Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2010
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 1

Afternoon Session
Thursday 12 August 2010

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided as a separate page
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

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Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) A is the point $(-2,-1)$ and B is the point $(1,5)$. Find the coordinates of the point Q which divides AB externally in the ratio $5:2$. **2**

(b) Show that $\dfrac{\sin 2x}{1+\cos 2x}=\tan x$. **2**

(c) Solve the inequality $\dfrac{2x}{x-1}\geq 1$. **3**

(d) Evaluate $\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right)$ exactly. **2**

(e) Using the substitution $u=\ln 3x$, find $\displaystyle\int\frac{dx}{x(\ln 3x)^2}$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Ten students are seated around a circular table.

(i) In how many ways can they be arranged? **1**

(ii) What is the probability that three particular students, Gemma, Pasha and Ricky, are not sitting together when the seats are randomly assigned. **2**

(b) A ball in the shape of a sphere has radius r centimetres at time t seconds. The surface area is changing as the radius changes over time. At a particular time, t seconds, the rate of change of the surface area is equal to the rate of change of the radius. **2**

Find the exact radius at this time.

(c) Find an expression for y in terms of x if for $x > 0$ and $y > 0$, **3**

$\tan^{-1} x = \tan^{-1} y + \frac{\pi}{4}$.

(d) A heated metal bar has a temperature of 1340°C when it is removed from a furnace. Its temperature T after t minutes in a room with a constant temperature of 25°C satisfies the equation $\frac{dT}{dt} = -k(T-25)$, where k is a constant.

(i) Show that the equation $T = 25 + 1315e^{-kt}$ satisfies this information. **1**

(ii) The metal bar cools to 1010°C after 12 minutes. **3**

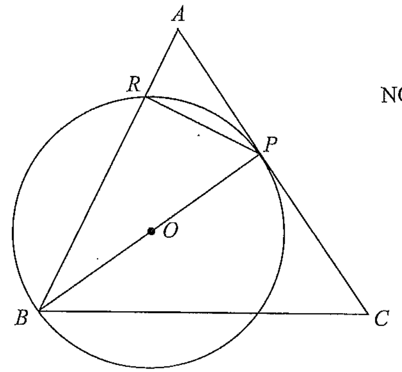
Find how long it will take for the bar to cool to 60°C, giving your answer correct to the nearest minute.

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) The polynomial $P(x)$ is expressed as $P(x) = (2x^2 + x + 3)Q(x) + (4x - 1)$. **2**

If $Q(x)$ leaves a remainder of 1 when divided by $(x+2)$, show that $(x+2)$ is a factor of $P(x)$.

(b) The diagram shows an isosceles triangle ABC, with $AB = BC$. The point P lies on AC and the point O lies on BP. A circle with centre O passes through B and P and cuts AB at R.



NOT TO SCALE

Copy or trace the diagram into your writing booklet.

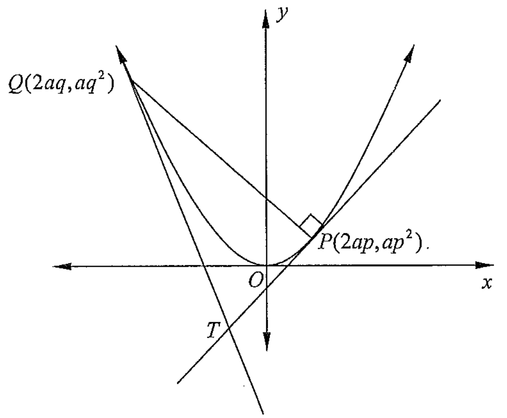
(i) Explain why $\angle RPA = \angle RBP$. **1**

(ii) Hence, or otherwise, prove that ΔBRP is similar to ΔBPC. **3**

Question 3 continues on page 5

Question 3 (continued)

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The equation of the tangent at the point P is $y = px - ap^2$ and the gradient of the chord PQ is $\frac{p+q}{2}$. The point T is the intersection of the tangents at P and Q.



NOT TO SCALE

(i) Show that the coordinates of T are $(a(p+q), apq)$. **2**

(ii) The chord PQ is also the normal at P. Show that $p + q + \frac{2}{p} = 0$. **2**

(iii) Hence, or otherwise, show that the equation of the locus of T as P moves on the parabola is $y = \frac{-4a^3}{x^2} - 2a$. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Harry and Bill are in a competition. Harry is the more skilful, having a probability of $\frac{2}{3}$ of winning any game against Bill. **3**

Find the probability that Harry wins 6 games to 4, taking into account that Harry wins the last game.

(b) (i) By considering the graph of $y = \sin^{-1} x$, or otherwise, show that the equation $\sin^{-1} x + x - \frac{\pi}{2} = 0$ has only one real and positive root. **2**

(ii) Taking $x = 0.7$ as the first approximation to this root, use one application of Newtown's method to find another approximation. Give your answer correct to 2 decimal places. **3**

(c) A series is given as $S_n = \tan^2 x - \tan^4 x + \tan^6 x - \ldots$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(i) Find the values of x for which this series has a limiting sum. **2**

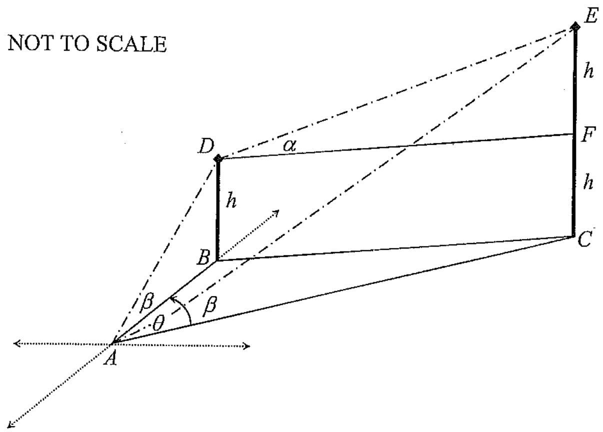
(ii) Express the limiting sum in simplest form. **2**

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) A man, standing on level ground at A, notices two vertical towers, BD and CE. The foot of tower BD, B, is due North of A and the foot of tower CE, C, is on a bearing of θ from A.

The height of BD is h metres and the height of CE is twice the height of BD.

The angle of elevation from A to the top of both towers is β. The angle of elevation to the top of CE from the top of BD is α.



(i) Show that $AC = 2h\cot\beta$. **1**

(ii) Find similar expressions for AB and BC. **2**

(iii) Use the cosine rule, or otherwise, to show that **2**

$$\cos\theta = \frac{5\cot^2\beta - \cot^2\alpha}{4\cot^2\beta}.$$

Question 5 continues on page 8

Question 5 (continued)

(b) Find the range of values of b for which the seventh term in the expansion of $(2+bx)^{11}$ has the largest coefficient. **3**

(c) A particle is moving in simple harmonic motion in a straight line between $x = a$ and $x = -a$. Its acceleration is given by $\ddot{x} = -n^2 x$, where x cm is its displacement from the origin at time $t \geq 0$ seconds and n is a constant.

(i) Show that the velocity of the particle, v, is given by $v^2 = n^2(a^2 - x^2)$. **2**

(ii) Find the extremities of the motion given that the particle has a velocity of $6\,\text{cms}^{-1}$ when $x = 4$ cm and its maximum velocity is $10\,\text{cms}^{-1}$. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that ${}^nC_k = {}^nC_{n-k}$. 1

(ii) Use the identity $(1+x)^n(1+x)^n \equiv (1+x)^{2n}$ to show that 2

$$\sum_{k=0}^{n}\left({}^nC_k\right)^2 = \frac{(2n)!}{(n!)^2}$$

(b) A function is defined as $f(x) = x^3 + x + 1$.

(i) Show that $f(x)$ has an inverse function, $f^{-1}(x)$, for all x. 1

(ii) Find the point of intersection of $f(x)$ and $f^{-1}(x)$. 2

Question 6 continues on page 10

(c) The position coordinates of any point on the path of a projectile at time $t \geq 0$, in seconds, with initial velocity v ms^{-1} at an angle of projection θ, and acceleration downwards due to gravity, g, are:

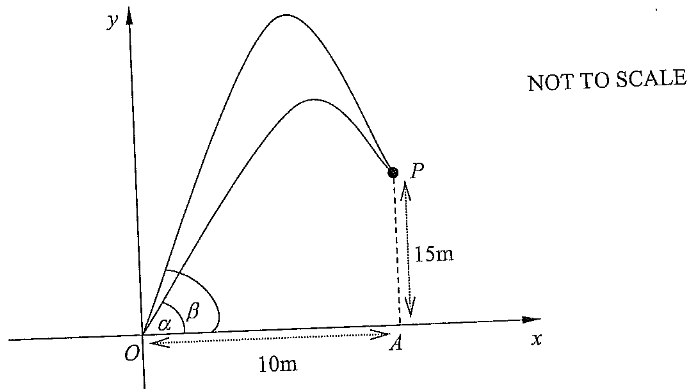
$$x = vt\cos\theta \text{ and } y = vt\sin\theta - \frac{1}{2}gt^2$$

(i) Show that the equation of the path of a projectile is given by 2

$$y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta.$$

Nicholas throws a small pebble from a fixed point O on level ground, with a velocity $v = 7\sqrt{10}$ ms^{-1} at an angle β with the horizontal. Shortly afterwards he throws another small pebble from the same point at the same speed but at a different angle to the horizontal, α, where $\alpha < \beta$ as shown.

The pebbles collide at a point P, vertically above the point A on the ground, where $OA = 10$ m and $AP = 15$ m. The acceleration downwards due to gravity is $g = 9.8$ ms^{-2}.



(ii) Show that $\tan\alpha = 2$ and $\tan\beta = 8$. 2

(iii) Show that the time elapsed between when the pebbles were thrown was $\dfrac{\sqrt{650}-\sqrt{50}}{7}$ seconds. 2

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Use Mathematical Induction to prove that $2n^2 > n^2 + n + 1$ for positive integers $n > 1$. **3**

(b) By definition $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ and $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$.

(i) Show that $2\sinh x \cosh x = \sinh(2x)$. **2**

(ii) Show that the equation $p\cosh x + q\sinh x = r$ can be written as $(p+q)e^{2x} - 2re^x + (p-q) = 0$ where p, q and r are constants. **2**

(iii) The constants p, q and r are all positive and $p^2 = q^2 + r^2$. **3**

Show that the equation $p\cosh x + q\sinh x = r$ has only one solution.

(iv) Solve the equation $13\cosh x + 5\sinh x = 12$. **2**

Give your answer in the form $\ln k$, where k is rational.

End of paper

CATHOLIC SECONDARY SCHOOLS ASSOCIATION

2010 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 1

Question 1 (12 marks)
(a) (2 marks)
Outcomes assessed: PE2
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses correct formula for division of interval or progress using other correct method	1
• finds correct coordinates from working	1

Sample Answer:

$A(x_1, y_1) = (-2, -1)$ and $B(x_2, y_2) = (1, 5)$; Q divides AB externally ie $m:n = 5:-2$

$$x_Q = \frac{-2\times -2+5\times 1}{5+(-2)} = \frac{9}{3} = 3 \qquad y_Q = \frac{-2\times -1+5\times 5}{5+(-2)} = \frac{27}{3} = 9$$

$\therefore Q$ has coordinates $(3, 9)$

(b) (2 marks)
Outcomes assessed: PE2
Targeted Performance Bands: E2-E3

Criteria	Marks
• correct trigonometric substitution	1
• completes the proof	1

Sample Answer:

$$\begin{aligned}\text{LHS} &= \frac{\sin 2x}{1+\cos 2x}\\ &= \frac{2\sin x\cos x}{1+2\cos^2 x-1}\\ &= \frac{2\sin x\cos x}{2\cos^2 x}\\ &= \frac{\sin x}{\cos x}\\ &= \tan x\\ &= \text{RHS}\end{aligned}$$

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(c) (3 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes correct quadratic or other correct significant step towards solution	1
• further progress towards solution	1
• finds correct solution	1

Sample Answer:

$$\frac{2x}{x-1} \ge 1 \quad \text{multiply by } (x-1)^2 \text{ with } x \ne 1$$

$$2x(x-1) \ge (x-1)^2$$

$$2x(x-1)-(x-1)^2 \ge 0$$

$$(x-1)(2x-(x-1)) \ge 0$$

$$(x-1)(x+1) \ge 0$$

Solution is $x \le -1$ or $x > 1$

(d) (2 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E2-E3

Criteria	Marks
• gives correct exact trigonometric value	1
• correctly evaluates exact inverse trigonometric value	1

Sample Answer:

$$\sin^{-1}\left(\sin\frac{7\pi}{6}\right) = \sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$$

(e) (3 marks)
Outcomes assessed: HE6
Targeted Performance Bands: E2-E3

Criteria	Mark
• rewrites the integral using the substitution	1
• finds the correct primitive	1
• gives final result	1

Sample Answer:

$$\int \frac{dx}{x(\ln 3x)^2} = \int \frac{du}{u^2} \qquad u = \ln 3x$$

$$= -\frac{1}{u} + C \qquad \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$$

$$= \frac{-1}{\ln 3x} + C$$

Question 2 (12 marks)
(a) (i) (1 mark)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• gives correct result (correct numerical equivalence)	1

Sample Answer:

$$(n-1)! = 9!$$
$$= 362880$$

(a) (ii) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards result	1
• gives correct result (correct numerical equivalence)	1

Sample Answer:

Number of arrangements without restrictions 9!
Number of arrangements if Gemma, Pasha and Ricky sit together is $3! \times 7!$
If Gemma, Pasha and Ricky sit separately then:
P(all 3 separate) = 1 − P(all together)

$$= 1 - \frac{3! \times 7!}{9!}$$
$$= 1 - \frac{6}{9 \times 8}$$
$$= \frac{11}{12}$$

(b) (2 marks)
Outcomes assessed: PE5, HE7
Targeted Performance Bands: E3-E4

Criteria	Mark
• establishes correct differential relationship or progress toward result	1
• finds correct radius from working	1

Sample Answer

$$A = 4\pi r^2 \quad \Rightarrow \frac{dA}{dr} = 8\pi r$$
$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$
$$= 8\pi r \frac{dr}{dt} \quad \text{but } \frac{dA}{dt} = \frac{dr}{dt}$$
$$\therefore 1 = 8\pi r$$
$$r = \frac{1}{8\pi} \text{ cm}$$

(d) (ii) (3 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses information to establish value for k or other progress towards solution	1
• further progress towards solution	1
• finds the correct time	1

Sample Answer:

When $t = 12$, $T = 1010$ ie $1010 = 25 + 1315e^{-12k}$

$$985 = 1315e^{-12k}$$
$$\frac{985}{1315} = e^{-12k}$$
$$-12k = \ln\left(\frac{197}{263}\right)$$
$$k = \frac{-1}{12}\ln\left(\frac{197}{263}\right)$$
$$= 0.024079...$$

When $T = 60$; $60 = 25 + 1315e^{-kt}$

$$35 = 1315e^{-0.024t}$$
$$\frac{35}{1315} = e^{-0.024t}$$
$$-0.024t = \ln\left(\frac{7}{263}\right)$$
$$t = \frac{-1}{0.024}\ln\left(\frac{7}{263}\right)$$
$$= 150.5965769... \qquad \text{OR } t = 151.09349... \text{ if using } k = 0.024$$
$$\therefore t = 151 \text{ minutes}$$

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Question 3 (12 marks)
(a) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses remainder theorem or other progress towards solution	1
• establishes correct conclusion	1

Sample Answer:

$P(x) = (2x^2 + x + 3)Q(x) + (4x - 1)$

$Q(x)$ has remainder 1 when divided by $(x+2)$

ie $Q(-2) = 1$

$\therefore\ P(-2) = \left(2\times(-2)^2 + (-2) + 3\right)\times 1 + \left(4\times(-2) - 1\right)$

$= (9) + (-9)$

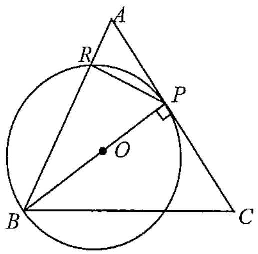
$= 0$

Since $P(-2) = 0$ by the Factor Theorem $(x+2)$ is a factor of $P(x)$.

(b) (i) (1 mark)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• applies theorem correctly	1

Sample Answer:



AC is a tangent to the circle with diameter BP.
$\therefore\ \angle RPA = \angle RBP$ (angle between tangent and chord is equal to the angle in the alternate segment)

(b) (ii) (3 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly identifies one pair of angles	1
• correctly identifies second pair of angles or other progress towards the proof	1
• completes the proof	1

Sample Answer:

In ΔBRP and ΔBPC

$\angle BPC = 90^\circ$ (tangent is perpendicular to the radius drawn from the point of contact)

$\angle BRP = 90^\circ$ (angle in a semi-circle is a right angle)

$\therefore\ \angle BRP = \angle BPC$

$\angle RBP = \angle PBC$ (PB bisects $\angle RBC$ given ΔABC is isosceles and $BP \perp AC$)

$\therefore\ \Delta BRP$ is similar to ΔBPC (equiangular)

(c) (i) (2 marks)
Outcomes assessed: PE3, PE4
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards finding the coordinates	1
• derives correct coordinates	1

Sample Answer:

Equation of tangent at P is $y = px - ap^2$ and equation of tangent at Q is $y = qx - aq^2$

solve simultaneously $\quad (p-q)x = a(p^2 - q^2)$

$$x = \frac{a(p-q)(p+q)}{(p-q)}$$

$$x = a(p+q)$$

Substitute for x: $y = p\left(a(p+q)\right) - ap^2$

$y = apq$

$\therefore\ T$ is $\left(a(p+q), apq\right)$

(c) (ii) (2 marks)
Outcomes assessed: PE3, PE4
Targeted Performance Bands: E3-E4

Criteria	Marks
• states the gradient of the normal	1
• equates gradients to show the result	1

Sample Answer:

The gradient of the tangent at P is p $\therefore$ gradient of normal is $\dfrac{-1}{p}$.

The gradient of the chord PQ is $\dfrac{p+q}{2}$ $\therefore\ \dfrac{p+q}{2} = -\dfrac{1}{p}$

ie $p + q + \dfrac{2}{p} = 0$

(c) (iii) (2 marks)

Outcomes assessed: PE3, PE4

Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards result	1
• establishes the correct locus	1

Sample Answer:

At T $x=a(p+q)$, $y=apq$ and from (ii) $p+q=\frac{-2}{p}$

$$\therefore x=\frac{-2a}{p} \text{ ie } p=\frac{-2a}{x}$$

$$\text{also } q=\frac{x}{a}-p \text{ ie } q=\frac{x}{a}-\frac{-2a}{x}=\frac{x^2+2a^2}{ax}$$

$$y=a\times\frac{-2a}{x}\times\frac{x^2+2a^2}{ax}$$

$$=\frac{-2a^2x^2-4a^4}{ax^2}$$

$$\therefore y=\frac{-4a^3}{x^2}-2a$$

Question 4 (12 marks)

(a) (3 marks)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards solution	1
• further progress towards solution	1
• finds correct answer (correct numerical equivalence)	1

Sample Answer:

Total of 10 games – Harry wins 5 out of the first 9 and the last game

For the first 9 games consider the binomial probability of winning 5 from 9 with $p=\frac{2}{3}$

$$P(\text{winning } 5)={}^9C_5\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^5$$

$$=\frac{9!}{5!4!}\times\frac{1}{3^4}\times\frac{2^5}{3^5}$$

$$=\frac{9\times7\times2^6}{3^9}$$

$$=\frac{448}{2187}$$

Harry wins the last game $\therefore$ probability of winning 6 games to 4 is $\frac{448}{2187}\times\frac{2}{3}=\frac{896}{6561}$

(b) (i) (2 marks)

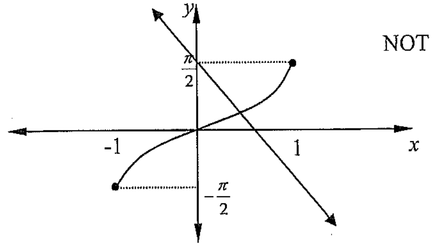
Outcomes assessed: PE2, HE7

Targeted Performance Bands: E2-E3

Criteria	Marks
• draws correct graph or other progress towards solution	1
• explains the conclusion	1

Sample Answer:

To solve $\sin^{-1}x+x-\frac{\pi}{2}=0$ consider the graphs of $y=\sin^{-1}x$ and $y=-x+\frac{\pi}{2}$



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There is only one point of intersection of the two graphs at a point where x is positive.

$\therefore$ $\sin^{-1}x+x-\frac{\pi}{2}=0$ has only one real positive root.

(b) (ii) (3 marks)

Outcomes assessed: PE3, HE7

Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards solution using Newton's Method	1
• further progress towards solution	1
• finds correct approximation (correct numerical equivalence)	1

Sample Answer:

$$f(x)=\sin^{-1}x+x-\frac{\pi}{2} \quad \therefore f'(x)=\frac{1}{\sqrt{1-x^2}}+1$$

For $x_1=0.7$ $\quad f(x_1)=\sin^{-1}0.7+0.7-\frac{\pi}{2}=-0.09539883...$

$$f'(x_1)=\frac{1}{\sqrt{1-0.7^2}}+1=2.40028008...$$

$$\therefore x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

$$=0.7-\frac{-0.09539883...}{2.40028008...}$$

$$=0.73974...$$

$$=0.74$$

(c) (i) (2 marks)
Outcomes assessed: PE2, PE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct relationship	1
• finds correct values (correct numerical equivalence)	1

Sample Answer:

Series is geometric with $r=-\tan^2 x,\ -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$

For a limiting sum $|r|<1$, ie consider $-1<-\tan^2 x<1$

If $-1<-\tan^2 x<1$ then $1>\tan^2 x>-1$, ie $-1<\tan^2 x<1$

Since $\tan^2 x\geq 0$, solve $0\leq \tan^2 x<1$ for x

$\tan x$ is an increasing function for $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$

and $\tan\left(\dfrac{-\pi}{4}\right)=-1,\ \tan(0)=0,\ \tan\left(\dfrac{\pi}{4}\right)=1$

ie for $-\dfrac{\pi}{4}<x<\dfrac{\pi}{4},\ 0\leq\tan^2 x<1$

$\therefore$ for a limiting sum $-\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

(c) (ii) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• applies correct formula	1
• correctly simplifies the expression	1

Sample Answer:

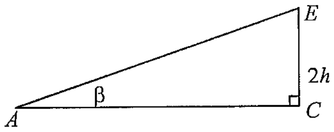
$$S_\infty=\frac{a}{1-r}$$
$$=\frac{\tan^2 x}{1-\left(-\tan^2 x\right)}$$
$$=\frac{\tan^2 x}{1+\tan^2 x}$$
$$=\frac{\tan^2 x}{\sec^2 x}$$
$$=\frac{\sin^2 x}{\cos^2 x}\times\cos^2 x$$
$$=\sin^2 x$$

(a) (i) (1 mark)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• derives correct result	1

Sample Answer:

In $\Delta ACE,\ \tan\beta=\dfrac{2h}{AC}\ \Rightarrow\ AC=2h\cot\beta$



(a) (ii) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• derives correct result for AB	1
• derives correct result for BC	1

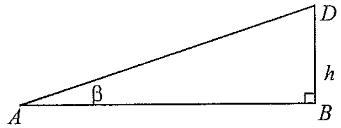
Sample Answer:

In $\Delta ABD,\ \tan\beta=\dfrac{h}{AB}\ \Rightarrow\ AB=h\cot\beta$

Also $BC=DF$

In $\Delta DEF,\ \tan\alpha=\dfrac{h}{DF}\ \Rightarrow\ DF=h\cot\alpha$

$\therefore BC=h\cot\alpha$



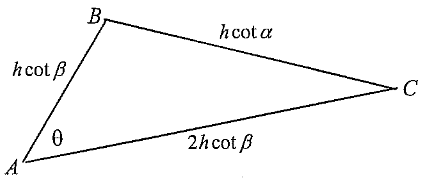
(a) (iii) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• applies the Cosine Rule to correct triangle	1
• shows correct result	1

Sample Answer:

In ΔABC

$$\cos\theta=\frac{AC^2+AB^2-BC^2}{2AC\times AB}$$
$$=\frac{4h^2\cot^2\beta+h^2\cot^2\beta-h^2\cot^2\alpha}{4h^2\cot^2\beta}$$
$$=\frac{5h^2\cot^2\beta-h^2\cot^2\alpha}{4h^2\cot^2\beta}$$
$$=\frac{h^2(5\cot^2\beta-\cot^2\alpha)}{4h^2\cot^2\beta}$$
$$=\frac{(5\cot^2\beta-\cot^2\alpha)}{4\cot^2\beta}$$



(b) (3 marks)
Outcomes assessed: PE2, HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct terms or coefficients for comparison or other progress towards result	1
• significant progress toward the result	1
• finds correct values (correct numerical equivalence)	1

Sample Answer:

Consider the 6^{th}, 7^{th} and 8^{th} terms of the expansion of $(2+bx)^{11}$

$T_6 = {}^{11}C_5 \times 2^6 \times (bx)^5$

$T_7 = {}^{11}C_6 \times 2^5 \times (bx)^6$

$T_8 = {}^{11}C_7 \times 2^4 \times (bx)^7$

Take coefficients of T_6 and T_8, and compare to T_7

Consider $T_7 > T_6$ ie ${}^{11}C_6 \times 2^5 \times b^6 > {}^{11}C_5 \times 2^6 \times b^5$

$$\therefore b > \frac{{}^{11}C_5 \times 2}{{}^{11}C_6} = 2$$

Similarly for $T_7 > T_8$ ie ${}^{11}C_6 \times 2^5 \times b^6 > {}^{11}C_7 \times 2^4 \times b^7$

$$\therefore b < \frac{{}^{11}C_6 \times 2}{{}^{11}C_7} = 2.8$$

$\therefore$ seventh term has the largest coefficient for $2 < b < 2.8$

(c) (i) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E2-E3

Criteria	Mark
• uses correct formula or progress using other correct method	1
• establishes correct result	1

Sample Answer:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2 x$$

$$\frac{1}{2}v^2 = \frac{-n^2x^2}{2} + C$$

at $x = a$, $v = 0$ since velocity is zero at the extremities

$$0 = \frac{-n^2a^2}{2} + C \Rightarrow C = \frac{n^2a^2}{2}$$

$$\frac{1}{2}v^2 = \frac{-n^2x^2}{2} + \frac{n^2a^2}{2} \quad \text{ie } v^2 = n^2a^2 - n^2x^2$$

$$\therefore v^2 = n^2(a^2 - x^2)$$

(c) (ii) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• uses correct substitutions or other progress towards result	1
• finds correct values	1

Sample Answer:

$v = 6$ when $x = 4 \quad \Rightarrow \quad 36 = n^2(a^2 - 16)$

maximum velocity is at the centre of the motion

$\therefore v = 10$ when $x = 0 \quad \Rightarrow \quad 100 = n^2a^2$

solving simultaneously $\Rightarrow \quad 36 = 100 - 16n^2$

$$16n^2 = 64$$

$$n^2 = 4$$

hence $a^2 = 25$

$\therefore$ extremities of motion are $a = 5$ and $a = -5$

Question 6 (12 marks)
(a) (i) (1 mark)
Outcomes assessed: HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly justifies the result	1

Sample Answer:

$${}^nC_k = \frac{n!}{k!(n-k)!} \qquad {}^nC_{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!}$$

$$\therefore {}^nC_k = {}^nC_{n-k}$$

(a) (ii) (2 marks)
Outcomes assessed: HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• consider terms in binomial expansion or other progress towards solution	1
• establishes the result	1

Sample Answer:

Consider terms in the expansion of $(1+x)^{2n}$

$$T_{k+1} = {}^{2n}C_k 1^{2n-k} x^k$$
$$= {}^{2n}C_k x^k$$

coefficient of x^n is: $^{2n}C_n = \dfrac{(2n)!}{n!(2n-n)!}$

$$= \frac{(2n)!}{(n!)^2}$$

Consider the expansion of $(1+x)^n(1+x)^n$

$$(1+x)^n(1+x)^n = \left[{}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n\right]\times\left[{}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n\right]$$

coefficients of x^n are: ${}^nC_0{}^nC_n + {}^nC_1{}^nC_{n-1} + {}^nC_2{}^nC_{n-2} + \ldots\ldots + {}^nC_{n-1}{}^nC_1 + {}^nC_n{}^nC_0$

$$= \left({}^nC_0\right)^2 + \left({}^nC_1\right)^2 + \left({}^nC_2\right)^2 + \ldots + \left({}^nC_{n-1}\right)^2 + \left({}^nC_n\right)^2 \quad \text{using (i)}$$

$$= \sum_{k=0}^{n}\left({}^nC_k\right)^2$$

$\therefore$ Since $(1+x)^n(1+x)^n = (1+x)^{2n}$ equating coefficients gives $\displaystyle\sum_{k=0}^{n}\left({}^nC_k\right)^2 = \frac{(2n)!}{(n!)^2}$

(b) (i) (1 mark)
Outcomes assessed: HE4
Targeted Performance Bands: E3-E4

Criteria	Marks
• differentiates or other method to explain correct conclusion	1

Sample Answer:

$f(x) = x^3 + x + 1$

$f'(x) = 3x^2 + 1 > 0$ for all x since $x^2 \ge 0$

$\therefore$ $f(x)$ is monotonic increasing and thus has an inverse function, $f^{-1}(x)$, for all x.

(b) (ii) (2 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E3-E4

Criteria	Marks
• identifies that curves intersect on $y = x$ or other progress towards result	1
• finds correct point of intersection	1

Sample Answer:

$f(x)$ and $f^{-1}(x)$ intersect on $y = x$

$\therefore$ solve $x^3 + x + 1 = x$

$x^3 + 1 = 0$

$x^3 = -1$

$x = -1$

$\therefore$ Point of intersection is $(-1, -1)$

(c) (i) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards result	1
• shows correct result	1

Sample Answer:

$x = vt\cos\theta$

$\therefore t = \dfrac{x}{v\cos\theta}$

substitute into $y = vt\sin\theta - \dfrac{1}{2}gt^2$

$$y = \frac{x}{v\cos\theta}(v\sin\theta) - \frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$$

$$= x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta$$

(c) (ii) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• progress toward solution	1
• substitutes and simplifies to obtain desired result	1

Sample Answer:

At the point P, $x=10$, $y=15$ and given $g=9.8$, $v=7\sqrt{10}$

Using (i) $\quad 15=10\times\tan\theta-\dfrac{9.8(10)^2}{2\left(7\sqrt{10}\right)^2}\sec^2\theta$

$$15=10\tan\theta-\frac{9.8\times100}{2\times49\times10}\left(1+\tan^2\theta\right)$$

$$15=10\tan\theta-1-\tan^2\theta$$

$$\tan^2\theta-10\tan\theta+16=0$$

$$(\tan\theta-8)(\tan\theta-2)=0$$

$$\therefore\tan\theta=8 \text{ or } \tan\theta=2$$

since $\alpha<\beta$, $\tan\beta=8$ and $\tan\alpha=2$

(c) (iii) (2 marks)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards solutions	1
• shows correct solution	1

Sample Answer:

Consider the two paths and find time travelled to reach P

Pebble 1: $v=7\sqrt{10}$, $\theta=\beta$, $\tan\beta=8$ and $x=10$

$$t_1=\frac{10}{7\sqrt{10}\cos\beta}=\frac{10\sec\beta}{7\sqrt{10}} \text{ and } \sec^2\beta=1+\tan^2\beta=65$$

$$\therefore t_1=\frac{10\sqrt{65}}{7\sqrt{10}}$$

$$=\frac{\sqrt{650}}{7}$$

Pebble 2: $v=7\sqrt{10}$, $\theta=\alpha$, $\tan\alpha=2$ and $x=10$

$$t_2=\frac{10}{7\sqrt{10}\cos\alpha}=\frac{10\sec\alpha}{7\sqrt{10}} \text{ and } \sec^2\alpha=1+\tan^2\alpha=5$$

$$\therefore t_2=\frac{\sqrt{50}}{7}$$

$$\therefore t_1-t_2=\frac{\sqrt{650}-\sqrt{50}}{7} \text{ seconds}$$

Question 7 (12 marks)

(a) (3 marks)

Outcomes assessed: HE2

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes the truth of $S(2)$	1
• establishes the correct relationship between $S(k)$ and $S(k+1)$	1
• deduces the required result	1

Sample Answer:

Let $S(n)$ be the statement $2n^2>n^2+n+1$ for $n>1$

Consider $S(2)$: $2\times2^2=8$ and $2^2+2+1=7$

$\therefore 2n^2>n^2+n+1$ for $n=2$ and hence $S(2)$ is true

Assume $S(k)$ is true: $2k^2>k^2+k+1 \quad *$

RTP: $S(k+1)$ is true, ie prove $2(k+1)^2>(k+1)^2+(k+1)+1$

$2(k+1)^2=2k^2+4k+2$

$>k^2+k+1+4k+2 \quad$ if $S(k)$ is true using *

$=k^2+2k+1+3k+2$

$=(k+1)^2+(k+1)+1+2k$

$\therefore 2(k+1)^2>(k+1)^2+(k+1)+1$ since $k>0$

Hence if $S(k)$ is true then $S(k+1)$ is also true. Thus since $S(2)$ is true it follows by induction that $S(n)$ is true for positive integers $n>1$.

(b) (i) (2 marks)

Outcomes assessed: PE2, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• substitutes correctly	1
• shows correct result	1

Sample Answer:

$\cosh x=\frac{1}{2}\left(e^x+e^{-x}\right)$ and $\sinh x=\frac{1}{2}\left(e^x-e^{-x}\right)$

LHS $=2\sinh x\cosh x$

$$=2\times\frac{1}{2}\left(e^x+e^{-x}\right)\times\frac{1}{2}\left(e^x-e^{-x}\right)$$

$$=\frac{1}{2}\left(\left(e^x\right)^2-\left(e^{-x}\right)^2\right)$$

$$=\frac{1}{2}\left(e^{2x}-e^{-2x}\right)$$

$$=\sinh(2x)$$

$=$ RHS

(b) (ii) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct equation using given substitutions	1
• shows correct result	1

Sample Answer:

$p\cosh x + q\sinh x = r$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$$p \times \frac{1}{2}(e^x + e^{-x}) + q \times \frac{1}{2}(e^x - e^{-x}) = r$$
$$\frac{pe^x}{2} + \frac{pe^{-x}}{2} + \frac{qe^x}{2} - \frac{qe^{-x}}{2} = r$$
$$\frac{e^x}{2}(p+q) + \frac{1}{2e^x}(p-q) = r$$
$$e^{2x}(p+q) + (p-q) = 2re^x$$
$$(p+q)e^{2x} - 2re^x + (p-q) = 0$$

(b) (iii) (3 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• recognises that the equation is a quadratic or other progress towards the solution	1
• uses the discriminant or other progress towards the solution	1
• establishes correct conclusion	1

Sample Answer:

From (ii) the equation $p\cosh x + q\sinh x = r$ is equivalent to

$(p+q)e^{2x} - 2re^x + (p-q) = 0$, which is a quadratic in e^x

$$\begin{aligned}\Delta &= 4r^2 - 4(p+q)(p-q)\\ &= 4r^2 - 4(p^2 - q^2)\\ &= 4(r^2 - p^2 + q^2)\\ &= 0 \quad \text{since } p^2 = q^2 + r^2\end{aligned}$$

$\therefore$ the equation has only one solution

(b) (iv) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes the correct equation or other progress towards solution	1
• finds the correct solution	1

Sample Answer:

For the equation $13\cosh x + 5\sinh x = 12 \Rightarrow p = 13, \quad q = 5, \quad r = 12$

$\therefore (p+q)e^{2x} - 2re^x + (p-q) = 0$ becomes $18e^{2x} - 24e^x + 8 = 0$

ie solve $9e^{2x} - 12e^x + 4 = 0$

$$(3e^x - 2)^2 = 0$$
$$3e^x = 2$$
$$e^x = \frac{2}{3}$$
$$\therefore \quad x = \ln\left(\frac{2}{3}\right)$$

OR

Let $e^{2x} = y$ ie solve $9y^2 - 12y + 4 = 0$

$$(3y-2)^2 = 0$$
$$y = \frac{2}{3} \quad \Rightarrow \quad e^x = \frac{2}{3}$$
$$\therefore \quad x = \ln\left(\frac{2}{3}\right)$$

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