension of 900N. Find the maximum number of revolutions per second. **2**

Question 4 continues on page 7

(d) (i) Show that $2\cos A\sin B=\sin(A+B)-\sin(A-B)$. **1**

(ii) The region R describes the area bounded by the graphs $y=-\cos x$ and $y=\sin\left(\frac{x}{2}\right)$ in the interval $0\leq x\leq \pi$ as shown below. **4**



A solid is formed with base R and cross-sections that are equilateral triangles perpendicular to the x-axis.

Calculate the volume of the solid formed.

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) Find all x in the domain $0 \le x \le \pi$, such that $\sin x = \cos 5x$. **3**

(b) Consider the expansions $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$ and $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

(i) Show that $\cot 4\theta = \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$. **2**

(ii) Hence show that one of the roots of the equation $x^2 - 6x + 1 = 0$ is $\cot^2\left(\dfrac{\pi}{8}\right)$. **3**

(iii) Hence find the value of $\operatorname{cosec}^2\left(\dfrac{\pi}{8}\right) + \operatorname{cosec}^2\left(\dfrac{3\pi}{8}\right)$. **2**

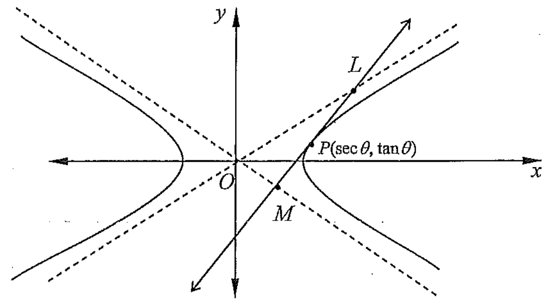
(c) Let $I_n = \int_1^2 (\ln x)^n \, dx$ where n is a positive integer.

(i) Show that $I_n = 2(\ln 2)^n - nI_{n-1}$. **2**

(ii) Hence evaluate $I_4 = \int_1^2 (\ln x)^4 \, dx$. Write your answer in exact form. **3**

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the hyperbola with equation $x^2 - y^2 = 1$.



(i) Show that the equation of the tangent to this hyperbola at the point $P(\sec\theta, \tan\theta)$ is given by $x\sec\theta - y\tan\theta = 1$. **2**

(ii) This tangent cuts the asymptotes at L and M. Show that $LP = PM$. **3**

(iii) Show that the area of ΔOLM, where O is the origin, is independent of the position of P. **3**

(b) Julie is standing on the middle of a bridge across a ravine. One of her earrings falls off and drops vertically into the ravine with acceleration reduced by air resistance proportional to its velocity, $v \text{ ms}^{-1}$. Ignore any horizontal forces.

(i) Explain why its acceleration is given by $\ddot{x} = g - kv$ where k is a constant and $g \text{ ms}^{-2}$ is acceleration due to gravity. **1**

(ii) Show that the velocity of the earring is given by $v = \dfrac{g}{k}\left(1 - e^{-kt}\right) \text{ ms}^{-1}$ where t seconds is the time it takes to fall x metres. **2**

(iii) Hence, find the terminal velocity of the earring. **1**

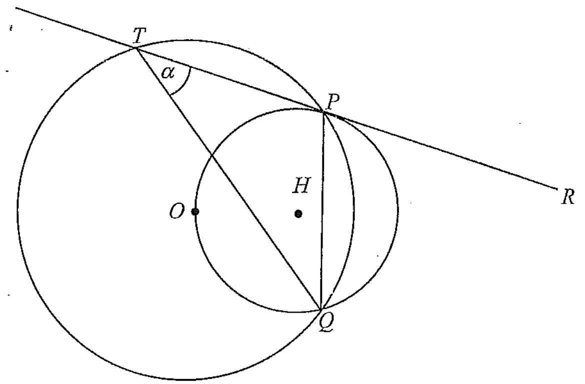
(iv) Derive an expression for the distance travelled, x, as a function of the velocity, v, at any time. **3**

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) Five identical rings are to be placed on four fingers of one hand. **3**

In how many ways can this be done?

(b) Two circles intersect at P and Q as shown. The centre of the smaller circle is H and the centre of the larger circle is O. The smaller circle passes through the centre of the larger circle. The tangent to the smaller circle at P, RPT, cuts the larger circle at T. The chord in the smaller circle, PQ, bisects $\angle RQO$. Let $\angle PTQ = \alpha$.



(i) Prove that ΔPQT is isosceles. **3**

(ii) Prove that P is the midpoint of RT. **4**

(c) (i) Show that if $a,\ b > 0$, $\frac{a}{b} + \frac{b}{a} \geq 2$. **1**

(ii) Show that if $a,\ b > 0$, $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$. **1**

(iii) Hence, or otherwise, show that if $a,\ b,\ c > 0$, $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$. **3**

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Use mathematical induction to show that if $f(x) = e^{2x}$ then the n-th derivative $f^{(n)}(x) = 2^n e^{2x}$ for all integers, $n > 0$. **3**

(ii) A power series of a function can be expressed as **2**

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots .$$

Show that $e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots$.

(iii) Given $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$, find $\lim_{x \to 0} \frac{1 + 2x + 2x^2 - e^{2x}}{\tan x - x}$. **2**

(b) (i) Show that for $k \geq 0$, $2k + 3 \geq 2\sqrt{(k+1)(k+2)}$. **2**

(ii) Hence show that for $n \geq 1$, $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right)$. **4**

(iii) Discuss the validity of the statement $\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10}$ for all positive integers. **2**

End of paper

CATHOLIC SECONDARY SCHOOLS ASSOCIATION

2010 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 2

Question 1 (15 marks)
(a) (i) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses an appropriate substitution	1
• finds correct primitive (+C not essential)	1

Sample Answer:

$$\int \frac{e^x - e^{-x}}{\left(e^x + e^{-x}\right)^2}\,dx = \int \frac{1}{u^2}\,du \qquad \text{let } u = e^x + e^{-x}$$

$$= \int u^{-2}\,du \qquad \frac{du}{dx} = e^x - e^{-x}$$

$$= -u^{-1} + C$$

$$= \frac{-1}{\left(e^x + e^{-x}\right)} + C$$

(a) (ii) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• simplifies integral	1
• finds correct primitive (+C not essential)	1

Sample Answer:

$$\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{9 - (x^2 - 6x + 9)}}$$

$$= \int \frac{dx}{\sqrt{9 - (x-3)^2}}$$

$$= \sin^{-1}\left(\frac{x-3}{3}\right) + C$$

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(b) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly applies integration by parts or significant progress towards solution	1
• further progress towards solution	1
• finds correct primitive (+C not essential)	1

Sample Answer:

$$\int \ln\left(x^2+1\right)dx = x\ln\left(x^2+1\right) - \int \frac{2x^2}{x^2+1}\,dx \qquad u = \ln\left(x^2+1\right) \quad \frac{dv}{dx} = 1$$

$$= x\ln\left(x^2+1\right) - 2\int \frac{x^2+1-1}{x^2+1}\,dx \qquad \frac{du}{dx} = \frac{2x}{x^2+1} \quad v = x$$

$$= x\ln\left(x^2+1\right) - 2\int\left(1 - \frac{1}{x^2+1}\right)dx$$

$$= x\ln\left(x^2+1\right) - 2x + 2\tan^{-1}x + C$$

(c) (4 marks)
Outcomes assessed: E8
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress using the substitution	1
• establishes integral using the substitution or other progress towards solution	1
• further progress towards solution	1
• evaluates correctly (correct numerical equivalence)	1

Sample Answer:

$$3 - x = 3 - (2 + \sin^2\theta) \qquad \text{and} \qquad x - 2 = (2 + \sin^2\theta) - 2$$

$$= 1 - \sin^2\theta \qquad\qquad = \sin^2\theta$$

$$= \cos^2\theta$$

$$2\frac{1}{2} = 2 + \sin^2\theta \qquad \text{and} \qquad 2\frac{1}{4} = 2 + \sin^2\theta$$

$$\text{for } 0 \le \theta \le \frac{\pi}{2} \quad \sin^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4} \quad \text{and} \quad \sin^2\theta = \frac{1}{4} \Rightarrow \theta = \frac{\pi}{6}$$

$$\int_{2\frac{1}{4}}^{2\frac{1}{2}} \frac{dx}{\sqrt{(3-x)(x-2)}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{2\sin\theta\cos\theta}{\sqrt{\cos^2\theta\sin^2\theta}}\,d\theta \qquad x = \sin^2\theta \Rightarrow \frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\,d\theta$$

$$= \left[2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{2\pi}{4} - \frac{2\pi}{6}$$

$$= \frac{\pi}{6}$$

(d) (i) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards solution using the given result	1
• shows correct result	1

Sample Answer:

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx \qquad \text{using } \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\tan x}{1+\tan x}+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$$

(d) (ii) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards solution	1
• evaluates correctly (correct numerical equivalence)	1

Sample Answer:

$$\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx = \int_0^{\frac{\pi}{4}} \left(\ln 2 - \ln(1+\tan x)\right)dx \qquad \text{using logarithmic laws}$$

$$\therefore \text{ From (i) } \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \left(\ln 2 - \ln(1+\tan x)\right)dx$$

$$\text{ie } 2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,dx$$

$$= \left[x\ln 2\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4}\ln 2 - 0\ln 2$$

$$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi \ln 2}{8} \quad \text{ie} \quad \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx = \frac{\pi\ln 2}{8}$$

Question 2 (15 marks)
(a) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Mark
• uses complex conjugate and equates expressions correctly	1
• finds correct complex number	1

Sample Answer:

$$z = x+iy \quad \therefore \ \bar{z} = x-iy$$
$$2z-\bar{z} = 2(x+iy)-(x-iy) = x+3iy$$
$$\text{Also } 2z-\bar{z} = 1+6i$$
$$\therefore x+3iy = 1+6i$$
$$\text{ie } x=1, y=2$$
$$\therefore \ z = 1+2i$$

(b) (i) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds correct modulus and argument	1

Sample Answer:

$$z = 1+\sqrt{3}\,i$$
$$|z| = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2$$
$$\arg z = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$
$$\therefore \ z = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

(b) (ii) (2 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses De Moivre's Theorem	1
• shows correct conclusion	1

Sample Answer:

By De Moivre's Theorem $z^7 = r^7(\cos 7\theta + i\sin 7\theta)$

$$\therefore z^7 = 2^7\left(\cos\frac{7\pi}{3}+i\sin\frac{7\pi}{3}\right)$$
$$= 128\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$
$$\therefore \ z^7 - 64z = 128\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) - 64\times 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$
$$= 0 \quad \text{as required}$$

(c) (3 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

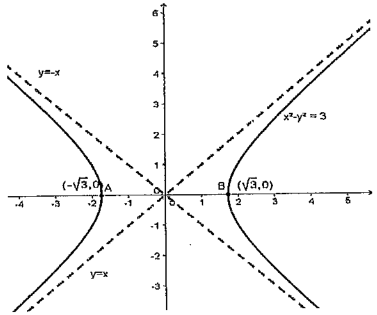
Criteria	Mark
• progress towards finding the equation of the curve	1
• correctly identifies the equation of the curve	1
• correctly sketches the curve	1

Sample Answer:

$z^2 = (x^2 - y^2) + 2ixy$

$\text{Re}(z^2 - 3) = x^2 - y^2 - 3 \therefore$ curve is $x^2 - y^2 - 3 = 0$

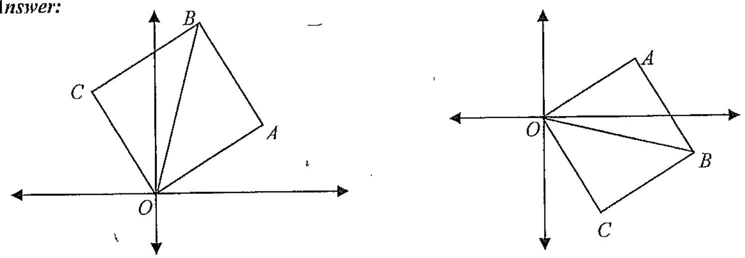
ie the hyperbola $x^2 - y^2 = 3$; x-intercepts are $x = \pm\sqrt{3}$ and asymptotes are $y = \pm x$



(d) (3 marks)
Outcomes assessed: E3
Targeted Performance Bands: E3-E4

Criteria	Mark
• establishes relationship using geometrical and vector properties	1
• finds one possible answer or other progress towards answers	1
• gives correct answers	1

Sample Answer:



$OABC$ is a square $\therefore$ If A is z then C is iz or $-iz$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ and $\overrightarrow{OC} = \overrightarrow{AB}$

Let Z be B

$\therefore Z = z + iz \qquad$ or $\quad Z = z - iz$

$\quad = (1+i)z \qquad\qquad = (1-i)z$

(e) (i) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Mark
• gives correct solution	1

Sample Answer:

$z^4 = 1 \quad \Rightarrow z^4 - 1 = 0$

$(z^2 - 1)(z^2 + 1) = 0$

$(z-1)(z+1)(z-i)(z+i) = 0$

$z = \pm 1, \ \pm i$

(e) (ii) (3 marks)
Outcomes assessed: E3
Targeted Performance Bands: E3-E4

Criteria	Mark
• uses result from (i) or other progress towards solution	1
• finds a solution or further progress towards solutions	1
• gives correct solutions	1

Sample Answer:

$z^4 = (z-1)^4 \quad \Rightarrow \quad \left(\frac{z}{z-1}\right)^4 = 1$

ie $\frac{z}{z-1} = \pm 1, \ \pm i$

If $\frac{z}{z-1} = 1 \quad \Rightarrow \quad z = z - 1$

no solution

If $\frac{z}{z-1} = -1 \quad \Rightarrow \quad z = -z + 1$

$z = \frac{1}{2}$

If $\frac{z}{z-1} = i \quad \Rightarrow \quad z = iz - i$

$z - iz = -i$

$z = \frac{-i}{1-i}$

If $\frac{z}{z-1} = -i \quad \Rightarrow \quad z = -iz + i$

$z + iz = i$

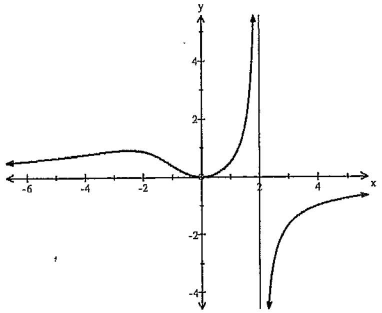
$z = \frac{i}{1+i}$

$z = \frac{1}{2}, \frac{-i}{1-i}, \frac{i}{1+i}$

Question 3 (15 marks)
(a) (i) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards the graph	1
• correctly sketches the graph	1

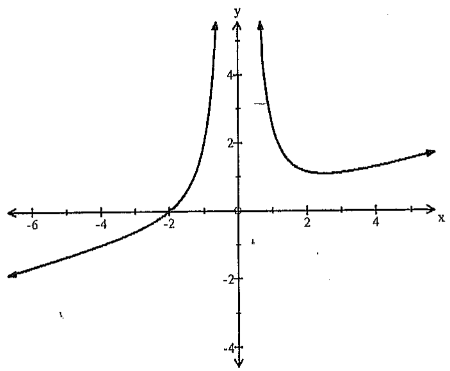
Sample Answer:



(a) (ii) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards the graph	1
• correctly sketches the graph	1

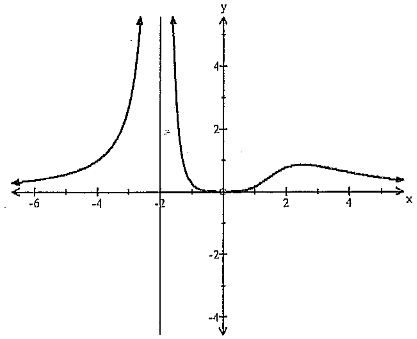
Sample Answer:



(a) (iii) (2 marks)
Outcomes assessed: E6
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards the graph	1
• correctly sketches the graph	1

Sample Answer:



(b) (3 marks)
Outcomes assessed: E6
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses implicit differentiation correctly	1
• further progress towards result	1
• determines equation of normal	1

Sample Answer:

$$x^2+3xy+4y^2=58$$

$$2x+3x\frac{dy}{dx}+3y+8y\frac{dy}{dx}=0$$

$$(3x+8y)\frac{dy}{dx}=-(2x+3y)$$

$$\frac{dy}{dx}=\frac{-(2x+3y)}{(3x+8y)}$$

At (2, 3) $\frac{dy}{dx}=\frac{-(4+9)}{(6+24)}$ ie. $m_T=-\frac{13}{30}$

$\therefore$ gradient of the normal $=\frac{30}{13}$

Hence the equation of the normal is:

$$y-3=\frac{30}{13}(x-2)$$

$$13y-39=30x-60$$

$$30x-13y-21=0$$

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(c) (3 marks)
Outcomes assessed: E4
Targeted Performance Bands: E2-E3

Criteria	Mark
• applies property of double root	1
• determines the double root	1
• finds correct solutions	1

Sample Answer:

Let $P(x)=x^3+3x^2-24x+28$

$P'(x)=3x^2+6x-24=0$ for a double root

$x^2+2x-8=0$

$(x+4)(x-2)=0$

$x=-4$ or $x=2$

$P(-4)=(-4)^3+3(-4)^2-24(-4)+28=108\neq 0$

$\therefore x=-4$ is not a solution

$P(2)=(2)^3+3(2)^2-24(2)+28=0$

$\therefore x=2$ is a solution

$\therefore P(x)=(x-2)^2(x+b)$ and by inspection $b=7$

$\therefore$ solutions are 2, 2 and -7

(d) (3 marks)
Outcomes assessed: E7
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards volume of shell	1
• establishes a correct integral for volume	1
• evaluates correctly (correct numerical equivalence)	1

Sample Answer:

Consider a slice parallel to the line $x=2$ with thickness Δx

The slice is rotated about $x=2$ to form a cylindrical shell with radius $2-x$ and height $x-x^2$

Volume of shell $\Delta V=2\pi(2-x)(x-x^2)\Delta x$

$$\therefore V=\lim_{\Delta x\to 0}\sum_{x=0}^{1}2\pi(2-x)(x-x^2)\Delta x$$

$$=2\pi\int_0^1\left(2x-3x^2+x^3\right)dx$$

$$=2\pi\left[x^2-x^3+\frac{x^4}{4}\right]_0^1$$

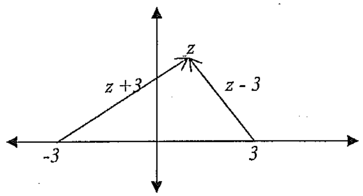
$$=2\pi\left(1-1+\frac{1}{4}-(0-0+0)\right)$$

$$=\frac{\pi}{2}\text{ cubic units}$$

Question 4 (15 marks)
(a) (i) (3 marks)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards solution using definition of ellipse	1
• shows locus is an ellipse	1
• finds correct equation	1

Sample Answer:



In the diagram let 3 and -3 represent the foci, S and S', and z represent the point P

By definition of an ellipse $SP+S'P=2a$.

Since $|z+3|+|z-3|=10$ then $SP+S'P=10$ $\therefore$ locus of z is an ellipse.

Hence $a=5$ and $ae=3$ $\therefore e=\frac{3}{5}$.

$b^2=a^2(1-e^2)=25\left(1-\frac{9}{25}\right)$

$\therefore b=4$

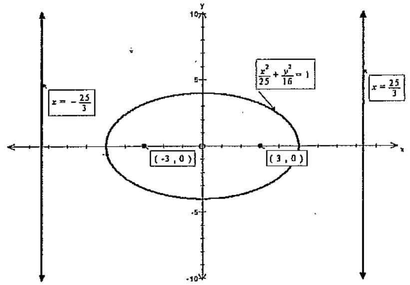
$\therefore$ Equation is $\frac{x^2}{25}+\frac{y^2}{16}=1$

(a) (ii) (1 mark)
Outcomes assessed: E3
Targeted Performance Bands: E2-E3

Criteria	Marks
• draws correct graph	1

Sample Answer:

Foci $(\pm 3,0)$ and directricies $x=\pm\frac{25}{3}$



(b) (2 marks)
Outcomes assessed: E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• determines the sum of roots	1
• shows final result	1

Sample Answer:

$P(x)=x^4-2Ax^3+B$, where $A\neq 0$ and the roots of $P(x)=0$ are α,β,γ and $\alpha+\beta+\gamma$.

sum of roots: $\alpha+\beta+\gamma+\alpha+\beta+\gamma=2A$

$\therefore\ \alpha+\beta+\gamma=A$

but $\alpha+\beta+\gamma$ is a root of $P(x)$ $\therefore A^4-2A\times A^3+B=0$

ie $B=A^4$

(c) (i) (2 marks)
Outcomes assessed: E5
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards solution	1
• finds the tension (correct numerical equivalence)	1

Sample Answer:

2 revolutions per second $\therefore\ \omega=4\pi$ rad/sec

$T=mr\omega^2$

$=5\times 0\cdot 6\times 16\pi^2$

$=48\pi^2 N$

$\approx 474N$

(c) (ii) (2 marks)
Outcomes assessed: E5
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards solution	1
• finds the correct answer (correct numerical equivalence)	1

Sample Answer:

$T<900$ ie $mr\omega^2<900$

$5\times 0\cdot 6\times\omega^2<900$

$\omega^2<300$

ie $0<\omega<10\sqrt{3}$

But $\omega=2\pi n$ where n is the number of revolutions per second

$\therefore n<\dfrac{10\sqrt{3}}{2\pi}\approx 2.7566$

ie approximately 2.8 revolutions per second

(d) (i) (1 mark)
Outcomes assessed: E2
Targeted Performance Bands: E2-E3

Criteria	Marks
• shows correct result	1

Sample Answer:

$RHS=\sin(A+B)-\sin(A-B)$

$=\sin A\cos B+\cos A\sin B-(\sin A\cos B-\cos A\sin B)$

$=2\cos A\sin B$

(d) (ii) (4 marks)
Outcomes assessed: E7
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct expression for area or other progress towards solution	1
• establishes correct expression for volume or significant progress towards solution	1
• further progress towards solution	1
• finds the volume (correct numerical equivalence)	1

Sample Answer:

Cross-section is an equilateral triangle with side $\sin\dfrac{x}{2}+\cos x$

$A(x)=\dfrac{1}{2}\left(\sin\dfrac{x}{2}+\cos x\right)\left(\sin\dfrac{x}{2}+\cos x\right)\sin 60^\circ$

$=\dfrac{\sqrt{3}}{4}\left(\sin\dfrac{x}{2}+\cos x\right)^2$

$=\dfrac{\sqrt{3}}{4}\left(\sin^2\dfrac{x}{2}+2\sin\dfrac{x}{2}\cos x+\cos^2 x\right)$

$V=\lim\limits_{\Delta x\to 0}\sum\limits_{x=0}^{\pi}\dfrac{\sqrt{3}}{4}\left(\sin^2\dfrac{x}{2}+2\sin\dfrac{x}{2}\cos x+\cos^2 x\right)\Delta x$

$=\dfrac{\sqrt{3}}{4}\displaystyle\int_0^{\pi}\left(\sin^2\dfrac{x}{2}+2\sin\dfrac{x}{2}\cos x+\cos^2 x\right)dx$

$=\dfrac{\sqrt{3}}{4}\displaystyle\int_0^{\pi}\left(\dfrac{1}{2}(1-\cos x)+\left(\sin\dfrac{3x}{2}-\sin\dfrac{x}{2}\right)+\dfrac{1}{2}(1+\cos 2x)\right)dx$

$=\dfrac{\sqrt{3}}{8}\left[x-\sin x-\dfrac{4}{3}\cos\dfrac{3x}{2}+4\cos\dfrac{x}{2}+x+\dfrac{\sin 2x}{2}\right]_0^{\pi}$

$=\dfrac{\sqrt{3}}{8}\left(\left(\pi-\sin\pi-\dfrac{4}{3}\cos\dfrac{3\pi}{2}+4\cos\dfrac{\pi}{2}+\pi+\dfrac{\sin 2\pi}{2}\right)-\left(0-\sin 0-\dfrac{4}{3}\cos 0+4\cos 0+0+\dfrac{\sin 0}{2}\right)\right)$

$=\dfrac{\sqrt{3}}{8}\left(2\pi-\dfrac{8}{3}\right)$

$=\dfrac{\sqrt{3}\,(3\pi-4)}{12}$ cubic units

Question 5 (15 marks)

(a) (3 marks)

Outcomes assessed: E2

Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes correct relationship or progress towards general solution	1
• finds simplified general solutions or further progress towards solutions	1
• finds the solutions	1

Sample Answer:

$$\sin x = \cos 5x \quad \Rightarrow \quad \cos\left(\frac{\pi}{2} - x\right) = \cos 5x$$

$$\therefore 5x = 2n\pi \pm \left(\frac{\pi}{2} - x\right)$$

$$\text{ie } 6x = 2n\pi + \frac{\pi}{2} \quad \text{or} \quad 4x = 2n\pi - \frac{\pi}{2}$$

$$x = \frac{(4n+1)\pi}{12} \quad \text{or} \quad x = \frac{(4n-1)\pi}{8}$$

$$\text{For } 0 \le \theta \le \pi;\ x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{3\pi}{8}, \frac{7\pi}{8}$$

(b) (i) (2 marks)

Outcomes assessed: E4

Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards result	1
• gives correct result	1

Sample Answer:

$$\cot 4\theta = \frac{\cos 4\theta}{\sin 4\theta}$$

$$= \frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta} \quad \text{divide by } \sin^4\theta$$

$$= \frac{\dfrac{\cos^4\theta}{\sin^4\theta} - \dfrac{6\cos^2\theta\sin^2\theta}{\sin^4\theta} + \dfrac{\sin^4\theta}{\sin^4\theta}}{\dfrac{4\sin\theta\cos^3\theta}{\sin^4\theta} - \dfrac{4\sin^3\theta\cos\theta}{\sin^4\theta}}$$

$$= \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$$

(b) (ii) (3 marks)

Outcomes assessed: E4

Targeted Performance Bands: E2-E3

Criteria	Marks
• equating to zero and solving or other progress towards result	1
• further progress towards result using (i)	1
• deduces correct result	1

Sample Answer:

$$\text{Solving } \cot 4\theta = 0 \text{ gives } 4\theta = \frac{\pi}{2} + k\pi,\ k = 0, \pm 1, \pm 2, \ldots$$

$$\therefore \theta = \frac{\pi}{8}, \frac{3\pi}{8}, \ldots$$

$$\text{If } \cot 4\theta = 0 \Rightarrow \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta} = 0$$

$$\text{ie } \cot^4\theta - 6\cot^2\theta + 1 = 0$$

$$\text{Let } x = \cot^2\theta \text{ to give } x^2 - 6x + 1 = 0$$

$$\text{ie } x = \cot^2\left(\frac{\pi}{8}\right) \text{ is a solution of } x^2 - 6x + 1 = 0$$

(b) (iii) (2 marks)

Outcomes assessed: E4

Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards result using (ii)	1
• deduces correct result	1

Sample Answer:

From (ii) the solutions to $x^2 - 6x + 1 = 0$ are $x = \cot^2\left(\frac{\pi}{8}\right)$ and $x = \cot^2\left(\frac{3\pi}{8}\right)$

$$\text{Sum of roots: } \cot^2\left(\frac{\pi}{8}\right) + \cot^2\left(\frac{3\pi}{8}\right) = 6$$

$$\text{ie } \operatorname{cosec}^2\left(\frac{\pi}{8}\right) - 1 + \operatorname{cosec}^2\left(\frac{3\pi}{8}\right) - 1 = 6$$

$$\therefore \operatorname{cosec}^2\left(\frac{\pi}{8}\right) + \operatorname{cosec}^2\left(\frac{3\pi}{8}\right) = 8$$

(c) (i) (2 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3-E4

Criteria	Marks
• correctly applies integration by parts	1
• shows the correct result	1

Sample Answer:

$$I_n = \int_1^2 (\ln x)^n\, dx \qquad u = (\ln x)^n \qquad \frac{dv}{dx} = 1$$

$$= \left[x(\ln x)^n\right]_1^2 - n\int_1^2 (\ln x)^{n-1}\, dx \qquad \frac{du}{dx} = \frac{n(\ln x)^{n-1}}{x} \qquad v = x$$

$$= \left(2(\ln 2)^n - (\ln 1)^n\right) - nI_{n-1}$$

$$= 2(\ln 2)^n - nI_{n-1}$$

(c) (ii) (3 marks)
Outcomes assessed: E8
Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards result using (i)	1
• evaluates one integral correctly	1
• gives correct answer	1

Sample Answer:

$$I_4 = \int_1^2 (\ln x)^4\, dx = 2(\ln 2)^4 - 4I_3$$

$$I_3 = 2(\ln 2)^3 - 3I_2$$

$$I_2 = 2(\ln 2)^2 - 2I_1$$

$$I_1 = 2(\ln 2) - I_0$$

$$I_0 = \int_1^2 (\ln x)^0\, dx$$

$$= \int_1^2 1\, dx$$

$$= [x]_1^2$$

$$= 1$$

$$\therefore I_1 = 2(\ln 2) - 1$$

$$I_2 = 2(\ln 2)^2 - 2(2(\ln 2) - 1)$$

$$I_3 = 2(\ln 2)^3 - 3\left(2(\ln 2)^2 - 2(2(\ln 2) - 1)\right)$$

$$I_4 = 2(\ln 2)^4 - 4\left(2(\ln 2)^3 - 3\left(2(\ln 2)^2 - 2(2(\ln 2) - 1)\right)\right)$$

$$= 2(\ln 2)^4 - 8(\ln 2)^3 + 24(\ln 2)^2 - 48\ln 2 + 24$$

Question 6 (15marks)
(a) (i) (2 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds correct gradient or progress towards result	1
• shows correct result	1

Sample Answer:

$$x^2 - y^2 = 1$$

$$2x - 2y \times \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{x}{y}$$

$$= \frac{\sec\theta}{\tan\theta}$$

equation of tangent: $y - \tan\theta = \dfrac{\sec\theta}{\tan\theta}(x - \sec\theta)$

$$y\tan\theta - \tan^2\theta = x\sec\theta - \sec^2\theta$$

$$x\sec\theta - y\tan\theta = \left(\sec^2\theta - \tan^2\theta\right) = 1$$

$$\therefore x\sec\theta - y\tan\theta = 1$$

(a) (ii) (3 marks)
Outcomes assessed: E3, E4
Targeted Performance Bands: E3-E4

Criteria	Marks
• solves equations simultaneously or significant progress towards solution	1
• shows correct coordinates or similar progress towards result	1
• uses midpoint (or other method) to show result	1

Sample Answer:

asymptotes are $y = \pm x$

For L on $y = x$, $x\sec\theta - x\tan\theta = 1$

ie $x = \dfrac{1}{\sec\theta - \tan\theta}$ and $y = \dfrac{1}{\sec\theta - \tan\theta}$

$$L = \left(\frac{1}{\sec\theta - \tan\theta}, \frac{1}{\sec\theta - \tan\theta}\right)$$

Similarly for M on $y = -x$, $x\sec\theta + x\tan\theta = 1$

ie $x = \dfrac{1}{\sec\theta + \tan\theta}$ and $y = \dfrac{-1}{\sec\theta + \tan\theta}$

$$M = \left(\frac{1}{\sec\theta + \tan\theta}, \frac{-1}{\sec\theta + \tan\theta}\right)$$

$$\text{midpoint } x_{LM} = \frac{\frac{1}{\sec\theta - \tan\theta} + \frac{1}{\sec\theta + \tan\theta}}{2}$$

$$= \frac{\sec\theta + \tan\theta + \sec\theta - \tan\theta}{2(\sec^2\theta - \tan^2\theta)}$$

$$= \frac{2\sec\theta}{2}$$

$= \sec\theta$ which is the x-coordinate of P

$$\text{midpoint } y_{LM} = \frac{\frac{1}{\sec\theta - \tan\theta} - \frac{1}{\sec\theta + \tan\theta}}{2}$$

$$= \frac{\sec\theta + \tan\theta - \sec\theta + \tan\theta}{2(\sec^2\theta - \tan^2\theta)}$$

$= \tan\theta$ which is the y-coordinate of P

P is the midpoint of LM, $\therefore\ LP = PM$

(a) (iii) (3 marks)

Outcomes assessed: E3, E4

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes expression for one length or other similar progress	1
• establish expression for both lengths or other similar progress	1
• gives correct conclusion	1

Sample Answer:

Area $\Delta OLM = \frac{1}{2} \times OL \times OM$ since ΔOLM is right-angled

$$OL = \sqrt{\left(\frac{1}{\sec\theta - \tan\theta}\right)^2 + \left(\frac{1}{\sec\theta - \tan\theta}\right)^2}$$

$$= \frac{\sqrt{2}}{\sec\theta - \tan\theta}$$

$$OM = \sqrt{\left(\frac{1}{\sec\theta + \tan\theta}\right)^2 + \left(\frac{-1}{\sec\theta + \tan\theta}\right)^2}$$

$$= \frac{\sqrt{2}}{\sec\theta + \tan\theta}$$

$$\therefore A = \frac{1}{2} \times \frac{\sqrt{2}}{\sec\theta - \tan\theta} \times \frac{\sqrt{2}}{\sec\theta + \tan\theta}$$

$$= \frac{1}{\sec^2\theta - \tan^2\theta}$$

$= 1$ which is independent of θ

(b) (i) (1 mark)

Outcomes assessed: E5, E9

Targeted Performance Bands: E2-E3

Criteria	Marks
• gives appropriate explanation	1

Sample Answer:

$F = m\ddot{x}$ and the forces acting on the particle are acceleration due to gravity downwards and resistance upwards where resistance is proportional to velocity, ie $-kv$

$\therefore m\ddot{x} = mg - mkv$ ie $\ddot{x} = g - kv$

(b) (ii) (2 marks)

Outcomes assessed: E5, E9

Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards solution using integration	1
• shows correct result	1

Sample Answer:

$$\ddot{x} = g - kv \text{ ie } \frac{dv}{dt} = g - kv$$

$$\frac{dt}{dv} = \frac{1}{g - kv}$$

$$t = \int \frac{1}{g - kv}\, dv$$

$$= -\frac{1}{k}\ln(g - kv) + C$$

When $t = 0, v = 0 \Rightarrow\ 0 = -\frac{1}{k}\ln(g) + C$ ie $C = \frac{1}{k}\ln g$

$$t = -\frac{1}{k}[\ln(g - kv) - \ln(g)]$$

$$= -\frac{1}{k}\ln\left(\frac{g - kv}{g}\right)$$

$$-kt = \ln\left(\frac{g - kv}{g}\right)$$

$$\frac{g - kv}{g} = e^{-kt}$$

$$1 - \frac{kv}{g} = e^{-kt}$$

$$\frac{kv}{g} = 1 - e^{-kt}$$

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

(b) (iii) (1 mark)
Outcomes assessed: E5, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• gives correct answer	1

Sample Answer:

As $t \to \infty$, $e^{-kt} \to 0$

$\therefore v = \frac{g}{k}$ ms^{-1} is the terminal velocity

(b) (iv) (3 marks)
Outcomes assessed: E5
Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards solution using correct relationship	1
• significant towards solution	1
• finds a correct expression	1

Sample Answer:

$$\ddot{x} = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = g - kv$$

$$\frac{dv}{dx} = \frac{g-kv}{v}$$

$$\frac{dx}{dv} = \frac{v}{g-kv}$$

$$x = \int \frac{v}{g-kv}\,dv$$

$$= -\frac{1}{k}\int\left(\frac{g-kv}{g-kv} - \frac{g}{g-kv}\right)dv$$

$$= -\frac{1}{k}\int\left(1 - \frac{g}{g-kv}\right)dv$$

$$= -\frac{v}{k} - \frac{g}{k^2}\ln(g-kv) + C$$

Initially $x = 0$ and $v = 0$

$$\therefore 0 = -\frac{g}{k^2}\ln(g) + C$$

$$C = \frac{g}{k^2}\ln(g)$$

$$\therefore x = -\frac{v}{k} - \frac{g}{k^2}\left(\ln(g-kv) - \ln(g)\right)$$

$$= -\frac{v}{k} + \frac{g}{k^2}\left(\ln\left(\frac{g}{g-kv}\right)\right) \quad \text{OR} \quad x = -\frac{v}{k} - \frac{g}{k^2}\left(\ln\left(\frac{g-kv}{g}\right)\right)$$

Question 7 (15 marks)
(a) (3 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards result finding one case	1
• significant progress towards result finding several cases	1
• gives correct answer (correct numerical equivalence)	1

Sample Answer:

all rings on one finger $\Rightarrow$ 4 possibilities; ${}^4C_1$

4 rings on one finger and 1 ring on another finger $\Rightarrow$ 12 possibilities; ${}^4C_1\,{}^3C_1$

3 rings on one finger and 2 rings on another finger $\Rightarrow$ 12 possibilities; ${}^4C_1\,{}^3C_1$

3 rings on one finger and 1 ring on each of two other fingers $\Rightarrow$ 12 possibilities; ${}^4C_1\,{}^3C_2$

2 rings on one, 2 rings on another and one on another finger $\Rightarrow$ 12 possibilities; ${}^4C_2\,{}^2C_1$

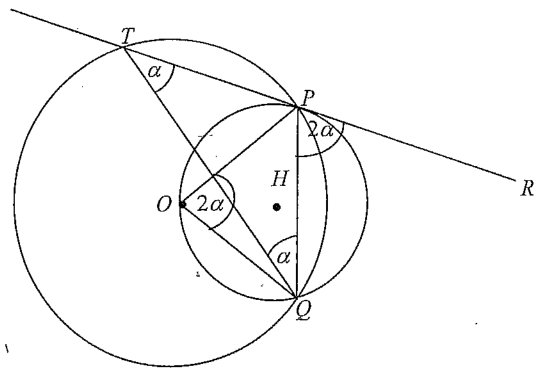
2 rings on one finger and 1 ring on each of the three other fingers $\Rightarrow$ 4 possibilities; ${}^4C_1\,{}^3C_3$

Total number of ways is 56.

(b) (i) (3 marks)
Outcomes assessed: E2
Targeted Performance Bands: E2-E3

Criteria	Marks
• a correct circle property for progress towards result	1
• a further circle property for progress towards result	1
• completes the proof	1

Sample Answer:



Construct PO and QO

$\angle POQ = 2\alpha$ (angle at the centre is twice the angle at the circumference; larger circle)

$\angle QPR = 2\alpha$ (angle between tangent and chord is equal to angle in alternate segment; smaller circle)

$\therefore \angle TQP = \alpha$ (exterior angle of ΔTQP is equal to the sum of the two interior opposite angles)

$\therefore \Delta TQP$ is isosceles (base angles equal)

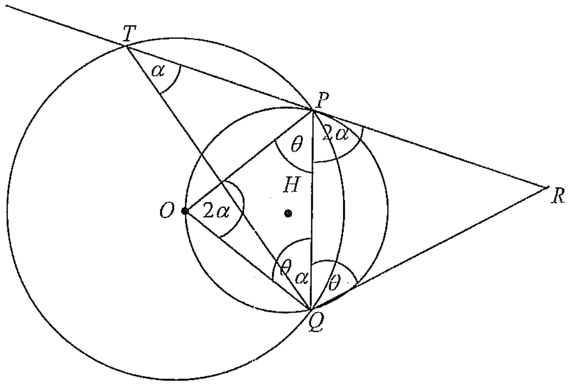
(b) (ii) (4 marks)

Outcomes assessed: E2

Targeted Performance Bands: E3-E4

Criteria	Marks
• identifying equal angles or similar progress towards result	1
• further progress towards result identifying relationships between angles	1
• significant progress towards result using angle properties	1
• proves correct result	1

Sample Answer:



Construct RQ and let $\angle RQP = \theta$

PQ bisects $\angle RQO$ $\therefore \angle PQO = \theta$

$\therefore \angle OPQ = \theta$ (ΔOPQ is isosceles, equal radii)

$2\alpha + 2\theta = 180^\circ$ (angle sum ΔOPQ)

In ΔPQR, $\angle PRQ = \theta$ given $2\alpha + 2\theta = 180^\circ$

Hence ΔPQR is isosceles (base angles equal) with $PR = PQ$

Also $PT = PQ$ (from (i))

Hence $PT = PR$ ie P is the midpoint of RT

(c) (i) (1 mark)

Outcomes assessed: E2, E9

Targeted Performance Bands: E2-E3

Criteria	Marks
• shows the correct result	1

Sample Answer:

Assume $\dfrac{a}{b} + \dfrac{b}{a} < 2$

$\therefore a^2 + b^2 < 2ab \Rightarrow a^2 - 2ab + b^2 < 0$

ie $(a-b)^2 < 0$ which is not true

$\therefore \dfrac{a}{b} + \dfrac{b}{a} \geq 2$

(c) (ii) (1 mark)

Outcomes assessed: E2, E9

Targeted Performance Bands: E3-E4

Criteria	Marks
• applies result from (i) to show expression	1

Sample Answer:

$$\text{LHS} = (a+b)\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$= a \times \frac{1}{a} + a \times \frac{1}{b} + b \times \frac{1}{a} + b \times \frac{1}{b}$$

$$= 2 + \frac{a}{b} + \frac{b}{a}$$

$$\geq 4 \text{ using (i)}$$

(c) (iii) (3 marks)

Outcomes assessed: E2, E9

Targeted Performance Bands: E3-E4

Criteria	Marks
• applies the result from (i) correctly	1
• shows progress toward the final result	1
• shows correct result	1

Sample Answer:

Using (i) three times: $\dfrac{a}{b} + \dfrac{b}{a} + \dfrac{b}{c} + \dfrac{c}{b} + \dfrac{c}{a} + \dfrac{a}{c} \geq 2 + 2 + 2$

$$\therefore \frac{a+c}{b} + \frac{b+c}{a} + \frac{a+b}{c} \geq 6$$

$$1 + \frac{a+c}{b} + 1 + \frac{b+c}{a} + 1 + \frac{a+b}{c} \geq 9$$

$$\frac{a+b+c}{b} + \frac{a+b+c}{a} + \frac{a+b+c}{c} \geq 9$$

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$$

Question 8 (15 marks)
(a) (i) (3 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes the truth of $S(1)$	1
• establishes the correct relationship between $S(k)$ and $S(k+1)$	1
• deduces the required result	1

Sample Answer:

Let $S(n)$ be the statement $f^{(n)}(x)=2^n e^{2x}$ for $n>0$

Consider $S(1)$: If $f(x)=e^{2x}$ then $f'(x)=2e^{2x}$

ie $f^{(1)}(x)=2^1 e^{2x}$

Hence $S(1)$ is true

Assume $S(k)$ is true: $f^{(k)}(x)=2^k e^{2x}$ *

RTP: $S(k+1)$ is true, ie prove $f^{(k+1)}(x)=2^{k+1}e^{2x}$

$f^{(k)}(x)=2^k e^{2x}$ if $S(k)$ is true using *

$\therefore f^{(k+1)}(x)=2^k\times 2e^{2x}$

$=2^{k+1}e^{2x}$

Hence if $S(k)$ is true then $S(k+1)$ is also true. Thus since $S(1)$ is true it follows by induction that $S(n)$ is true for positive integers $n>0$.

(a) (ii) (2 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses (i) for progress towards result	1
• derives power series	1

Sample Answer:

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots$$

$f(x)=e^{2x}$ and from (i) $f'(x)=2e^{2x}\Rightarrow f''(x)=2^2e^{2x}\Rightarrow f'''(x)=2^3e^{2x}\Rightarrow f^{(4)}(x)=2^4e^{2x}$ etc

$$\therefore e^{2x}=e^0+2xe^0+\frac{x^2}{2!}2^2e^0+\frac{x^3}{3!}2^3e^0+\frac{x^4}{4!}2^4e^0+\dots$$

$$\text{ie } e^{2x}=1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\frac{16x^4}{4!}+\dots$$

(a) (iii) (2 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards result using (ii)	1
• finds correct limit	1

Sample Answer:

$$\frac{1+2x+2x^2-e^{2x}}{\tan x-x}=\frac{1+2x+2x^2-\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\frac{16x^4}{4!}+\dots\right)}{x+\frac{x^3}{3}+\frac{2x^5}{15}+\dots-x}$$

$$=\frac{-\frac{8x^3}{6}-\frac{16x^4}{24}+\dots}{\frac{x^3}{3}+\frac{2x^5}{15}+\dots}$$

$$=\frac{x^3\left(-\frac{4}{3}-\frac{2x}{3}+\dots\right)}{x^3\left(\frac{1}{3}+\frac{2x^2}{5}+\dots\right)}$$

$$=\frac{-\frac{4}{3}-\frac{2x}{3}+\dots}{\frac{1}{3}+\frac{2x^2}{5}+\dots}$$

$$\therefore \lim_{x\to 0}\frac{1+2x+2x^2-e^{2x}}{\tan x-x}=\frac{-\frac{4}{3}}{\frac{1}{3}}=-4$$

(b) (i) (2 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• some progress towards result using expansions	1
• shows the result	1

Sample Answer:

$$(2k+3)^2-4(k+1)(k+2)=4k^2+12k+9-4k^2-12k-8$$
$$=1$$

$\therefore (2k+3)^2>4(k+1)(k+2)$

For $k\ge 0$, $2k+3>0$ and $(k+1)(k+2)>0$

$\therefore 2k+3>2\sqrt{(k+1)(k+2)}$

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(b) (ii) (4 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• some progress towards solution using (i) or by mathematical induction	1
• further progress towards establishing relationship or next induction steps	1
• significant progress towards solution using relationship or further induction steps using (i)	1
• derives the correct solution	1

Sample Answer:

From (i) $2(k+1)+1>2\sqrt{(k+1)(k+2)}$

Dividing by $\sqrt{k+1}$ gives $2\sqrt{k+1}+\frac{1}{\sqrt{k+1}}>2\sqrt{k+2}$

$\therefore \frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+2}-\sqrt{k+1}\right)$ (*)

Consider $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}=\sum_{k=0}^{n-1}\frac{1}{\sqrt{k+1}}$

Hence using (*) gives $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}>2\sum_{k=0}^{n-1}\left(\sqrt{k+2}-\sqrt{k+1}\right)$

$=2\left(\sqrt{2}-1+\sqrt{3}-\sqrt{2}+\ldots+\sqrt{n+1}-\sqrt{n}\right)$

$=2\left(\sqrt{n+1}-1\right)$

OR

Proof by mathematical induction

Let $S(n)$ be the statement $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-1\right)$

Consider $S(1): 1>2\left(\sqrt{2}-1\right) \Rightarrow 3>2\sqrt{2}$ which is true $(9>8)$

Hence $S(1)$ is true.

Assume $S(k)$ is true: $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}>2\left(\sqrt{k+1}-1\right)$ (*)

RTP: $S(k+1)$ is true ie prove $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+2}-1\right)$

$LHS=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$

$>2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}}$ if S(k) is true using (*)

$=\frac{2\left(\sqrt{k+1}-1\right)\sqrt{k+1}+1}{\sqrt{k+1}}$

$=\frac{2(k+1)-2\sqrt{k+1}+1}{\sqrt{k+1}}$

$=\frac{2k+3-2\sqrt{k+1}}{\sqrt{k+1}}$

$>\frac{2\sqrt{(k+1)(k+2)}-2\sqrt{k+1}}{\sqrt{k+1}}$ using (i)

$=\frac{2\sqrt{k+1}\left(\sqrt{k+2}-1\right)}{\sqrt{k+1}}$

$=2\left(\sqrt{k+2}-1\right)$ as required

Hence if $S(k)$ is true then $S(k+1)$ is also true. Thus since $S(1)$ is true it follows by induction that $S(n)$ is true for positive integral n.

(b) (iii) (2 marks)
Outcomes assessed: E2, E9
Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards result by establishing contradiction or other method	1
• gives correct conclusion	1

Sample Answer:

Consider $\sum_{k=1}^{N}\frac{1}{\sqrt{k}}>10^{10}$

If $2\left(\sqrt{N+1}-1\right)>10^{10}$

$\sqrt{N+1}>\frac{10^{10}}{2}+1$

$N>\left(\frac{10^{10}}{2}+1\right)^2-1$ (**)

From (ii) $\sum_{k=1}^{N}\frac{1}{\sqrt{k}}>2\left(\sqrt{N+1}-1\right)>10^{10}$ for any integer satisfying (**)

$\therefore$ The result $\sum_{k=1}^{N}\frac{1}{\sqrt{k}}<10^{10}$ is not valid.